National University of Singapore

Department of Mathematics

08/09 Semester I   MA5205   Graduate Analysis I   **Assignment 5**

1. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on a measurable set E. If $f_k \to f$ and $f_k \le f$ almost everywhere on E, show that $\int_E f_k \to \int_E f$. Give an example of a sequence $\{f_k\}$ such that $f_k \ge 0$ and $f_k \to f$ almost everywhere but $\int_E f_k$ does not converge to $\int_E f$.

2. Let $f \in L([0,1])$, show that $x^k f \in L([0,1])$ for all k and $\int_0^1 x^k f(x)dx \to 0$.

3. Let $f(x,y)$ with $0 \le x, y \le 1$ satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y and $(\partial f/\partial x)$ is a bounded function of (x,y). Show that $\partial f/\partial x$ is a measurable function of y for each x and
$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy.$$

4. Given a non-Lebesgue integrable function such that its improper Riemann integral exists and is finite.

5. For $p > 0$ and $\int_E |f_k - f|^p \to 0$ as $k \to \infty$, show that $f_k \to f$ in measure on E. And further show that there exists a subsequence f_{k_i} such that $f_{k_i} \to f$ almost everywhere on E.

6. Given an example of a bounded continuous function f on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$, but $f \notin L^p(0,\infty)$ for any $p > 0$.

7. Suppose $f \ge 0$ and $\omega(\alpha) \le c(1+\alpha)^{-p}$ for all $\alpha > 0$, show that $f \in L^r$ for $0 < r < p$.

8. Suppose $f \ge 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{pk}\omega(2^k) < \infty$.

9. If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

10. Let $f(x)$ be an integrable function on $(-\infty,\infty)$. Show that
$$g(x) = \int_{-\infty}^{\infty} e^{-ixt} f(t)dt$$

is a continuous function on $(-\infty, \infty)$ where $i = \sqrt{-1}$ and

$$g(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-itx} - 1}{it} f(t)dt.$$

11. Suppose $f(x), f_n(x)$ are integral functions on a measurable set E, $f_n \to f$ a.e. on E and $\int_E |f_n| \to \int_E |f|$ as $n \to \infty$. Show that for every measurable subset A, $\int_A |f_n| \to \int_A |f|$ as $n \to \infty$.

12. Suppose f_n are integrable functions on a measurable set E and $f_n \to f$ a.e. on E as $n \to \infty$. And also assume that there exists a constant $K > 0$ such that $\int_E |f_n| < K$. Show that f is integrable on E.

13. Let (a, b) be a finite open interval and f an integrable function on (a, b). Show that $\lim_{t\to\infty} \int_a^b e^{itx} f(x)dx = 0$.

14. Suppose f is an integrable function on a measurable set E and for every bounded measurable function $\phi(x)$ on E, we have $\int_E f(x)\phi(x) = 0$. Show that $f = 0$ a.e. on E.

15. Suppose f is monotone increasing function on the interval $[a, b]$. Show that $f'(x)$ is finite a.e. on E.

16. Suppose f is monotone increasing function on $[a, b]$. Show that $f'(x)$ is integrable and

$$\int_a^b f'(x) \leq f(b) - f(a).$$