Rostock. Math. Kolloq. **63**, 63–77 (2008)

Subject Classification (AMS)
35J60, 35J65

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Existence of non-negative Solutions for cooperative elliptic Systems involving Schrödinger Operators in the whole Space

ABSTRACT. In this paper, we obtain some new results on the existence of non-negative solutions for systems of the form

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } \mathbb{R}^N,\ i = 1, ..., n,$$

where each of the q_i are positive potentials satisfying $\lim_{|x|\to+\infty} q_i(x) = +\infty$, each of the m_i are bounded positive weights, each of the a_{ij}, $i \neq j$, are bounded non-negative weights and each of the μ_i are real parameters. Depending upon the hypotheses on f_i, we obtain some new results by using sub- and super-solution methods and the Schauder Fixed Point Theorem.

1 Introduction

In this paper, we are interested in the existence of non-negative solutions of the following cooperative elliptic system

$$(-\Delta + q_i)u_i = \mu_i m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, \ldots, u_n), \text{ in } \mathbb{R}^N,\ i = 1, ..., n. \tag{1.1}$$

We consider the following hypotheses for each $i = 1, \ldots, n$ and $j = 1, \ldots, n$

($\mathbf{h_1}$) $q_i \in L^2_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$ $(p > \frac{N}{2})$ such that $\lim_{|x|\to+\infty} q_i(x) = +\infty$ and $q_i \geq \text{const} > 0$.

($\mathbf{h_2}$) $a_{ij} \in L^\infty(\mathbb{R}^N)$ and $a_{ij} \geq 0$ if $i \neq j$.

($\mathbf{h_3}$) $m_i \in L^\infty(\mathbb{R}^N)$ and there exists $\alpha_i > 0$ such that $m_i(x) \geq \alpha_i > 0$ for all $x \in \mathbb{R}^N$.

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Note that our system is cooperative since $a_{ij} \geq 0$ if $i \neq j$. We will specify later the hypotheses on each f_i and we denote by μ_i real parameters for $i = 1, \dots, n$.

The variational space is denoted by $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ where for $i = 1, \dots, n$, $V_{q_i}(\mathbb{R}^N)$ is the completion of $D(\mathbb{R}^N)$, the set of C^∞ functions with compact support, under the norm

$$\|u\|_{q_i} = \sqrt{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + q_i u^2\right)}. \tag{1.2}$$

We recall that each of the embedding $V_{q_i}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is compact. We denote by

$$\|u\|_{m_i} = \sqrt{\int_\Omega m_i u^2} \quad \text{for all } u \in L^2(\Omega).$$

According to the hypothesis (**h₃**), $\|\cdot\|_{m_i}$ is a norm in $L^2(\mathbb{R}^N)$, equivalent to the usual norm so the embedding $V_{q_i}(\mathbb{R}^N) \hookrightarrow (L^2(\mathbb{R}^N), \|\cdot\|_{m_i})$ is still compact.

We denote by M_i the operator of multiplication by m_i in $L^2(\mathbb{R}^N)$. The operator

$$(-\Delta + q_i)^{-1} M_i : (L^2(\mathbb{R}^N), \|\cdot\|_{m_i}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{m_i})$$

is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence tending to 0. We denote by λ_i the first eigenvalue and by ϕ_i the corresponding eigenfunction which satisfy

$$(-\Delta + q_i)\phi_i = \lambda_i m_i \phi_i \text{ in } \mathbb{R}^N,\ \lambda_i > 0 \tag{1.3}$$

and $\|\phi_i\|_{m_i} = 1$. We recall that λ_i is simple and that $\phi_i > 0$ (see for examples [1, 2, 4, 5, 15, 16]). By the Courant-Fischer formulas, λ_i is given by

$$\lambda_i = \inf\left\{ \frac{\int_{\mathbb{R}^N} \left(|\nabla \phi|^2 + q_i \phi^2\right)}{\int_{\mathbb{R}^N} m_i \phi^2}, \phi \in D(\mathbb{R}^N) \right\}. \tag{1.4}$$

The aim of this paper is to study the existence of non-negative solutions for the system (1.1). This extends earlier results obtained for the Laplacian operator in a bounded domain (see [12, 13]), for an operator of divergence form in a bounded domain (see [9]), for equations or systems involving Schrödinger operators $-\Delta + q_i$ in $\mathbb{R}^N$ (see [3, 6–8, 10, 11]).

Our paper is organized as follows. Section 2 provides some preliminaries and notations before stating our main result which is given in Section 3. In Section 4, we give some remarks on our hypotheses for a two-by-two system.

