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NEW GENERALIZATIONS OF AN INTEGRAL INEQUALITY

Abstract

In this short paper an integral inequality posed in the 11^{th} International Mathematical Competition for University Students is further generalized.

1 Introduction.

Problem 2 of the 11^{th} International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004 (see [1]) reads as follows:

Proposition 1. *Let $f, g : [a, b] \to [0, \infty)$ be two continuous and non-decreasing functions such that*

$$\int_a^x \sqrt{f(t)}\,\mathrm{d}t \le \int_a^x \sqrt{g(t)}\,\mathrm{d}t \tag{1}$$

for $x \in [a, b]$ and

$$\int_a^b \sqrt{f(t)}\,\mathrm{d}t = \int_a^b \sqrt{g(t)}\,\mathrm{d}t. \tag{2}$$

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Then

$$\int_a^b \sqrt{1+f(t)}\,\mathrm{d}t \geq \int_a^b \sqrt{1+g(t)}\,\mathrm{d}t.$$

Considering (2), it is clear that (1) can be rewritten as $\int_x^b \sqrt{f(t)}\,\mathrm{d}t \geq \int_x^b \sqrt{g(t)}\,\mathrm{d}t$. By replacing $f(x)$ with $\sqrt{f(x)}$ and $g(x)$ with $\sqrt{g(x)}$, Proposition 1 can be simplified into the following Proposition 2.

Proposition 2. *Let $f, g : [a, b] \to [0, \infty)$ be two continuous, non-decreasing functions such that*

$$\int_x^b f(t)\,\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t \tag{3}$$

for $x \in [a, b]$ and

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b g(t)\,\mathrm{d}t. \tag{4}$$

Then

$$\int_a^b \sqrt{1+f^2(t)}\,\mathrm{d}t \geq \int_a^b \sqrt{1+g^2(t)}\,\mathrm{d}t. \tag{5}$$

Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$ and $G(x) = \int_a^x g(t)\,\mathrm{d}t$. Then $F(x) \leq G(x)$ for $x \in [a, b]$ and $G(x)$ is a convex function on $[a, b]$. On the other hand, since $F(a) = G(a)$ and $F(b) = G(b)$, then inequality (5) is apparently valid because the length of the curve $y = F(x)$ is not less than that of the curve $y = G(x)$. This explains the geometric meaning of Proposition 2 and gives a solution to Proposition 1.

In [3], Proposition 1 and Proposition 2 were generalized as follows:

Theorem A. *Let $f : [a, b] \to [0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a continuous, non-decreasing function satisfying* (3) *and* (4)*. Then*

$$\int_x^b h(f(t))\,\mathrm{d}t \geq \int_x^b h(g(t))\,\mathrm{d}t \tag{6}$$

for every convex function h on $[0, \infty)$.

Theorem B. *Let $f : [a, b] \to [0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a continuous, non-increasing function satisfying the reversed inequalities of* (3) *and* (4)*. Then inequality* (6) *holds true for every convex function h on $[0, \infty)$.*

The main aim of this paper is to further generalize Proposition 1 and Proposition 2. Our main results are included in the two theorems below.

Theorem 1. *Let $f : [a, b] \to [0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a continuous, non-decreasing function satisfying* (3)*. Then inequality* (6) *is valid for every convex function h such that $h' \geq 0$ and h' is integrable on $[0, \infty)$.*

Corollary 1. *Let $f : [a, b] \to [0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a continuous, non-decreasing function satisfying* (3)*. Then $\int_a^b f^\alpha(t)\,\mathrm{d}t \geq \int_a^b g^\alpha(t)\,\mathrm{d}t$ for every $\alpha > 1$.*

Theorem 2. *Let $f : [a, b] \to [0, \infty)$ be a continuous function and $g : [a, b] \to [0, \infty)$ a continuous, non-increasing function satisfying the reversed inequality of* (3)*. Then inequality* (6) *holds true for every convex function h such that $h' \leq 0$ and h' is integrable on $[0, \infty)$.*

2 Proofs of Theorem 1 and Theorem 2.

In order to prove our theorems, the well known second mean value theorem for integrals will be available.

