National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 7**

1. Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(x) \geq c|x|^{-n}$.

2. Let $\phi(x), x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For any $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. If $f \in L(\mathbb{R}^n)$, show that $\lim_{\epsilon\to 0}(f * \phi_\epsilon)(x) = f(x)$ in the Lebesgue set of f. [Note that $\int \phi_\epsilon = 1$ for any $\epsilon > 0$, so that $f * \phi_\epsilon(x) - f(x) = \int[f(x-y) - f(x)]\phi_\epsilon(y)dy$. Use the corollary of Hardy-Littlewood theorem.]

3. If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$ with $|E_1| > 0$ and $|E_2| > 0$. Show that the set $\{x \in \mathbb{R}^1 : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$ contains an open interval with positive length.

4. Let f be of bounded variation on $[a, b]$. If $f = g + h$ where g is absolutely continuous and h is singular, show that, for any function ϕ, we have
$$\int_a^b \phi df = \int_a^b \phi f' dx + \int_a^b \phi dh.$$

5. Show that if $\alpha > 0$, x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

6. Show that f is absolutely continuous on $[a, b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_k) - f(a_k)]| < \epsilon$ for any finite collection $\{[a_k, b_k]\}$ of non-overlapping subintervals of $[a, b]$ with $\sum(b_k - a_k) < \delta$.

7. Show that if f is of bounded variation on $[a, b]$ and if the function $V(x) = V(f, a, x)$ is absolutely continuous, then f is absolutely continuous on $[a, b]$.

8. Show that if f is of bounded variation on $[a, b]$, then $\int_a^b |f'| \leq V(f, a, b)$. And also show that if the equality holds, then f is absolutely continuous on $[a, b]$.

9. Show that an absolutely continuous function maps a measure zero set into a measure zero set.

10. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be bounded and measurable on $[a, b]$ with $a = g(\alpha)$ and $b = g(\beta)$. Show that $f(g(t))g'(t)$ is a measurable function on $[\alpha, \beta]$ and $\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt$.

11. Show that if ϕ is convex if and only if it is continuous and $\phi(\frac{x_1+x_2}{2}) \leq \frac{\phi(x_1)+\phi(x_2)}{2}$, for all $x_1, x_2 \in (a, b)$.

12. Show that if $\phi(x) = \int_a^x f(t)dt + \phi(a)$ in (a, b) and f is monotone increasing, then ϕ is convex in (a, b).