2 Preliminaries and Notations

2.1 Review of results for the scalar case $(i = 1)$

We consider here the following equation, in a variational sense,

$$(-\Delta + q)u = \lambda mu + g \text{ in } \mathbb{R}^N. \tag{e}$$

We assume the following: The potential q satisfies $(\mathbf{h_1})$, the weight m satisfies $(\mathbf{h_3})$, the constant λ is a real parameter and finally $g \in L^2(\mathbb{R}^N)$. We denote by $(\lambda_{m,q}, \phi_{m,q})$ the eigenpair of eigenvalue and eigenfunction which satisfy

$$(-\Delta + q)\phi_{m,q} = \lambda_{m,q} m \phi_{m,q} \quad , \quad \lambda_{m,q} > 0, \phi_{m,q} > 0.$$

We recall the following results on the existence of solutions and the Maximum Principle.

Theorem 2.1 (see [11]; Theorems 1.1 and 1.2) *Assume that $\lambda < \lambda_{m,q}$. Then there exists a unique solution $u \in V_q(\mathbb{R}^N)$ for the equation (e). Moreover,*

(i) *the weak Maximum Principle holds: if $g \geq 0$, then this solution u satisfies $u \geq 0$,*

(ii) *the strong Maximum Principle holds too: if $g \geq 0$, $g \neq 0$ then $u > 0$.*

2.2 Notations and Hypotheses

We recall that for each $i = 1, \ldots, n$ the eigenpair (λ_i, ϕ_i) is defined by (1.3)-(1.4). Let us denote by Φ the vector defined by

$$^t\Phi = (\phi_1, \ldots, \phi_n). \tag{2.1}$$

We assume that for each $i = 1, \ldots, n$, the nonlinear term f_i of the system (1.1) satisfies the following hypotheses

$(\mathbf{h_4})$ For each $i = 1, \ldots, n$

(i) $f_i(x, k_1\phi_1, \ldots, k_n\phi_n) \in L^2(\mathbb{R}^N)$ for all positive numbers $k_1, \ldots, k_n$.

(ii) $0 \leq f_i(x, u_1, \ldots, u_n)$ for all $u_1 \geq 0, \ldots, u_n \geq 0$.

(iii) For all $0 \leq u_1 \leq v_1, \ldots, 0 \leq u_n \leq v_n$,

$$0 \leq f_i(x, u_1, \ldots, u_n) \leq f_i(x, v_1, \ldots, v_n).$$

(iv) For all positive real numbers $k_1, \dots, k_n$,

$$\frac{f_i(x, \eta k_1\phi_1, \dots, \eta k_n\phi_n)}{\eta\phi_i} \to 0 \text{ as } \eta \to +\infty, \text{ uniformly in } x.$$

(v) f_i is Lipschitz respect to $(u_1, \dots, u_n)$ uniformly in x.

For instance, the function $f_i(x, u_1, \dots, u_n) = \sqrt{u_1 + \dots + u_n}\ 1_K$ (where 1_K denotes the indicator function on a compact $K \subset \mathbb{R}^N$) satisfies ($\mathbf{h_4}$)(iv).

We denote by $L = (l_{ij})$ the following $n \times n$ matrix be defined by

$$l_{ij} := \begin{cases} (\lambda_i - \mu_i)\alpha_i & \text{if } i = j, \\ -\|a_{ij}\|_{L^\infty(\mathbb{R}^N)} & \text{if } i \neq j, \end{cases} \tag{2.2}$$

for all $i, j = 1, \dots, n$. We also assume that the following hypothesis holds for some $\beta > 0$

($\mathbf{h_5}$) $(L - \beta I)\Phi \geq 0$

where I is the $n \times n$ identity matrix. Here $(L-\beta I)\Phi \geq 0$ means that the entries of $(L-\beta I)\Phi$ are non-negative functions. Note that the hypothesis ($\mathbf{h_5}$) forces that the coupling is very weak, i. e., with small coefficients a_{ij} and with eigenfunctions ϕ_i which have the same behaviour at infinity: $\frac{\phi_i(x)}{\phi_j(x)}$ is bounded for all $i, j = 1, \dots, n$ and all $x \in \mathbb{R}^N$. We can now develop our main result in the next section.

3 Existence of solutions

We begin stating our main result, obtained by considering a sub- and a super-solution of the system (1.1) and using the Schauder Fixed Point Theorem. We recall that $(v_1, \dots, v_n)$ is a sub-solution (resp. a super-solution) of the system (1.1) if for each $i = 1, \dots, n$, we have

$$(-\Delta + q_i)v_i \leq \mu_i m_i v_i + \sum_{j=1; j\neq i}^{n} a_{ij}v_j + f_i(x, v_1, \dots, v_n) \text{ in } \mathbb{R}^N \tag{3.1}$$

(resp. $\geq$).

