Department of Mathematics, Mechanics and Informatics

The Compactness from Mountain Pass to Saddle Point

Ngo Quoc Anh

November 8, 2007

Ngo Quoc Anh, VNU

Outline

In this talk, we study the presence of compactness condition on some minimax theorems. Step by step, we shall discuss the geometric meaning and then state the theorem.

- Mountain Pass Theorem (MPT)
  - ⋆ A visual approach
  - ⋆ A counter-example
  - ⋆ The necessary of compactness condition
  - ⋆ State of the theorem
- Saddle Point Theorem (SPT)
  - ⋆ A visual approach
  - ⋆ A counter-example
  - ⋆ The necessary of compactness condition
  - ⋆ State of the theorem
- Some discussion

Idea of MPT

Consider a real function of two real independent variables $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which is continuously differentiable and satisfies the following condition: there exist $r>0$, $\mathbf{e}\in\mathbb{R}^{\mathbf{2}}$, $\|\mathbf{e}\|>r$ such that

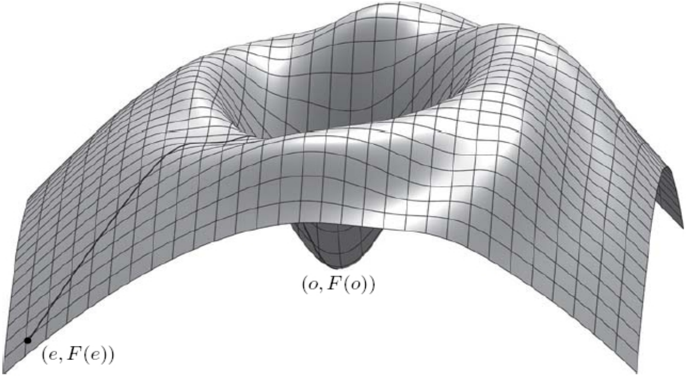
$$\inf_{\|\mathbf{x}\|=r} F(\mathbf{x}) > F(\mathbf{o}) \geqslant F(\mathbf{e}). \tag{1}$$

The first inequality and the Extreme Value Theorem immediately imply that F has a local minimum and thus a critical point in the set

$$\left\{\mathbf{x}\in\mathbb{R}^2 : \|\mathbf{x}\|<r\right\}.$$

The graph of such a function is sketched in the next frame .

Geometric meaning of MPT

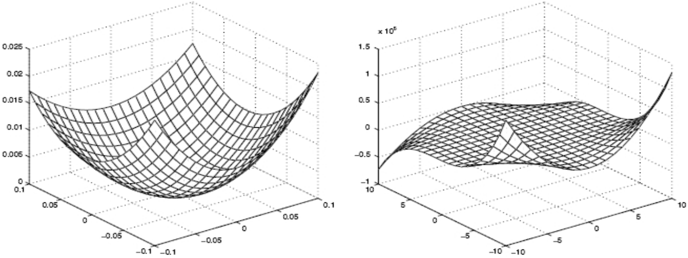


It seems that if we select the "highest" point with the "lowest" altitude corresponds to the value $c := \inf\limits_{\gamma \in \Gamma} \sup\limits_{u \in \gamma([0,1])} F(u)$, we have found a critical point of F where

$$\Gamma = \{\gamma \in C([0,1], \mathbb{R} \times \mathbb{R}) : \gamma(0) = \mathbf{o}, \gamma(1) = \mathbf{e}\}.$$

A counter-example by Nirenberg & Brézis

Let $F(x,y) = x^2 - (x-1)^3 y^2$, $r = \frac{1}{2}$, $e = (2,2)$ then $F(o) = F(e) = 0$ and $\inf\limits_{\|(x,y)\|=r} F(x,y) > 0$. This means that critical point obtained by the considerations made in previous frames is positive.



One easily sees that $(0,0)$ is the only critical point of F. So, it is natural to ask why this happens!!!

Reasons...

Such a situation corresponds, roughly speaking, to the fact that the altitude of the "highest" points approaches the critical value but the distance of these points from the origin diverges to $+\infty$.

More precisely, if $\mathbf{x}_n \in \mathbb{R}^2$ are such that

$$F(\mathbf{x}_n) = \max_{0 \leqslant t \leqslant 1} F(\gamma_n(t))$$

and

$$F(\mathbf{x}_n) \to c \tag{2}$$

then

$$\|\mathbf{x}_n\| \to +\infty.$$

It follows that the existence of r and $\mathbf{e}$ is not sufficient to guarantee the existence of a critical point which is different from the local minimum in $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < r\}$.

