National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 8**

1. Prove the converse of Hölder inequality for $1 \leq p \leq \infty$(namely Theorem 6 in lecture notes). Show also that for real valued $f \notin L^p(E)$, there exists a function $g \in L^{p'}(E)$, $1/p + 1/p' = 1$, such that $fg \notin L^1(E)$.

2. Prove Minkowski's inequality in both L^p and l^p with $p \geq 1$. And show that for l^p space, this inequality does not hold if $0 < p < 1$.

3. Let $f, f_k \in L^p$. Show that if $\|f - f_k\|_{L^p} \to 0$ as $k \to \infty$, then $\|f_k\|_{L^p} \to \|f\|_{L^p}$. And conversely if $f_k \to f$ a.e. and $\|f_k\|_{L^p} \to \|f\|_{L^p}$ as $k \to \infty$, show that $\|f - f_k\|_{L^p} \to 0$ provided $1 \leq p < \infty$.

4. If $f_k \to f$ in $L^p(E)$ as $k \to \infty$ and $1 \leq p < \infty$, $g_k \to g$ point-wise and $\|g_k\|_{L^\infty} \leq M$ is bounded for all k, show that $f_k g_k \to fg$ in L^p.

5. If $f \in L^p(\mathbb{R}^n)$ and $0 < p < \infty$, show that

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)| dy = 0 \ \ a.e..$$

6. Show that for $0 < p < 1$, the unit open ball with center at 0 is not convex.

7. Assume $p \geq 1$. A sequence $\{f_k\} \in L^p$ is said to weakly converge to f in L^p if for any function $g \in L^{p'}, 1/p + 1/p' = 1$, $\int_E f_k g \to \int_E fg$. Show that if $f_k \to f$ in measure and $\|f_k\|_{L^p} \leq M$ for some constant M, then $f_k \to f$ weakly.

8. Suppose $f \in L^p(E)$ and $A \subset E$ is a measurable subset of E. Show that

$$\{\int_E |f|^p\}^{1/p} \leq \{\int_A |f|^p\}^{1/p} + \{\int_{E \setminus A} |f|^p\}^{1/p}.$$

9. Let p, q, r be three positive numbers such that $1/p + 1/q + 1/r = 1$. Suppose $f \in L^p(E), g \in L^q(E)$ and $h \in L^r(E)$ for a measurable set E. Show that

$$|\int_E fgh| \leq \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$