Theorem 3.1 *Assume that the hypotheses* ($\mathbf{h_1}$)-($\mathbf{h_5}$) *are satisfied. Then the system* (1.1) *has at least one non-negative solution in* $V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$.

Proof: First, note that $u_0 = (0, \dots, 0)$ is a sub-solution of the system (1.1). Then, by hypothesis ($\mathbf{h_5}$), we have $(L - \beta I)\Phi \geq 0$ and so we get for each $i = 1, \dots, n$,

$$(\lambda_i - \mu_i)m_i\phi_i - \beta\phi_i - \sum_{j=1; j\neq i}^{n} a_{ij}\phi_j \geq 0.$$

Since, by hypothesis (**h$_4$**)(iv), we have for η sufficiently large

$$0 \leq \frac{f_i(x, \eta\phi_1, \ldots, \eta\phi_n)}{\eta\phi_i} \leq \beta,$$

we can write for η sufficiently large

$$(\lambda_i - \mu_i)m_i\eta\phi_i - \sum_{j=1;j\neq i}^{n} a_{ij}\eta\phi_j \geq \eta\beta\phi_i \geq f_i(x, \eta\phi_1, \ldots, \eta\phi_n).$$

Thus we have

$$(\lambda_i - \mu_i)m_i\eta\phi_i - \sum_{j=1;j\neq i}^{n} a_{ij}\eta\phi_j \geq f_i(x, \eta\phi_1, \ldots, \eta\phi_n). \tag{3.2}$$

Therefore $u^0 := \eta\Phi = (\eta\phi_1, \ldots, \eta\phi_n)$ is a super-solution of the system (1.1).

Now, we define the set $\sigma = [u_0, u^0]$. Let α be a positive real such that for all i, $\mu_i + \alpha > 0$. Let

$$T : (L^2(\mathbb{R}^N))^n \to (L^2(\mathbb{R}^N))^n$$
$$(u_1, \ldots, u_n) = u \mapsto v = (v_1, \ldots, v_n)$$

where for each $i = 1, \ldots, n$

$$(-\Delta + q_i + \alpha m_i)v_i = (\mu_i + \alpha)m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, \ldots, u_n) \text{ in } \mathbb{R}^N. \tag{3.3}$$

Note that, by the scalar case, T is well defined for all $u \in \sigma$.

As in [3], we prove now that $T(\sigma) \subset \sigma$. Let $u = (u_1, \ldots, u_n) \in \sigma$ and $T(u) = v = (v_1, \ldots, v_n)$. By the weak Maximum Principle for the scalar case, since the system (1.1) is a cooperative one and $\alpha > 0$, we get $v_i \geq 0$ for each $i = 1, \ldots, n$. Moreover, we have

$$\begin{aligned}(-\Delta + q_i + \alpha m_i)(\eta\phi_i - v_i) =& (\lambda_i + \alpha)\eta m_i\phi_i - (\mu_i + \alpha)m_i u_i \\ &- \sum_{j=1;j\neq i}^{n} a_{ij}u_j - f_i(x, u_1, \ldots, u_n).\end{aligned}$$

By (3.2), we get

$$\begin{aligned}&(-\Delta + q_i + \alpha m_i)(\eta\phi_i - v_i) \\ &\qquad \geq (\mu_i + \alpha)m_i(\eta\phi_i - u_i) + \sum_{j=1;j\neq i}^{n} a_{ij}(\eta\phi_j - u_j) \\ &\qquad\quad + f_i(x, \eta\phi_1, \ldots, \eta\phi_n) - f_i(x, u_1, \ldots, u_n) \\ &\qquad \geq 0.\end{aligned}$$

By the weak Maximum Principle for the scalar case, we obtain $v_i \le \eta\phi_i$ for each $i = 1, \ldots, n$. Therefore $T(\sigma) \subset \sigma$.

As in [3], we prove now that T is continuous and that $T(\sigma)$ is compact. Let $(u_k)_k$, where $u_k = (u_{1k}, \ldots, u_{nk})$, be a sequence in σ and define $T(u_k) = v_k$ where $v_k = (v_{1k}, \ldots, v_{nk})$. First, if (u_k) converges to $u = (u_1, \ldots, u_n)$ for $\|\cdot\|_{(L^2(\Omega))^n}$, and if $T(u) = v = (v_1, \ldots, v_n)$, from (3.3) we have for each $i = 1, \ldots, n$ and for all k

$$\begin{aligned}(-\Delta + q_i + \alpha m_i)(v_{ik} - v_i) =& (\mu_i + \alpha) m_i (u_{ik} - u_i) + \sum_{j=1; j\neq i}^{n} a_{ij}(u_{jk} - u_j) \\ &+ f_i(x, u_{1k}, \ldots, u_{nk}) - f_i(x, u_1, \ldots, u_n). \end{aligned} \tag{3.4}$$