Reasons...

Such a situation corresponds, roughly speaking, to the fact that the altitude of the "highest" points approaches the critical value but the distance of these points from the origin diverges to $+\infty$.

More precisely, if $\mathbf{x}_n \in \mathbb{R}^2$ are such that

$$F(\mathbf{x}_n) = \max_{0 \leqslant t \leqslant 1} F(\gamma_n(t))$$

and

$$F(\mathbf{x}_n) \to c \tag{2}$$

then

$$\|\mathbf{x}_n\| \to +\infty.$$

It follows that the existence of r and $\mathbf{e}$ is not sufficient to guarantee the existence of a critical point which is different from the local minimum in $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < r\}$.

Reasons...

Such a situation corresponds, roughly speaking, to the fact that the altitude of the "highest" points approaches the critical value but the distance of these points from the origin diverges to $+\infty$.

More precisely, if $\mathbf{x}_n \in \mathbb{R}^2$ are such that

$$F(\mathbf{x}_n) = \max_{0 \leqslant t \leqslant 1} F(\gamma_n(t))$$

and

$$F(\mathbf{x}_n) \to c \tag{2}$$

then

$$\|\mathbf{x}_n\| \to +\infty.$$

It follows that the existence of r and $\mathbf{e}$ is not sufficient to guarantee the existence of a critical point which is different from the local minimum in $\left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < r\right\}$.

...and Palais-Smale condition

In fact, (1) guarantees the existence of a sequence $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ such that

$$F(\mathbf{x}_n) \to c \quad \text{and} \quad \nabla F(\mathbf{x}_n) \to o. \tag{3}$$

This leads us to assume for a moment that F satisfies the following compact condition (so-called Palais-Smale or (PS) for short).

Let $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ be such that $\{F(\mathbf{x}_n)\}_n$ is a bounded sequence in $\mathbb{R}$ and $\nabla F(\mathbf{x}_n) \to o$. Then $\{\mathbf{x}_n\}_n$ is a bounded sequence in $\mathbb{R}^2$.

Now, the situation described cannot occur. Indeed, by (3) and (PS) condition, there exists $\{\mathbf{x}_{n_k}\}_k \subset \{\mathbf{x}_n\}_n$ such that $\mathbf{x}_{n_k} \to \mathbf{x}_c$. Since $F \in C^1$ then $\nabla F(\mathbf{x}_c) = \mathbf{o}$.

...and Palais-Smale condition

In fact, (1) guarantees the existence of a sequence $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ such that

$$F(\mathbf{x}_n) \to c \quad \text{and} \quad \nabla F(\mathbf{x}_n) \to o. \tag{3}$$

This leads us to assume for a moment that F satisfies the following compact condition (so-called Palais-Smale or (PS) for short).

Let $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ be such that $\{F(\mathbf{x}_n)\}_n$ is a bounded sequence in $\mathbb{R}$ and $\nabla F(\mathbf{x}_n) \to o$. Then $\{\mathbf{x}_n\}_n$ is a bounded sequence in $\mathbb{R}^2$.

Now, the situation described cannot occur. Indeed, by (3) and (PS) condition, there exists $\{\mathbf{x}_{n_k}\}_k \subset \{\mathbf{x}_n\}_n$ such that $\mathbf{x}_{n_k} \to \mathbf{x}_c$. Since $F \in C^1$ then $\nabla F(\mathbf{x}_c) = \mathbf{o}$.

...and Palais-Smale condition

In fact, (1) guarantees the existence of a sequence $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ such that

$$F(\mathbf{x}_n) \to c \quad \text{and} \quad \nabla F(\mathbf{x}_n) \to o. \tag{3}$$

This leads us to assume for a moment that F satisfies the following compact condition (so-called Palais-Smale or (PS) for short).

Let $\{\mathbf{x}_n\}_n \subset \mathbb{R}^2$ be such that $\{F(\mathbf{x}_n)\}_n$ is a bounded sequence in $\mathbb{R}$ and $\nabla F(\mathbf{x}_n) \to o$. Then $\{\mathbf{x}_n\}_n$ is a bounded sequence in $\mathbb{R}^2$.

Now, the situation described cannot occur. Indeed, by (3) and (PS) condition, there exists $\{\mathbf{x}_{n_k}\}_k \subset \{\mathbf{x}_n\}_n$ such that $\mathbf{x}_{n_k} \to \mathbf{x}_c$. Since $F \in C^1$ then $\nabla F(\mathbf{x}_c) = \mathbf{o}$.

