National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 3**

1. Construct a two-dimensional unmeasurable set.
2. Show that the Borel $\sigma-$algebra $\mathcal{B}$ in $\mathbf{R}^n$ is the smallest $\sigma-$algrbra containing the closed sets in $\mathbf{R}^n$.
3. If $\{E_k\}, k = 1, 2, \cdots$ is a sequence of sets with $\sum_{k=1}^{\infty} |E_k|_e < \infty$, show that both $\limsup E_k$ and $\liminf E_k$ have measure zero.
4. If E_1 and E_2 are measurable sets, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.
5. If E_1 and E_2 are measurable subsets of $\mathbf{R}^1$, then $E_1 \times E_2$ is a measurable subset of $\mathbf{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$.
6. We can define the inner measure of a set E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that (i) $|E|_i \leq |E|_e$ and (ii) if $|E|_e < \infty$, then E is measurable if and only if $|E|_i = |E|_e$. (Use Lemma 3 in section 3 according to Lecture notes).
7. Show that the second part of previous question is false if $|E|_e = \infty$.
8. If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E - A|_e$.
9. Give an example which shows that the image of a measurable set under a continuous transformation may not be measurable.
10. Show that there exist sets $E_1, E_2, \cdots, E_k, \cdots$ such that $E_k \searrow E$, and $|E_k|_e < \infty$ and $\lim_{k\to\infty} |E_k|_e > |E|_e$ with strict inequality.
11. Show that there exist disjoint $E_1, E_2, \cdots, E_k, \cdots$ such that $| \cup E_k|_e < \sum |E_k|_e$ with strict inequality.
12. Let E_1 and E_2 be open sets in $\mathbf{R}^1$ and E_1 be a proper subset of E_2. Does this imply that $|E_1| < |E_2|$?
13. Construct a closed subset $F \subset [0, 1]$ such that F does not contain any open interval and $|F| = \frac{1}{2}$.