Multiplying (3.4) by $v_{ik} - v_i$ and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned}\|v_{ik} - v_i\|^2_{q_i + \alpha m_i} =& (\mu_i + \alpha) \int_{\mathbb{R}^N} m_i (u_{ik} - u_i)(v_{ik} - v_i) \\ &+ \sum_{j=1, j\neq i}^{n} \int_{\mathbb{R}^N} a_{ij}(u_{jk} - u_j)(v_{ik} - v_i) \\ &+ \int_{\mathbb{R}^N} \big(f_i(x, u_{1k}, \ldots, u_{nk}) - f_i(x, u_1, \ldots, u_n)\big)(v_{ik} - v_i).\end{aligned}$$

Using the hypothesis (**h$_4$**)(v) and the Cauchy-Schwartz inequality, since the coefficients m_i and a_{ij} are bounded, we deduce that there exists a constant $C_1 > 0$ such that

$$\|v_{ik} - v_i\|_{q_i + \alpha m_i} \le C_1 \sum_{j=1}^{n} \|u_{jk} - u_j\|_{L^2(\Omega)}.$$

Therefore T is continuous. We prove now that $T(\sigma)$ is compact. Multiplying (3.3) by v_{ik} we have also

$$\|v_{ik}\|^2_{q_i + \alpha m_i} = (\mu_i + \alpha) \int_{\mathbb{R}^N} m_i u_{ik} v_{ik} + \sum_{j=1, j\neq i}^{n} \int_{\mathbb{R}^N} a_{ij} u_{jk} v_{ik} + \int_{\mathbb{R}^N} f_i(x, u_{1k}, \ldots, u_{nk}) v_{ik}.$$

Since the coefficients m_i and a_{ij} are bounded, then by the hypothesis (**h$_4$**), we see that $f_i(x, u_{1k}, \ldots, u_{nk})$ is bounded too and we can deduce the existence of a constant $C_2 > 0$ and of a constant $C_3 > 0$ such that

$$\|v_{ik}\|_{q_i + \alpha m_i} \le C_2 \, (\sum_{j=1}^{n} \|u_{jk}\|_{L^2(\mathbb{R}^N)} + C_3).$$

But $u \in \sigma$. Therefore the sequence $(v_k)_k$ is bounded in $V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ and since each of the embedding $V_{q_i}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is compact, we can find a subsequence of $(v_k)_k$ which is convergent in $(L^2(\mathbb{R}^N))^n$. Therefore $T(\sigma)$ is compact.

By the Schauder Fixed Point Theorem, we deduce the existence of $u \in \sigma$ such that $T(u) = u$. Clearly u is a non-negative solution of the system (1.1). $\square$

As in [14], we can relax the hypotheses on the increasing of each function f_i but assuming stronger hypotheses on the regularity of f_i. For the next result, we will suppose that each function f_i of the system (1.1) satisfies the following hypothesis

($\mathbf{h'_4}$) (i) $f_i(x, u_1, \dots, \eta\phi_i, \dots, u_n) \in L^2(\mathbb{R}^N)$ for all $\eta > 0$ and all $0 \le u_1 \le \eta\phi_1, \dots, 0 \le u_n \le \eta\phi_n$.

(ii) $0 \le f_i(x, u_1, \dots, u_n)$ for all $u_1 \ge 0, \dots, u_n \ge 0$.

(iii) f_i is of class C^1.

(iv) For all $0 \le u_1 \le \eta\phi_1, \dots, 0 \le u_n \le \eta\phi_n$,

$$\frac{f_i(x, u_1, \dots, \eta\phi_i, \dots, u_n)}{\eta\phi_i} \to 0 \text{ as } \eta \to +\infty, \text{ uniformly in } x.$$

(v) f_i is Lipschitz respect to $(u_1, \dots, u_n)$ uniformly in x.

Following [14], we say that a couple $(u_{01}, \dots, u_{0n}) - (u_1^0, \dots, u_n^0)$ is a sub−super-solution of the system (1.1) (Müller type conditions) if for each $i = 1, \dots, n$,

$$u_{0i} \le u_i^0$$

and moreover

$$\begin{cases} 0 \ge (-\Delta + q_i)u_{0i} - \mu_i m_i u_{0i} - \sum_{j=1;j\neq i}^n a_{ij}u_j - f_i(x, u_1, \dots, u_{0i}, \dots, u_n), \\ 0 \le (-\Delta + q_i)u_i^0 - \mu_i m_i u_i^0 - \sum_{j=1;j\neq i}^n a_{ij}u_j - f_i(x, u_1, \dots, u_i^0, \dots, u_n), \end{cases} \tag{3.5}$$

for any $u_j \in [u_{0j}, u_j^0]$.

