National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 2**

1. Let $\{f_k\}$ be a sequence of functions of bounded variation on $[a,b]$. If $V(f_k;a,b) \leq M < \infty$, for all k and $f_k \to f$ point-wise on $[a,b]$ as $k \to \infty$, show that f is of bounded variation and that $V(f;a,b) \leq M$. Give an example of a convergent sequence of functions of bounded variation whose limit is not of bounded variation.

2. Suppose f is of bounded variation on $[a,b]$. If f is continuous on $[a,b]$, show that $V(f;a,x)$, $P(f;a,x)$ and $N(f;a,x)$ are all continuous on $[a,b]$.

3. Show that if f is of bounded variation on $(-\infty,\infty)$, then f is the difference of two increasing bounded functions.

4. Let C be a curve with parametric equations $x = \phi(t)$ and $y = \psi(t)$ for $a \leq t \leq b$. Show that (i) if ϕ and ψ are both of bounded variation and continuous, then $L = \lim_{|\Gamma|\to 0} l(\Gamma)$; (ii) if ϕ and ψ are continuously differentiable, show that $L = \int_a^b [(\phi'(t))^2 + (\psi'(t))^2]^{1/2} dt$.

5. Show that $\int_a^b f d\phi$ exists if and only if for any given $\epsilon > 0$, there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \epsilon$ if $|\Gamma| < \delta$ and $|\Gamma'| < \delta$.

6. Give an example for f and ϕ such that $\int_a^c f d\phi$ and $\int_c^b f d\phi$ both exist, but $\int_a^b f d\phi$ does not exist.

7. Suppose f is continuous and ϕ is of bounded variation on $[a,b]$. Show that the function $\psi(x) = \int_a^x f d\phi$ is of bounded variation. Also show that if g is a continuous function on $[a,b]$, then $\int_a^b g d\psi = \int_a^b g f d\phi$.

8. Let ϕ be of bounded variation on $(-\infty,\infty)$ and $f(x)$ be continuous on $(-\infty,\infty)$ such that $\lim_{|x|\to\infty} f(x) = 0$. Show that $\int_{-\infty}^{\infty} f d\phi$ exists.

9. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ be finite sequence and $-\infty < s < \infty$. If $\{a_k\}$ for $k = 1, 2, \cdots, m$ is another finite sequence, write $\sum_{k=1}^m a_k e^{-s\lambda_k}$ as a Riemann-Stieltjes integral.