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NOTES ON AN INTEGRAL INEQUALITY

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ABSTRACT. In this paper, some integral inequalities are presented by analytic approach. An open question will be proposed later on.

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1. INTRODUCTION

Let $f(x)$ be a continuous function on $[0, 1]$ satisfying

$$\int_x^1 f(t)\,dt \geq \frac{1-x^2}{2}, \quad \forall x \in [0,1]\,. \tag{1.1}$$

Firstly, we consider an integral inequality below.

Lemma 1.1. *If (1.1) holds then we have*

$$\int_0^1 [f(x)]^2\,dx \geq \int_0^1 xf(x)\,dx. \tag{1.2}$$

The aim of this paper is to generalize (1.2) in order to obtain some new integral inequalities. In the first part of this paper, we will prove Lemma 1.1 and present some preliminary results. Our main results are Theorem 2.1, Theorem 2.2 which will be proved in Section 2 and Theorem 3.2, Theorem 3.3 which will be proved in Section 3. Finally, an open question is proposed. And now, we begin with a proof of Lemma 1.1.

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Proof of Lemma 1.1. It is known that

$$0 \leq \int_0^1 (f(x) - x)^2 \, dx = \int_0^1 f^2(x) \, dx - 2\int_0^1 x f(x) \, dx + \int_0^1 x^2 dx,$$

which yields

$$\int_0^1 f^2(x) \, dx \geq 2\int_0^1 x f(x) \, dx - \frac{1}{3}.$$

Let $A := \int_0^1 \left(\int_x^1 f(t) \, dt\right)$. By using our assumption we have

$$A = \int_0^1 \left(\int_x^1 f(t) \, dt\right) \geq \int_0^1 \frac{1 - x^2}{2} dx = \frac{1}{3}.$$

On the other hand, integrating by parts, we also get

$$\begin{aligned} A &= \int_0^1 \left(\int_x^1 f(t) \, dt\right) \\ &= x \int_x^1 f(t) \, dt \Big|_0^1 + \int_0^1 x f(x) \, dx \\ &= \int_0^1 x f(x) \, dx. \end{aligned}$$

Thus

$$\int_0^1 x f(x) \, dx \geq \frac{1}{3},$$

which gives the conclusion. □

Remark 1.2. Condition (1.1) can be rewritten as

$$\int_x^1 f(t) \, dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1]. \tag{1.3}$$

Throughout this paper, we always assume that function f satisfies (1.1), moreover, we also assume that

$$f(x) \geq 0 \tag{1.4}$$

for every $x \in [0, 1]$.

Lemma 1.3. *$\int_0^1 x^{n+1} f(x) \, dx \geq \frac{1}{n+3}$ for all $n \in \mathbb{N}$.*

Proof. We have

$$\begin{aligned} \int_0^1 x^n \left(\int_x^1 f(t) \, dt\right) dx &= \frac{1}{n+1} \int_0^1 \left(\int_x^1 f(t) \, dt\right) d\left(x^{n+1}\right) \\ &= \frac{1}{n+1} x^{n+1} \int_x^1 f(t) \, dt \Big|_{x=0}^{x=1} \\ &\quad + \frac{1}{n+1} \int_0^1 x^{n+1} f(x) \, dx, \end{aligned}$$

which yields

$$\int_0^1 x^{n+1} f(x) \, dx = (n+1) \int_0^1 x^n \left(\int_x^1 f(t) \, dt\right) dx.$$

On the other hand

$$\int_0^1 x^n \left(\int_x^1 f(t)\,dt \right) dx \geq \int_0^1 x^n \frac{1-x^2}{2} dx.$$

Therefore

$$\int_0^1 x^{n+1} f(x)\,dx \geq (n+1) \int_0^1 x^n \frac{1-x^2}{2} dx = \frac{1}{n+3}.$$

The proof is completed. □

2. THE CASE OF NATURAL NUMBERS

Theorem 2.1. *Assume that (1.1) and (1.4) hold. Then*

$$\int_0^1 f^{n+1}(x)\,dx \geq \int_0^1 x^n f(x)\,dx$$

for every $n \in \mathbb{N}$.