It is clear that this definition is much more stringent than the natural definition (3.1) where (3.5) are only satisfied for $u_j = u_{0j}$ (for the sub-solution) and for $u_j = u_j^0$ (for the super-solution). Note that if f_i is increasing, both definitions coincide.

Theorem 3.2 *Assume that the hypotheses* ($\mathbf{h_1}$)-($\mathbf{h_3}$), ($\mathbf{h'_4}$) *and* ($\mathbf{h_5}$) *are satisfied. Then the system* (1.1) *has at least one non-negative solution in* $V_{q_1}(\mathbb{R}^N) \times \dots \times V_{q_n}(\mathbb{R}^N)$.

Proof: As in Theorem 3.1, we denote by $u_0 = (0, \dots, 0)$, $\Phi = (\phi_1, \dots, \phi_n)$ and by $u^0 = \eta\Phi$ for η sufficiently large positive real defined later. First, we prove that (u_0, u^0) is a couple of sub-super-solution in the sense of (3.5).

Indeed, proceeding as for Theorem 3.1, using hypotheses (**h_4'**) and (**h_5**) we have (for η sufficiently large)

$$(\lambda_i - \mu_i)m_i\eta\phi_i \geq \sum_{j=1;j\neq i}^{n} a_{ij}\eta\phi_j + f_i(x,u_1,\dots,\eta\phi_i,\dots,u_n) \text{ for any } 0 \leq u_j \leq \eta\phi_j$$

and therefore (since the system (1.1) is cooperative)

$$(\lambda_i - \mu_i)m_i\eta\phi_i \geq \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x,u_1,\dots,\eta\phi_i,\dots,u_n) \text{ for any } 0 \leq u_j \leq \eta\phi_j.$$

We define the set $\sigma = [u_0, u^0]$. Let α be a positive real such that for all i, $\mu_i + \alpha > 0$. Let

$$\begin{aligned} T_\sigma : \sigma &\to (L^2(\mathbb{R}^N))^n \\ (u_1,\dots,u_n) = u &\mapsto v = (v_1,\dots,v_n) \end{aligned}$$

where for each $i = 1,\dots,n$,

$$(-\Delta + q_i + \alpha m_i + \rho m_i)v_i = (\mu_i + \alpha)m_i u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x,u_1,\dots,u_n) + \rho m_i u_i \text{ in } \mathbb{R}^N$$

and where $\rho > 0$ is a constant such that $f_i(x,u_1,\dots,u_n) + \rho m_i u_i$ is increasing in u_i. We can find such ρ by hypotheses (**h_3**) and (**h_4'**) since the function f_i is C^1. Still by the scalar case, the operator T_σ is well defined and proceeding as for Theorem 3.1, we can prove that T_σ is continuous and $T_\sigma(\sigma)$ is compact.

Now we prove that $T_\sigma(\sigma) \subset \sigma$. Let $u = (u_1,\dots,u_n) \in \sigma$ and $T_\sigma(u) = v = (v_1,\dots,v_n)$. Note by the scalar case $v_i \geq 0$ for each $i = 1,\dots,n$. Moreover we have for each $i = 1,\dots,n$

$$\begin{aligned} (-\Delta + q_i + \alpha m_i + \rho m_i)(\eta\phi_i - v_i) \geq& (\mu_i + \alpha)m_i(\eta\phi_i - u_i) \\ &+ f_i(x,u_1,\dots,\eta\phi_i,\dots u_n) + \rho m_i\eta\phi_i \\ &- f_i(x,u_1,\dots,u_n) - \rho m_i u_i \\ \geq& 0. \end{aligned}$$

Since $u_i \leq \eta\phi_i$ and $w_i \mapsto f_i(x,u_1,\dots,w_i,\dots,u_n) + \rho m_i w_i$ is increasing, by the scalar case, we obtain $v_i \leq \eta\phi_i$ for each $i = 1,\dots,n$. Therefore $T_\sigma(\sigma) \subset \sigma$.