Lemma 1 ([2, p. 35])**.** *Let $f(x)$ be bounded and monotonic and let $g(x)$ be integrable on $[a, b]$. Then there exists some $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(a)\int_a^\xi g(x)\,\mathrm{d}x + f(b)\int_\xi^b g(x)\,\mathrm{d}x.$$

Proof of Theorem 1. Let $\phi(x) = -\int_x^b f(t)\,\mathrm{d}t$ and $\varphi(x) = -\int_x^b g(t)\,\mathrm{d}t$. Since h is convex, $h(t) \geq h(s) + (t - s)h'(s)$ for $a \leq s,\ t \leq b$. Therefore $h(\phi'(t)) \geq h(\varphi'(t)) + [\phi'(t) - \varphi'(t)]h'(\varphi'(t))$, which gives

$$\int_a^b h(\phi'(t))\,\mathrm{d}t \geq \int_a^b h(\varphi'(t))\,\mathrm{d}t + \int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t))\,\mathrm{d}t.$$

Now it is sufficient to prove that $\int_a^b [\phi'(t) - \varphi'(t)]h'(\varphi'(t))\,\mathrm{d}t \geq 0$. Since $h(t)$ is convex, the function $h'(t)$ is non-decreasing. Since $g(t)$ is non-decreasing, the function $\varphi'(t)$ is also non-decreasing. Thus the composite function $h'(\varphi'(t))$ is non-decreasing with respect to t. Using Lemma 1 for some $\xi \in [a,b]$ yields

$$\begin{aligned}
&\int_a^b [\phi'(t) - \varphi'(t)] h'(\varphi'(t))\,\mathrm{d}t \\
&= h'(\varphi'(a)) \int_a^\xi [\phi'(t) - \varphi'(t)]\,\mathrm{d}t + h'(\varphi'(b)) \int_\xi^b [\phi'(t) - \varphi'(t)]\,\mathrm{d}t \\
&= h'(g(a))[\phi(\xi) - \phi(a) - \varphi(\xi) + \varphi(a)] + h'(g(b))[\phi(b) - \phi(\xi) - \varphi(b) + \varphi(\xi)] \\
&\geq h'(g(a))[\phi(\xi) - \varphi(\xi)] + h'(g(b))[\phi(b) - \phi(\xi) - \varphi(b) + \varphi(\xi)] \\
&= [\phi(\xi) - \varphi(\xi)][h'(g(a)) - h'(g(b))] \geq 0,
\end{aligned}$$

since $\phi(\xi) \leq \varphi(\xi)$ and $0 \leq h'(g(a)) \leq h'(g(b))$. The proof of Theorem 1 is complete. □

PROOF OF THEOREM 2. Considering the proof of Theorem 1, it suffices to prove $\int_a^b [\phi'(t) - \varphi'(t)][-h'(\varphi'(t))]\,\mathrm{d}t \leq 0$. Since h is convex, $-h'$ is non-increasing. Since g is non-increasing, φ' is also non-increasing. Consequently, the composite function $-h'(\varphi'(t))$ is non-increasing with respect to t. Utilizing Lemma 1 for some $\xi \in [a,b]$ leads to

$$\begin{aligned}
&\int_a^b [\phi'(t) - \varphi'(t)] h'(\varphi'(t))\,\mathrm{d}t \\
&= h'(\varphi'(a)) \int_a^\xi [\phi'(t) - \varphi'(t)]\,\mathrm{d}t + h'(\varphi'(b)) \int_\xi^b [\phi'(t) - \varphi'(t)]\,\mathrm{d}t \\
&= h'(\varphi'(a))(\phi(\xi) - \phi(a) - \varphi(\xi) + \varphi(a)) + h'(\varphi'(b))(\phi(b) - \phi(\xi) - \varphi(b) + \varphi(\xi)) \\
&\geq [\phi(\xi) - \varphi(\xi)][h'(g(a)) - h'(g(b))] \geq 0,
\end{aligned}$$

since $\phi(\xi) \geq \varphi(\xi)$ and $0 \geq h'(g(a)) \geq h'(g(b)$. The proof is complete. □

References

[1] *The* 11^{th} *International Mathematical Competition for University Students, Skopje, Macedonia, 25–26 July 2004, Solutions for problems on Day 2*, Available online at `http://www.imc-math.org.uk/index.php?year=2004` or `http://www.imc-math.org.uk/imc2004/day2_solutions.pdf`.

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[3] Q.-A. Ngô & F. Qi, *Generalizations of an integral inequality*, preprint.