National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 4**

1. let f be a simple function, taking its distinct values on disjoint sets $E_1, E_2, \cdots, E_n$. Show that f is measurable if and only if $E_1, E_2, \cdots, E_n$ are measurable.

2. Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(Tx)$ is measurable.[If $E_1 = \{x : f(x) > a\}$ and $E_2 = \{x : f(Tx) > a\}$, show that $E_2 = T^{-1}(E_1)$.]

3. Give an example to show that $\phi(f(x))$ may not be measurable if ϕ and f are both measurable. [Let F be the Cantor-Lebesgue function and let f be its inverse suitably defined. Let ϕ be the characteristic function of a set of measure zero whose image under F is not measurable.]

4. Let f be U.S.C. and less than $+\infty$ on a compact set. Show that f is bounded above on E. Also show that there exists a point $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

5. If $\{f_k\}$ is a sequence of functions which are U.S.C. at x_0, show that $\inf_k f_k(x)$ is also U.S.C.

6. if f is defined and continuous on E, show that $\{a < f < b\}$ is relatively open and that $\{a \leq f \leq b\}$ and $\{f = a\}$ are both relatively closed.

7. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(x)| \leq M_x$ for all k and for each $x \in E$, show that given $\epsilon > 0$, there is a closed $F \subset E$ and finite M such that $|E - F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

8. Prove that $f_k \to f$ in measure on E if and only if given $\epsilon > 0$, there exists $K > 0$ such that $|\{|f_k - f| > \epsilon\}| < \epsilon$ if $k > K$.

9. If f is measurable and finite a.e. on $[a, b]$, show that given $\epsilon > 0$, there is a continuous function g on $[a, b]$ such that $|\{x : f(x) \neq g(x)\}| < \epsilon$.