By the Schauder Fixed Point Theorem, we deduce the existence of at least one fixed point of T_σ or equivalently, one weak non-negative solution of the system (1.1). □

4 Study of a two-by-two system $(i = 2)$

For a 2×2 cooperative system with constant coefficients a, b, c, d and the same potential q, if we rewrite the system (1.1) under the following form

$$\begin{cases} (-\Delta + q)u = au + bv + f(x, u, v) \text{ in } \mathbb{R}^N, \\ (-\Delta + q)v = cu + dv + g(x, u, v) \text{ in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

the hypothesis ($\mathbf{h_5}$), $(L - \beta I)\Phi \geq 0$, means that

$$(\lambda_q - a - b - \beta)\phi_q \geq 0 \text{ and } (\lambda_q - c - d - \beta)\phi_q \geq 0$$

where λ_q is the principal eigenvalue associated with the eigenfunction ϕ_q for the operator $-\Delta + q$ (q being a potential satisfying the hypothesis ($\mathbf{h_1}$)). Since $\phi_q > 0$, ($\mathbf{h_5}$) is equivalent, in this case, to $\lambda_q \geq a + b + \beta$ and $\lambda_q \geq c + d + \beta$. Therefore the hypothesis ($\mathbf{h_5}$) is stronger than the usual hypothesis in [3], [6–11], [12], [13] which is $\lambda_q > a$, $\lambda_q > d$ and $(\lambda_q - a)(\lambda_q - d) > bc$ or equivalently

$$\begin{pmatrix} \lambda_q - a & -b \\ -c & \lambda_q - d \end{pmatrix} \tag{4.2}$$

is a non-singular M-matrix. However, for the nonlinear terms of the system (1.1), we consider here in Theorem 3.1 another class of functions f_i (denoted by f and g for $n = 2$) than the one used in [3] or [6–11] (where in these papers, each function f_i satisfies

$$0 \leq f_i(x, u_1, \ldots, u_n) \leq \theta_i$$

for all $u_i \geq 0$ and with θ_i a fixed function in $L^2(\mathbb{R}^N)$).

Moreover, for the system (4.1) when the function g depends only of u ($g(u, v) := g(u)$), using a decoupling method, we can prove the existence of a non-negative solution assuming that the nonlinear term $f(x, u, v)$ satisfies ($\mathbf{h_4}$) and that the 2×2 matrix defined by (4.2) is a non-singular M-matrix (which is the usual condition and a weaker hypothesis than ($\mathbf{h_5}$)).

So we now consider the following cooperative system

$$\begin{cases} (-\Delta + q)u = au + bv + f(x, u, v) \text{ in } \mathbb{R}^N, \\ (-\Delta + q)v = cu + dv + g(x, u) \text{ in } \mathbb{R}^N. \end{cases} \tag{4.3}$$

Theorem 4.1 *Assume that the potential q satisfies the hypothesis ($\mathbf{h_1}$), the coefficients a, b, c, d are real parameters with $b \geq 0$ and $c \geq 0$, the function f satisfies the hypothesis ($\mathbf{h_4}$) respect to ϕ_q the principal eigenfunction associated with λ_q the first eigenvalue of the operator $-\Delta + q$. Assume also that the function g satisfies the following hypothesis*

($\mathbf{h_6}$) (i) *There exists a constant* $K > 0$ *such that* $0 \le g(u) \le Ku$ *for all* $u \in L^2(\mathbb{R}^N)$, $u \ge 0$.

(ii) $g(u_1) \le g(u_2)$ *if* $0 \le u_1 \le u_2$.

(iii) g *is Lipschitz respect to* u *uniformly in* x.

Assume that the 2×2 *matrix* Λ *be defined by*

$$\Lambda = \begin{pmatrix} \lambda_q - a & -b \\ -(c+K) & \lambda_q - d \end{pmatrix}$$

is a non-singular M-matrix. Then the system (4.3) *has at least one non-negative solution* $(u, v) \in (V_q(\mathbb{R}^N))^2$.

Proof: We use the decoupling method combined with the sub- and super-solution method. First, we recall that

$$(-\Delta + q)\phi_q = \lambda_q \phi_q \text{ in } \mathbb{R}^N \tag{4.4}$$

with $\lambda_q > 0$ and $\phi_q > 0$. We proceed as in [1] and we define for $u \ge 0$ the continuous and compact operator

$$Bu := (-\Delta + q - d)^{-1}(cu + g(u)). \tag{4.5}$$

Note that the operator B is well defined since $d < \lambda_q$. Therefore (u, v) is a solution of the system (4.3) if and only if $v = Bu$ and

$$(-\Delta + q - a)u = bBu + f(x, u, Bu) \text{ in } \mathbb{R}^N. \tag{4.6}$$

We denote by $\underline{u} := 0$. By the weak Maximum Principle for the scalar case, since $c \ge 0$ and $g(\underline{u}) \ge 0$, we have $B\underline{u} \ge 0$ and using the hypothesis ($\mathbf{h_4}$), we have also $f(x, \underline{u}, B\underline{u}) \ge 0$. Therefore $\underline{u}$ is a sub-solution of the equation (4.6).

