National University of Singapore

Department of Mathematics

08/09 Semester I MA5205 Graduate Analysis I **Assignment 9**

1. (a) Show that the function h defined by $h(x) = e^{-1/x^2}$ for $x > 0$ and $h(x) = 0$ for $x \le 0$ is in $C^\infty(\mathbb{R}^1)$.
   (b) Show that the function $g(x) = h(x-a)h(b-x)$, $a < b$ is C^∞ with support $[a, b]$.
   (c) Construct a function in $C_0^\infty(\mathbb{R}^n)$ whose support is a ball or an interval.

2. Let G and G_1 be bounded open subsets of $\mathbb{R}^n$ such that $\bar{G}_1 \subset G$. Construct a function $h \in C_0^\infty$ such that $h = 1$ in G_1 and $h = 0$ outside G.

3. Let $f \in L^p(-\infty, +\infty)$ with $1 \le p \le \infty$. Show that the Poisson integral of f is harmonic in the upper half-plane $y > 0$.

4. For $s, t \ge 0$, let $K(s,t)$ satisfy $K \ge 0$ and $K(\lambda s, \lambda t) = \lambda^{-1} K(s,t)$ for all $\lambda > 0$, and suppose that $\int_0^\infty t^{-1/p} K(1,t)dt = \gamma < \infty$ for some p with $1 \le p \le \infty$. Show that if
$$(Tf)(s) := \int_0^\infty f(t)K(s,t)dt \quad (f \ge 0),$$
then $\|Tf\|_{L^p} \le \gamma \|f\|_{L^p}$.

5. The maximal function is defined as $f^*(x) = \sup |Q|^{-1} \int_Q |f|$, where the supremum is taken over cubes with center x. Let f^{**} be defined similarly, but with the supremum taken over all Q containing x. Thus $f^* \le f^{**}$. Show that there is a positive constant c depending only on the dimension such that $f^{**}(x) \le cf^*(x)$.

6. Let $f_\epsilon = f * K_\epsilon$ where $K \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} K = \gamma$. If $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$, show that $\|f_\epsilon - \gamma f\|_{L^p} \to 0$ as $\epsilon \to 0$.

7. Let $f \in L^p(0,1)$, $1 \le p < \infty$, and for each $k = 1, 2, \cdots$, define a function f_k on $(0,1)$ by letting $I_{k,j} = \{x : (j-1)2^{-k} \le x < j2^{-k}\}$, $j = 1, 2, 3, \cdots, 2^k$, and setting $f_k(x) = |I_{k,j}|^{-1} \int_{I_{k,j}} f$ for $x \in I_{k,j}$. Show that $f_k \to f$ in $L^p(0,1)$.

8. Suppose $f \in L^1(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n)$ with $f \ge 0$. Define
$$Sf(x) = \int_{\mathbb{R}^n} [\ln \frac{|y|}{|x-y|}] f(y) dy.$$
Show that $Sf(x)$ is finite for all $x \in \mathbb{R}^n$ and $Sf \in L^1_{\text{loc}}(\mathbb{R}^n)$.