The Classical MPT

This is the Classical MPT due to Ambrosetti & Rabinowitz.

Theorem (Ambrosetti & Rabinowitz 1973)

Let V be a Banach space. Suppose $F \in C^1(V)$ satisfies (PS) condition. There exist $r > 0$ and $e \in V$ such that $\|e\| > r$ and

$$\inf_{\|u\|=r} F(u) > F(o) \geqslant F(e).$$

Define

$$\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = o, \gamma(1) = e\}.$$

Then

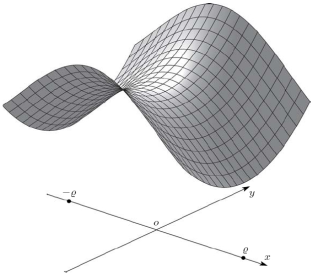
$$\beta = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} F(u)$$

is a critical value.

Idea of SPT

Consider a function $F \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Assume that there exist $\varrho > 0$ such that

$$\inf_{y \in \mathbb{R}} F(0, y) > \max\{F(-\varrho, 0), F(\varrho, 0)\}. \qquad (4)$$



The impression one can get from the graph is that

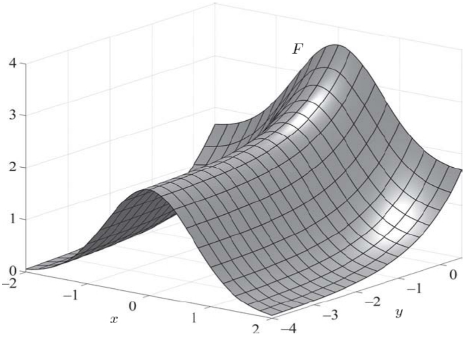
$$c = \inf_{\gamma \in \Gamma} \max_{t \in [-\varrho, \varrho]} F(\gamma(t))$$

is a critical value of F.

A counter-example...

Let $F(x,y) = 2e^{-x^2} + e^y$. Set $\varrho = 1$.

$$\inf_{y \in \mathbb{R}} F(0,y) = 2 > \frac{2}{e} + 1 = \max\{F(-1,0), F(1,0)\}.$$



The reason why the geometry condition (4) is not sufficient to guarantee the existence of a critical point of F is the same as in the MPT. This means that (PS) condition is needed.

...and the fact

This is the SPT due to Rabinowitz.

Theorem (Rabinowitz 1978)

Let $X = Y \oplus Z$ *be a Banach space with* Z *closed in* X *and* $\dim Y < \infty$*. For* $\varrho > 0$ *define*

$$\mathcal{M} := \{u \in Y : \|u\| \leqslant \varrho\} \text{ and } \mathcal{M}_0 := \{u \in Y : \|u\| = \varrho\}.$$

Let $F \in C^1(X, \mathbb{R})$ *be such that* $\inf\limits_{u \in Z} F(u) > \max\limits_{u \in \mathcal{M}_0} F(u)$*. If* F *satisfies (PS) condition with*

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} F(\gamma(u))$$

where

$$\Gamma := \{\gamma \in C(\mathcal{M}, X) : \gamma|_{\mathcal{M}_0} = I\}$$

then c *is a critical point of* F*.*

Here...

(PS) condition is crucial for the minimax theorems. It is called a compact condition in sense that if the set $\mathbb{K}_c$ of critical points of functional Φ at level c, that is,

$$\mathbb{K}_c = \left\{ u \in X : \Phi(u) = c, \Phi'(u) = 0 \right\}$$

then $\mathbb{K}_c$ is compact.

Even that $\mathbb{K}_c$ is compact, it has no influence on the size of critical points. In such situations, Morse theory and further advanced methods are needed to study.

Condition (PS) may seem rather restrictive.

...and beyond

Actually, for quite a while many mathematicians felt convinced that in spite of its success in dealing with 1D variational problems like geodesics, the (PS) could never play a role in the solution of "interesting" variational problems in higher dimensions.

Recent advances in the Calculus of Variations have changed this view and it has become apparent that the methods of Palais and Smale apply to many problems of physical and/or geometric interest and-in particular-that the (PS) will in general hold true for such problems in a broad range of energies.

Moreover, the failure of (PS) at certain levels reflects highly interesting phenomena related to internal symmetries of the systems under study, which geometrically can be described as separation of spheres, or mathematically as "singularities", respectively as "change in topology".

Ngo Quoc Anh, VNU

Thank you...