We construct now a super-solution of the equation (4.6) of the form $\overline{u} = \eta\phi_q$ where η will be a real positive parameter defined further on. Note that $\overline{u}$ is a super-solution of the equation if and only if

$$(\lambda_q - a)\eta\phi_q \ge bB\eta\phi_q + f(x, \eta\phi_q, B\eta\phi_q). \tag{4.7}$$

We have

$$bB\eta\phi_q = \frac{bc\eta}{\lambda_q - d}\phi_q + b(-\Delta + q - d)^{-1}(g(\eta\phi_q)).$$

By the hypothesis upon g, we have $0 \le g(\eta\phi_q) \le K\eta\phi_q$. Still using the weak Maximum Principle for the scalar case, we deduce that

$$(-\Delta + q - d)^{-1}(g(\eta\phi_q)) \le (-\Delta + q - d)^{-1}(K\eta\phi_q) = \frac{\eta K}{\lambda_q - d}\phi_q.$$

So we get

$$bB\eta\phi_q \leq \frac{b\eta(c+K)}{\lambda_q - d}\phi_q.$$

Moreover, from the hypothesis which assures that Λ is a non-singular M-matrix, we have

$$\frac{(\lambda_q - a)(\lambda_q - d) - b(c+K)}{\lambda_q - d} > 0.$$

Since

$$f(x, \eta\phi_q, B\eta\phi_q) \leq f(x, \eta\phi_q, \eta\frac{(c+K)}{\lambda_q - d}\phi_q),$$

by ($\mathbf{h_4}$), we can choose a positive real η sufficiently large such that

$$0 \leq \frac{f(x, \eta\phi_q, B\eta\phi_q)}{\eta\phi_q} < \frac{(\lambda_q - a)(\lambda_q - d) - b(c+K)}{\lambda_q - d}.$$

Therefore for η sufficiently large and now fixed, we have

$$bB\eta\phi_q + f(x, \eta\phi_q, B\eta\phi_q) \leq \eta\frac{b(c+K)}{\lambda_q - d}\phi_q + \frac{(\lambda_q - a)(\lambda_q - d) - b(c+K)}{\lambda_q - d}\eta\phi_q$$

and so (4.7) is satisfied or equivalently $\overline{u} = \eta\phi_q$ is a super-solution of the equation (4.6).

Now we define the operator T on $\sigma = [\underline{u}, \overline{u}]$ by

$$Tu := (-\Delta + q - a)^{-1}(bBu + f(x, u, Bu)). \tag{4.8}$$

Still again, the operator T is well defined since $a < \lambda_q$, $Bu \in L^2(\mathbb{R}^N)$ and $f(x, u, Bu) \in L^2(\mathbb{R}^N)$ for all $u \in \sigma$. We prove that $T(\sigma) \subset \sigma$. Let $u \in \sigma$. Since $u \geq 0$ then $Bu \geq 0$ and so $f(x, u, Bu) \geq 0$ by the weak Maximum Principle for the scalar case. Therefore $Tu \geq 0$. We have from (4.7) and (4.8)

$$(-\Delta + q - a)(Tu) = bBu + f(x, u, Bu)$$

and

$$(-\Delta + q - a)\eta\phi_q = (\lambda_q - a)\eta\phi_q \geq bB\eta\phi_q + f(x, \eta\phi_q, B\eta\phi_q).$$

So we get

$$(-\Delta + q - a)(\eta\phi_q - Tu) \geq b(B\eta\phi_q - Bu) + f(x, \eta\phi_q, B\eta\phi_q) - f(x, u, Bu) \text{ in } \mathbb{R}^N.$$

Moreover, since g is an increasing function with respect to u, by the weak Maximum Principle for the scalar case, we deduce that B is an increasing function with respect to u too. Therefore, using ($\mathbf{h_4}$) for f, we obtain

$$(-\Delta + q - a)(\eta\phi_q - Tu) \geq 0.$$

The weak Maximum Principle allows us to conclude that $Tu \leq \eta\phi_q$ since $a < \lambda_q$. As for Theorem 3.1, we can prove that T is a continuous operator for the $L^2(\mathbb{R}^N)$-norm and by the compact embedding $V_q(\mathbb{R}^{\mathbb{N}}) \hookrightarrow L^2(\mathbb{R}^N)$ we get that $T(\sigma)$ is compact.

