National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 1**

1. Find $\limsup E_k$ and $\liminf E_k$ for $E_k = [-\frac{1}{k}, 1]$ for k odd, and $E_k = [-1, \frac{1}{k}]$ for k even.

2. Show that any closed subset of a compact set is compact.

3. Give an example of two disjoint closed subsets F_1 and F_2 in $\mathbb{R}^n$ for which $\operatorname{dist}(F_1, F_2) = 0$.

4. Suppose A is a closed set of $\mathbb{R}$ such that for all rational numbers $0 \leq r \leq 1$, $r \in A$. Show that $[0, 1] \subset A$.

5. Let U be an open set in $\mathbb{R}^n$ and $C \subset U$ is proper compact set of U. Show that there exists a compact set D such that C is a subset of the interior of D and $D \subset U$.

6. Show that any linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a continuous transformation.

7. Let $A := \{(x, y) \in \mathbb{R}^2 | x > 0 \text{and } 0 < y < x^2\}$.

   (i) Show that any line through $(0, 0)$ contains an interval $\subset \mathbb{R}^2 \backslash A$;

   (ii) Let $f(x) = \chi_A(x)$; i.e., $f(x) = 0$ if $x \notin A$ and $f(x) = 1$ if $x \in A$. For any vector $h \in \mathbb{R}^2$, we define $g_h : \mathbb{R} \longrightarrow \mathbb{R}$ by $g_h(t) = f(th)$. Prove that for each fixed h, g_h is continuous at 0, but f is not continuous at $(0, 0)$.

8. Let $\{x_k\}$ be a bounded infinite sequence of $\mathbb{R}^n$. Show that $\{x_k\}$ has a limit point.

9. Give an example of a decreasing sequence of non-empty closed subsets of $\mathbb{R}^n$ whose intersection is empty.

10. If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.