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Generalization of a Well-known Inequality

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Dedicated to Djairo De Figueiredo on his seventieth birthday

Section 1.
The well-known inequality refers to a nonnegative C^2 function u defined on an interval $(-R, R)$. The inequality is in:

Proposition 1. *Assume*

$$|\ddot{u}| \le M.$$

Then,

$$|\dot{u}(0)| \le \sqrt{2u(0)M} \qquad \text{if } M \ge \frac{2u(0)}{R^2} \tag{1}$$

$$|\dot{u}(0)| \le \frac{u(0)}{R} + \frac{R}{2}M \qquad \text{if } M < \frac{2u(0)}{R^2}. \tag{2}$$

This is sometimes called Glaeser′s inequality, see [2]; there it is attributed to Malgrange. It was used by Nirenberg and Treves in [3], where it is said that the inequality was probably known to Cauchy.

Here is the simple

Proof. For x in $(-R, R)$,

$$u(x) = u(0) + x\dot{u}(0) + \int_0^x (x - y)\ddot{u}(y)dy.$$

So

$$|\dot{u}(0)| \le \frac{u(0)}{|x|} + \frac{|x|}{2}M. \tag{3}$$

If $R \ge \sqrt{\frac{2u(0)}{M}}$, minimize the right hand side of (3) for $|x|$ on $(0, R)$. This yields (1). If $R < \sqrt{\frac{2u(0)}{M}}$, simply take $|x| = R$ — to get (2). □

The function $u = (x + R)^2$ shows that the constant $\sqrt{2}$ in (1) cannot be improved. We present here several generalizations to higher dimensions. Our first generalization is for C^2, nonnegative function u defined on a ball $B_R = \{|x| \le R\}$ in $\mathbb{R}^n$.

Proposition 2. *Assume*

$$\max|\Delta u| = M.$$

Then there is a constant C depending only on n such that

$$|\nabla u(x)| \le C\sqrt{u(0)M} \qquad \text{if } R \ge \sqrt{\frac{u(0)}{M}} \ge 2|x|, \tag{4}$$

$$|\nabla u(x)| \le C(\frac{u(0)}{R} + RM) \qquad \text{if } 2|x| \le R < \sqrt{\frac{u(0)}{M}}. \tag{5}$$

Question 1. *What is the best constant C in (4) for $x = 0$?*

Proof of Proposition 2. For $0 < r < R$, let v be the function which is harmonic in $|x| \le r$, with

$$v = u \quad \text{on } |x| = r.$$

Then $w = u - v$ satisfies

$$\begin{aligned} |\Delta w| = |\Delta u| &\le M \quad \text{in } B_r \\ w &= 0 \quad \text{on } \partial B_r. \end{aligned}$$

A standard inequality is

$$r|\nabla w(x)| + |w(x)| \le CMr^2, \qquad \forall\ |x| \le \frac{r}{2}. \tag{6}$$

Here, and from now on in this proof, C represents different positive constants depending only on n. Now, by the gradient estimates and the Harnack inequality,

$$r|\nabla v(x)| \le C \sup_{B_{\frac{3r}{4}}} v \le Cv(0), \qquad \forall\ |x| \le \frac{r}{2}. \tag{7}$$

Combining (6) and (7) we find

$$r|\nabla u(x)| \le C(Mr^2 + v(0)) \le C(Mr^2 + u(0) + CMr^2), \qquad \forall\ |x| \le \frac{r}{2}.$$

Thus

$$|\nabla u(x)| \le C(\frac{u(0)}{r} + Mr), \qquad \forall\ |x| \le \frac{r}{2}.$$

If $R \ge \sqrt{\frac{u(0)}{M}}$, we take $r = \sqrt{\frac{u(0)}{M}}$, and we obtain (4). If $R < \sqrt{\frac{u(0)}{M}}$, we take $r = R$, and we obtain (5). $\square$

Section 2.
Here is another simple generalization for $u \geq 0$ in B_R.

