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NOTES ON AN OPEN PROBLEM OF F. QI AND Y. CHEN AND J. KIMBALL

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ABSTRACT. In this paper, an integral inequality is studied. An answer to an open problem proposed by Feng Qi and Yin Chen and John Kimball is given.

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1. INTRODUCTION

In [6], Qi studied an interesting integral inequality and proved the following result

Theorem 1.1 (Proposition 1.1, [6]). *Let $f(x)$ be continuous on $[a,b]$, differentiable on (a,b) and $f(a)=0$. If $f'(x)\geq 1$ for $x\in(a,b)$, then*

$$\int_a^b f^3(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^2. \tag{1.1}$$

If $0\leq f'(x)\leq 1$, then the inequality (1.1) reverses.

Qi extended this result to a more general case [6], and obtained the following inequality (1.2).

Theorem 1.2 (Proposition 1.3, [6]). *Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a)\geq 0$, where $0\leq i\leq n-1$,*

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and $f^{(n)}(x) \geq n!$, *then*

$$\int_a^b f^{n+2}(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^{n+1}. \tag{1.2}$$

Qi then proposed an open problem (Theorem 1.6, [6]): *Under what condition is the inequality* (1.2) *still true if* n *is replaced by any positive real number* r*?*

Some new results on this subject can be found in [1], [2], [3] and [4]. In [2], Chen and Kimball proposed a theorem

Theorem 1.3 (Theorem 5, [2])**.** *Suppose* $f(x)$ *has derivative of the* n*-th order on the interval* $[a,b]$ *such that* $f^{(i)}(a) = 0$ *for* $i = 0, 1, 2, \ldots, n-1$. *If* $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ *and* $f^{(n)}(x)$ *is increasing, then the inequality* (1.2) *holds. If* $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ *and* $f^{(n)}(x)$ *is decreasing, then the inequality* (1.2) *reverses.*

After proving the theorem, Chen and Kimball proposed a conjecture. The conjecture is that the above monotony assumption of Theorem 1.3 could be dropped. In this paper, we will prove that this conjecture holds. We use the same technique which was introduced by Qi in [6].

2. Main Results

At the beginning of this section, we consider the case $n = 2$ as the first step in the process.

Lemma 2.1. *Suppose* $f(x)$ *has continuous a derivative of the* 2*-nd order on the interval* $[a,b]$ *such that* $f^{(i)}(a) = 0$, *where* $i = 0, 1$, *and* $f^{(2)}(x) \geq \frac{2}{3}$, *then*

$$\int_a^b f^4(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^3. \tag{2.1}$$

Proof. It follows from $f^{(2)}(x) \geq \frac{2}{3} > 0$ that f' is (strictly) increasing in $[a,b]$. Since $f'(a) = 0$ then $f'(x) > f'(a) = 0$ for every $a < x \leq b$. Therefore f is also increasing in $[a,b]$. Let

$$H(x) = \int_a^x f^4(x)\,dx - \left(\int_a^x f(x)\,dx\right)^3, \quad x \in [a,b].$$

Direct calculation produces

$$H'(x) = \left(f^3(x) - 3\left(\int_a^x f(x)\,dx\right)^2\right) f(x) =: h_1(x) f(x),$$

which yields

$$h_1'(x) = 3\left(f(x) f'(x) - 2\int_a^x f(x)\,dx\right) f(x) =: 3h_2(x) f(x).$$

Then

$$h_2'(x) = (f'(x))^2 + f(x) f''(x) - 2f(x)$$

and

$$\begin{aligned} h_2'(x) &= (f'(x))^2 + f(x) f''(x) - 2f(x) \\ &\geq (f'(x))^2 + \left(\frac{2}{3} - 2\right) f(x) =: h_3(x). \end{aligned}$$

Thus

$$\begin{aligned} h_3'(x) &= 2f'(x)\,f''(x) - \frac{4}{3}f'(x) \\ &\geq 2f'(x)\left(f''(x) - \frac{2}{3}\right) \\ &\geq 0. \end{aligned}$$

Therefore $h_3(x)$, $h_2(x)$ and $h_1(x)$ are increasing and then $H(x)$ is also increasing. Hence $H(b) \geq H(a) = 0$ which completes this proof. □

Now we state our main result.

Theorem 2.2. *Let n be a positive integer. Suppose $f(x)$ has a continuous derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$, where $0 \leq i \leq n-1$, and $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$, then*

$$\int_a^b f^{n+2}(x)\,dx \geq \left(\int_a^b f(x)\,dx\right)^{n+1}. \tag{2.2}$$

Proof of Theorem 2.2. Letting

$$g(x) = \frac{(n+1)^{n-1}}{n!}f(x),$$

one can easily see that $g^{(n)}(x) \geq 1$ for all x.

The problem now is to show that the inequality below is true

$$\int_a^b g^{n+2}(x)\,dx \geq \frac{(n+1)^{n-1}}{n!}\left(\int_a^b g(x)\,dx\right)^{n+1}.$$

Let

$$G(x) = \int_a^x g^{n+2}(t)\,dt - \frac{(n+1)^{n-1}}{n!}\left(\int_a^x g(t)\,dt\right)^{n+1}.$$

One can find that

$$\begin{aligned} G'(x) &= g(x)\left(g^{n+1}(x) - \frac{(n+1)^n}{n!}\left(\int_a^x g(t)\,dt\right)^n\right) \\ &= g(x)\,g_1(x). \end{aligned}$$

We will prove $g_1(x) \geq 0$ by induction. According to Lemma 2.1, the case $n = 2$ is proved. Denote

$$g_2(x) = g^{\frac{n+1}{n}}(x) - \frac{(n+1)}{\sqrt[n]{n!}}\int_a^x g(t)\,dt.$$

It is easy to see that the function $h(x) := g'(x)$ satisfies the following conditions

a) $h^{(k)}(a) = 0$ for all $k \leq n-2$, and
b) $h^{(n-1)}(x) \geq 1$.

Therefore, by induction

$$h^n(x) \geq \frac{n^{n-1}}{(n-1)!}\left(\int_a^x h(t)\,dt\right)^{n-1}$$

or equivalently

$$g'(x) \geq \sqrt[n]{\frac{n^{n-1}}{(n-1)!}}g^{\frac{n-1}{n}}(x).$$

Hence,

$$\frac{n+1}{n} g^{\frac{1}{n}}(x) g'(x) \geq \frac{n+1}{n} \sqrt[n]{\frac{n^{n-1}}{(n-1)!}} g(x).$$

Thus,

$$g^{\frac{n+1}{n}}(x) \geq \frac{n+1}{n} \sqrt[n]{\frac{n^{n-1}}{(n-1)!}} \int_a^x g(x)\, dx.$$

Then, the conclusion $g_2(x) \geq 0$ follows from the fact that

$$\frac{n+1}{n} \sqrt[n]{\frac{n^{n-1}}{(n-1)!}} = \frac{n+1}{\sqrt[n]{n!}},$$

which yields $g_1(x) \geq 0$. Then $G(x) \geq 0$. Our proof is completed. □

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