Graduate Analysis I

Final Examination

Semester 1, 2008-2009

Department of Mathematics,
National University of Singapore

Time allowed: $2\frac{1}{2}$ hours

Question 1. [25 marks]

(i) Show that if f is ocntinuous over $(a, b]$ except at $x = a$ and if f is Lebesgue integrable over $(a, b]$, then f is improper integrable, i.e., the limit $\lim_{t\to a+} \int_t^b f(x)dx$ exists and is finite.

(ii) Show that $\frac{1}{x}$ is not Lebesgue integrable over $[0, 1]$.

(iii) Show that it $f(x)$ is Lebesgue integrable over $[0, a]$, then for any $k > 0$, $f(kx)$ is integrable over $[0, a/k]$.

(iv) Show that $f(x) = \frac{1}{x}\cos\left(\frac{2}{x}\right)$ if $x \in (0, 1]$ and $f(0) = 0$ is not Lebesgue integrable but its improper integral exists.

Question 2. [20 marks]

Define $f(x) = x^2$ if x is a rational number in the interval $[0, 1]$ and $f(x) = x^3$ if x is an irrational number in the same interval.

(i) Show that for all $t > 0$, $\ln(1 + e^t) < t + \ln 2$.

(ii) Use (i) or otherwise to show that

$$\lim_{n\to\infty} \frac{1}{n}\int_0^1 \ln\left(1 + e^{nf(x)}\right) dx = \frac{1}{4}.$$

Question 3. [20 marks]

Show that if $f \in L^p(\mathbb{R}^n)$ and $K \in L^{p'}(\mathbb{R}^n)$ with $1 \le p <\le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then the convolution $f * K$ is bounded and continuous in $\mathbb{R}^n$.

Question 4. [15 marks]

Suppose $f_k \to f$ in $L^p(E)$, $1 \le p < \infty$, for a measurable set E and $g_k \to g$ point-wise in E, $\|g\|_{L^\infty} \le M$ for all k. Show that $f_k g_k \to fg$ in $L^p(E)$.

Question 5. [20 marks]

Show that if $f \in L^1(\mathbb{R}^n)$, then the maximal function $f^\star \in L^q(E)$ for any $0 < q < 1$ and any measurable set E with finite measure.