Proposition 3. *Suppose*

$$\|\Delta u\|_{L^p(B_R)} = M \quad \textit{for some } p > n.$$

Then

$$|\nabla u(x)| \leq C u(0)^{\frac{p-n}{2p-n}} M^{\frac{p}{2p-n}} \qquad \textit{if } R \geq (1-\frac{n}{p})^{\frac{p}{n-2p}} (\frac{u(0)}{M})^{\frac{p}{2p-n}} \geq 2|x|, \tag{8}$$

$$|\nabla u(x)| \leq C(\frac{u(0)}{R} + MR^{1-\frac{n}{p}}) \qquad \textit{if } 2|x| \leq R < (1-\frac{n}{p})^{\frac{p}{n-2p}} (\frac{u(0)}{M})^{\frac{p}{2p-n}}. \tag{9}$$

Here C is a constant depending only on n and p.

Proof. For $0 < r < R$, let v and w be defined in B_r as in the preceding proof. First we have

$$r|\nabla v(x)| \leq C \sup_{B_{\frac{3}{4}r}} v \leq C v(0), \qquad \forall\ |x| \leq \frac{r}{2}.$$

Next, by standard estimates, for $p > n$,

$$r|\nabla w(x)| + |w(x)| \leq CMr^{2-\frac{n}{p}}, \qquad \forall\ |x| \leq \frac{r}{2}. \tag{10}$$

Here $C = C(n,p)$. Hence

$$r|\nabla u(x)| \leq CMr^{2-\frac{n}{p}} + Cv(0) \leq CMr^{2-\frac{n}{p}} + Cu(0), \qquad \forall\ |x| \leq \frac{r}{2}$$

i.e.

$$|\nabla u(x)| \leq C(\frac{u(0)}{r} + Mr^{1-\frac{n}{p}}), \qquad \forall\ |x| \leq \frac{r}{2}. \tag{11}$$

The minimum of the right hand side of (11), with respect to r, is achieved when

$$-\frac{u(0)}{r^2} + (1-\frac{n}{p})Mr^{-\frac{n}{p}} = 0$$

i.e. when

$$r = (1-\frac{n}{p})^{\frac{p}{n-2p}} (\frac{u(0)}{M})^{\frac{p}{2p-n}}.$$

Arguing then as before, we obtain (8) and (9). □

Section 3.
What can we say if $u \geq 0$ and

$$M = \|\Delta u\|_{L^p(B_R)} \quad \text{for some } p \leq n? \tag{12}$$

If $p \in (\frac{n}{2}, n)$ we can obtain a Hölder continuity result with exponent

$$\alpha = 2 - \frac{n}{p} \tag{13}$$

in a form like (9). Namely we have

Proposition 4. *Suppose $u \geq 0$ in B_R and (12) holds with some $p \in (\frac{n}{2}, n)$. Then, for $x, y \in B_{R/2}$, $x \neq y$, and $\alpha = 2 - \frac{n}{p}$,*

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(\frac{u(0)}{R^\alpha} + MR^{2-\frac{n}{p}-\alpha}) \tag{14}$$

where C depends on n and p.

Proof. Let v and w be defined as before, but in the entire ball B_R. By standard elliptic estimates and the Harnack inequality, since $\frac{n}{2} < p < n$, we have, for $x \neq y$ in $B_{R/2}$,

$$R^\alpha \frac{|v(x) - v(y)|}{|x - y|^\alpha} \leq C \sup_{B_{\frac{3R}{4}}} v \leq Cv(0).$$

Also

$$|w(0)| + R^\alpha \frac{|w(x) - w(y)|}{|x - y|^\alpha} \leq CMR^{2-\frac{n}{p}}. \tag{15}$$

Combining these, we find, as before,

$$R^\alpha \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq CMR^{2-\frac{n}{p}} + Cu(0),$$

namely, (14). □

Remark 1. *More generally, suppose $p > \frac{n}{2}$. Let $0 < \alpha < 1$ be such that $p > \frac{n}{2-\alpha}$. Then the inequality (15) still holds, with $C = C(n, p, \alpha)$. Thus we find that for $u \geq 0$ in B_R and*

$$\|\Delta u\|_{L^p(B_R)} = M, \qquad p > \frac{n}{2}$$

then, in B_R,

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(\frac{u(0)}{R^\alpha} + MR^{2-\frac{n}{p}-\alpha}),$$

where $C = C(n, p, \alpha)$.