Proof. By using the Cauchy inequality, we obtain

$$f^{n+1}(x) + n x^{n+1} \geq (n+1) x^n f(x).$$

Thus

$$\int_0^1 f^{n+1}(x)\,dx + n \int_0^1 x^{n+1} dx \geq (n+1) \int_0^1 x^n f(x)\,dx.$$

Moreover, by using Lemma 1.3, we get

$$\begin{aligned}(n+1) \int_0^1 x^n f(x)\,dx &= n \int_0^1 x^n f(x)\,dx + \int_0^1 x^n f(x)\,dx \\ &\geq \frac{n}{n+2} + \int_0^1 x^n f(x)\,dx,\end{aligned}$$

that is

$$\int_0^1 f^{n+1}(x)\,dx + \frac{n}{n+2} \geq \frac{n}{n+2} + \int_0^1 x^n f(x)\,dx,$$

which completes this proof. □

Theorem 2.2. *Assume that (1.1) and (1.4) hold. Then*

$$\int_0^1 f^{n+1}(x)\,dx \geq \int_0^1 x f^n(x)\,dx$$

for every $n \in \mathbb{N}$.

Proof. It is known that

$$(f^n(x) - x^n)(f(x) - x) \geq 0, \quad \forall x \in [0,1],$$

that is

$$f^{n+1}(x) + x^{n+1} \geq x^n f(x) + x f^n(x), \quad \forall x \in [0,1].$$

By integrating with some simple calculation we conclude that

$$\int_0^1 f^{n+1}(x)\,dx + \frac{1}{n+2} \geq \int_0^1 x^n f(x)\,dx + \int_0^1 x f^n(x)\,dx.$$

Once again, by Lemma 1.3, we obtain

$$\int_0^1 f^{n+1}(x)\,dx + \frac{1}{n+2} \geq \frac{1}{n+2} + \int_0^1 x f^n(x)\,dx,$$

which gives the conclusion. □

Remark 2.3. By the same argument, we see that the result of Lemma 1.3 also holds when n is a positive real number. That is

$$\int_0^1 x^{\alpha+1} f(x)\,dx \geq \frac{1}{\alpha+3}, \quad \forall\alpha > 0. \tag{2.1}$$

3. THE CASE OF REAL NUMBERS

In order to generalize our results, the case of positive real numbers, we recall another version of the Cauchy inequality as follows.

Theorem 3.1 (General Cauchy inequality)**.** *Let α and β be positive real numbers satisfying $\alpha+\beta=1$. Then for every positive real numbers x and y, we always have*

$$\alpha x + \beta y \geq x^\alpha y^\beta.$$

Theorem 3.2. *Assume that (1.1) and (1.4) hold. Then*

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x^\alpha f(x)\,dx$$

for every positive real number $\alpha > 0$.

Proof. Using Theorem 3.1 we get

$$\frac{1}{\alpha+1} f^{\alpha+1}(x) + \frac{\alpha}{\alpha+1} x^{\alpha+1} \geq x^\alpha f(x),$$

which gives

$$\frac{1}{\alpha+1}\int_0^1 f^{\alpha+1}(x)\,dx + \frac{\alpha}{\alpha+1}\int_0^1 x^{\alpha+1} dx \geq \int_0^1 x^\alpha f(x)\,dx.$$

By the same argument together with (2.1) we obtain

$$\begin{aligned} \frac{1}{\alpha+1}\int_0^1 f^{\alpha+1}(x)\,dx &+ \frac{\alpha}{(\alpha+1)(\alpha+2)} \\ &\geq \frac{1}{\alpha+1}\int_0^1 x^\alpha f(x)\,dx + \frac{\alpha}{\alpha+1}\int_0^1 x^\alpha f(x)\,dx \\ &\geq \frac{1}{\alpha+1}\int_0^1 x^\alpha f(x)\,dx + \frac{\alpha}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

Hence

$$\frac{1}{\alpha+1}\int_0^1 f^{\alpha+1}(x)\,dx \geq \frac{1}{\alpha+1}\int_0^1 x^\alpha f(x)\,dx.$$

The present proof is completed. □

Theorem 3.3. *Assume that (1.1) and (1.4) hold. Then*

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x f^\alpha(x)\,dx$$

for every positive real number $\alpha > 0$.

The proof of Theorem 3.3 is similar to the proof of Theorem 2.2 therefore, we omit it.

Lastly, we propose the following open problem.

Open Problem. *Let $f(x)$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \quad \forall x \in [0,1].$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 t^{\alpha} f^{\beta}(x)\,dx.$$

hold for α and β?

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