National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 6**

1. If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their Cartesian product $E_1 \times E_2 = \{(x,y) : x \in E_1, y \in E_2\}$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$ and $|E_1 \times E_2| = |E_1||E_2|$.

2. Let f be measurable on $(0,1)$. If $f(x) - f(y)$ is integrable over the square $0 \le x, y \le 1$, show that $f \in L(0,1)$.

3. Let f be measurable and periodic with period $1 : f(x+1) = f(x)$. Suppose that there is a finite c such that $\int_0^1 |f(a+t) - f(b+t)|dt \le c$ for all a and b. Show that $f \in L(0,1)$. (Hint: set $a = x$ and $b = -x$, integrate with respect to x and make the change of variable $\eta = x + t$ and $\xi = t - x$.)

4. Let F be a closed subset of $\mathbb{R}^1$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} dy$$

is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Marcinkiewiez's theorem.)

5. Under the hypotheses of Marcinkiewiez's theorem and assuming $b - a < 1$, show that the function

$$M_0(x) = \int_a^b [\log(1/\delta(y))]^{-1} |x-y|^{-1} dy$$

is finite a.e. in F.

6. Show that $M_\lambda(x, F) = \infty$ if $x \notin F, \lambda > 0$.

7. Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show, by Fubini's theorem, that $v_n = 2v_{n-1} \int_0^1 (1-t^2)^{(n-1)/2} dt$.

8. Use Fubini's theorem to prove that $\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$.