Section 4.

We extend Proposition 2 from Δ to second order elliptic operators with continuous coefficients. Consider

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x),$$

where a_{ij}, b_i, c are continuous functions in the unit ball B_1 of $\mathbb{R}^n$, $c(x) \leq 0$ for all $|x| < 1$, and, for some constants $0 < \lambda \leq \Lambda < \infty$,

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi^i\xi^j \leq \Lambda|\xi|^2, \qquad \forall\ |x| < 1, \forall\ \xi \in \mathbb{R}^n.$$

The extension of Proposition 2 concerns some $W^{2,p}$, $p > 1$, nonnegative function u defined on $B_R \subset \mathbb{R}^n$ for some $R \leq \frac{1}{2}$.

Proposition 5. *Assume the above and*

$$\max |Lu| = M.$$

Then there is a constant C depending only on $n, \lambda, \Lambda, \|b_i\|_{L^\infty(B_{\frac{3}{4}})}, \|c\|_{L^\infty(B_{\frac{3}{4}})}$, and the modulus of continuity of $a_{ij}(x)$ in $B_{\frac{3}{4}}$ such that (4) and (5) hold.

Proof. For $0 < r \leq R$, let v be the solution of

$$Lv = 0 \text{ in } B_r, \qquad v = u \text{ on } \partial B_r.$$

Then $w = v - u$ satisfies

$$|Lw| \leq M \text{ in } B_r, \qquad w = 0 \text{ on } \partial B_r.$$

By the $W^{2,p}$ estimates, (6) holds, where, and from now on in the proof, C denotes various positive constants depending only on $n, \lambda, \Lambda, \|b_i\|_{L^\infty(B_{\frac{3}{4}})}, \|c\|_{L^\infty(B_{\frac{3}{4}})}$, and the modulus of continuity of $a_{ij}(x)$ in $B_{\frac{3}{4}}$. Estimate (7) follows from the Harnack inequality of Krylov and Safonov, see [1]. The rest of the proof is identical to the corresponding part of the proof of Proposition 2. □

Section 5.

We extend Proposition 4 from Δ to operators L in Section 4. We assume $u \geq 0$ and

$$M = \|Lu\|_{L^p(B_R)}. \tag{16}$$

Proposition 6. *Let L be the operator in Section 4, we suppose $u \geq 0$ in B_R for some $R \leq \frac{1}{2}$ and (16) holds with some $p \in (\frac{n}{2}, n)$. Then for $x, y \in B_{\frac{R}{2}}$, $x \neq y$, and $\alpha = 2 - \frac{n}{p}$,*

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C(\frac{u(0)}{R^\alpha} + MR^{2-\frac{n}{p}-\alpha}) \tag{17}$$

where C depends on n, p, $\lambda, \Lambda, \|b_i\|_{L^\infty(B_{\frac{3}{4}})}, \|c\|_{L^\infty(B_{\frac{3}{4}})}$, and the modulus of continuity of $a_{ij}(x)$ in $B_{\frac{3}{4}}$.

Proof. It is similar to that of Proposition 4, with the help of $W^{2,p}$ estimates for L and the Harnack inequality of Krylov and Safonov. □

Remark 2. *If we take $p = n$ in Proposition 6, then by using the Hölder continuity estimate of Krylov and Safonov instead of the $W^{2,p}$ estimates in the proof of Proposition 6, inequality (17) holds for some positive constants α and C which depend on n, p, $\lambda, \Lambda, \|b_i\|_{L^\infty(B_{\frac{3}{4}})}$, and $\|c\|_{L^\infty(B_{\frac{3}{4}})}$, but independent of the modulus of continuity of a_{ij}.*

References

[1] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[2] G. Glaeser, Racine carrée d′une fonction différentiable, *Ann. Inst. Fourier* **13** (1963), 203-207.

[3] L. Nirenberg and F. Treves, Solvability of a first order linear partial differential equation, *Comm. Pure Appl. Math.* **16** (1963), 331-351.

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