By the Schauder Fixed Point Theorem, we deduce the existence of $u_0 \in V_q(\mathbb{R}^N)$ such that

$$(-\Delta + q)u_0 = au_0 + bBu_0 + f(x, u_0, Bu_0) \text{ in } \mathbb{R}^N.$$

Clearly, (u_0, Bu_0) is a non-negative solution of the system (4.3). □

Note that if we add an hypothesis on the nonlinear term g, then we can construct a sub-solution of the equation (4.6) of the form $\underline{u} = \epsilon\phi_q$ and consequently, we get the existence of a positive solution of the system (4.3). This is the following result.

Theorem 4.2 *Assume that the potential q satisfies the hypothesis ($\mathbf{h_1}$), the coefficients a, b, c, d are real parameters with $b > 0$ and $c \geq 0$, the function f satisfies the hypothesis ($\mathbf{h_4}$) respect to ϕ_q the principal eigenfunction associated with λ_q the first eigenvalue of the operator $-\Delta + q$. Assume also that the function g satisfies the hypothesis ($\mathbf{h_6}$) and the following hypothesis*

$$\lim_{s\to 0^+} \frac{g(s)}{s} \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b}.$$

Assume that the 2×2 matrix Λ be defined by

$$\Lambda = \begin{pmatrix} \lambda_q - a & -b \\ -(c + K) & \lambda_q - d \end{pmatrix}$$

is a non-singular M-matrix. Then the system (4.3) has at least one positive solution $(u, v) \in (V_q(\mathbb{R}^N))^2$.

Proof: We proceed as for Theorem 4.1. We construct a sub-solution of the equation (4.6) of the form $\underline{u} = \epsilon\phi_q$ such that $\underline{u} \leq s_0$ where s_0 is a positive real sufficiently small which satisfies

$$\frac{g(s)}{s} \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b}$$

for all $0 < s \leq s_0$. This is possible due to the boundedness of the function ϕ_q.

Indeed, since $0 < \epsilon\phi_q \leq s_0$, then

$$g(\epsilon\phi_q) \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b}\epsilon\phi_q.$$

Thus, by the Maximum Principle for the scalar case, we have:

$$(-\Delta + q - d)^{-1}g(\epsilon\phi_q) \geq \frac{(\lambda_q - a)(\lambda_q - d) - bc}{b(\lambda_q - d)}\epsilon\phi_q$$

and so

$$bB(\epsilon\phi_q) \geq \frac{bc\epsilon}{\lambda_q - d}\phi_q + \frac{(\lambda_q - a)(\lambda_q - d) - bc}{(\lambda_q - d)}\epsilon\phi_q = (\lambda_q - a)\epsilon\phi_q.$$

Since $f(x, \epsilon\phi_q, B\epsilon\phi_q) \geq 0$, we well deduce that $\underline{u} = \epsilon\phi_q$ is a sub-solution of the equation (4.6).

We can conclude as for Theorem 4.1 applying the Schauder Fixed Point Theorem for the operator T defined by (4.8) in the set $[\epsilon\phi_q, \eta\phi_q]$. We have just to verify that $T([\epsilon\phi_q, \eta\phi_q]) \subset [\epsilon\phi_q, \eta\phi_q]$ i.e. if $\epsilon\phi_q \leq u \leq \eta\phi_q$, then $Tu \geq \epsilon\phi_q$. Indeed, from (4.6), since $\epsilon\phi_q$ is a sub-solution of (4.6), we have:

$$(-\Delta + q - a)(Tu - \epsilon\phi_q) \geq b(Bu - B(\epsilon\phi_q)) + f(x, u, Bu) - f(x, \epsilon\phi_q, B\epsilon\phi_q).$$

By the Maximum Principle for the scalar case, since $a < \lambda_q$, we get $\epsilon\phi_q \leq Tu$. □

We conclude giving a uniqueness result. As in [3], we add for that the following hypothesis

- (**h₇**) There exists a concave function H such that $f(x, u, v) = b\frac{\partial H}{\partial u}(x, u, v)$ and $g(x, u) = c\frac{\partial H}{\partial v}(x, u, v)$ for all u, v.

Then, proceeding exactly as in [3], we have the following result.

Theorem 4.3 *Assume that the potential q satisfies the hypothesis ($\mathbf{h_1}$), the coefficients a, b, c, d are real parameters with $b > 0$ and $c > 0$, the function f satisfies the hypothesis ($\mathbf{h_4}$), the function g satisfies the hypothesis ($\mathbf{h_6}$). Assume that the 2×2 matrix Λ be defined by*

$$\Lambda = \begin{pmatrix} \lambda_q - a & -b \\ -(c + K) & \lambda_q - d \end{pmatrix}$$

is a non-singular M-matrix and the hypothesis ($\mathbf{h_7}$) is satisfied. Then the system (4.3) has a unique positive solution $(u, v) \in (V_q(\mathbb{R}^N))^2$.

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received: October 2, 2008

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