Analysis and Applications, Vol. 7, No. 2 (2009) 185–197

EXISTENCE RESULTS FOR A CLASS OF NON-UNIFORMLY ELLIPTIC EQUATIONS OF p-LAPLACIAN TYPE

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Received 5 May 2008
Accepted 23 October 2008

In this paper, we establish the existence of non-trivial weak solutions in $W_0^{1,p}(\Omega)$, $1 < p < \infty$, to a class of non-uniformly elliptic equations of the form

$$-\mathrm{div}(a(x, \nabla u)) = \lambda f(u) + \mu g(u)$$

in a bounded domain Ω of $\mathbb{R}^N$. Here a satisfies

$$|a(x, \xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1 \in L^1_{\mathrm{loc}}(\Omega)$, $h_0(x) \geqq 0$, $h_1(x) \geqq 1$ for a.e. x in Ω.

Keywords: p-Laplacian; non-uniform; elliptic; divergence form; minimum principle.

Mathematics Subject Classification 2000: 35J20, 35J60, 58E05

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. Various particular forms of the Dirichlet problem involving elliptic operators in divergence form

$$-\mathrm{div}(a(x, \nabla u)) = \lambda f(u) \qquad (\mathcal{P}_\lambda)$$

have been studied in the recent years. Here, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ fulfill certain structural conditions.

Recently, [11] studied problem $(\mathcal{P}_\lambda)$ when the potential a satisfies

$$|a(x, \xi)| \leqq c(1 + |\xi|^{p-1}), \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^N$$

for some constant $c > 0$. In [7], the authors extended the result in [11] to the non-uniform case in the sense that the functional associated with the problem may be infinity for some u by assuming the potential a satisfies

$$|a(x,\xi)| \leqq c(h_0(x) + h_1(x)|\xi|^{p-1}), \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^N,$$

where $h_1 \in L^1_{\text{loc}}(\Omega)$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_0(x) \geqq 0, h_1(x) \geqq 1$ for a.e. x in Ω.

In both papers [11, 7], the nonlinear term f verifies the Ambrosetti–Rabinowitz type condition: defining $F(t) = \int_0^t f(s)ds$, there exists $t_0 > 0$ and $\theta > p$ such that

$$0 < \theta F(t) \leqq t f(t), \quad \forall t \in \mathbb{R}, \quad |t| \geqq t_0. \tag{1.1}$$

From that, one can deduce that

$$|f(t)| \geqq c|t|^{\theta-1}, \quad \forall t \in \mathbb{R}, \quad |t| \geqq t_0.$$

This means that f is $(p-1)$-superlinear at infinity. It is worth mentioning that the inequality (1.1) which generalizes to p-Laplacian condition (p_5) in [1], appears for the first time in [5] (see also in [6]).

Very recently in [9], the authors studied problem $(\mathcal{P}_\lambda)$ when the nonlinear term f is continuous and $(p-1)$-sublinear at infinity, i.e.

(f_1) $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{p-1}} = 0$ ($(p-1)$-sublinear at infinity).

They also assume that

(f_2) There exists $s_0 \in \mathbb{R}$ such that $\int_0^{s_0} f(t)\mathrm{d}t > 0$.

With some more restrictive conditions, the authors obtained the existence of three weak solutions of problem $(\mathcal{P}_\lambda)$ via an abstract critical point result due to Bonanno and Ricceri (see [2, 14, 15] for details).

Next, we consider a perturbation of the problem $(\mathcal{P}_\lambda)$ of the form

$$-\mathrm{div}(a(x,\nabla u)) = \lambda f(u) + \mu g(u) \tag{$\mathcal{P}_{\lambda,\mu}$}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. We introduce the following hypothesis regarding function g.

(g) $\lim_{|t|\to+\infty} \frac{|g(t)|}{|t|^{p-1}} = l < +\infty$ (asymptotically $(p-1)$-linear at infinity).

Motivated by the above mentioned papers, in the present paper, by relaxing some conditions on f stated in [9] (we only assume (f_1), (f_2) and (g) hold in our problems), we shall obtain the existence of weak solutions of problem $(\mathcal{P}_\lambda)$ and $(\mathcal{P}_{\lambda,\mu})$ in two directions: one is from $(p-1)$-superlinear at infinity to $(p-1)$-sublinear at infinity together with the presence of the perturbation g and the other is into the non-uniform case. Actually, we shall prove that the corresponding energy functional is coercive and satisfies the usual Palais–Smale condition.

In order to state our main theorem, let us introduce our hypotheses on the structure of problem $(\mathcal{P}_\lambda)$. Assume that $N \geqq 1$ and $p > 1$. Let Ω be a bounded domain in $\mathbb{R}^N$ having C^2 boundary $\partial\Omega$. Consider $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x, \xi)$, as the continuous derivative with respect to ξ of the continuous function $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$, that is, $a(x, \xi) = \frac{\partial A(x,\xi)}{\partial \xi}$. Assume that there are a positive real number c_0 and two nonnegative measurable functions h_0, h_1 on Ω such that $h_1 \in L^1_{\text{loc}}(\Omega)$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1(x) \geqq 1$ for a.e. x in Ω.

Suppose that a and A satisfy the hypotheses below:

(A$_1$) $|a(x, \xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_2$) There exists a constant $k_1 > 0$ such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leqq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k_1 h_1(x)|\xi - \psi|^p$$

for all x, ξ, ψ, that is, A is p-uniformly convex.

(A$_3$) A is p-subhomogeneous, that is,

$$0 \leqq a(x, \xi)\xi \leqq pA(x, \xi)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_4$) There exists a constant $k_0 > 0$ such that

$$A(x, \xi) \geqq k_0 h_1(x)|\xi|^p$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_5$) $A(x, 0) = 0$ for all $x \in \Omega$.

We refer the reader to [7, 10, 11, 16] for various examples. Let $W^{1,p}(\Omega)$ be the usual Sobolev space. Next, we define $X := W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\| = (\int_\Omega |\nabla u|^p \, \mathrm{d}x)^{\frac{1}{p}}$. We now consider the following subspace of $W_0^{1,p}(\Omega)$

$$E = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega h_1(x)|\nabla u|^p \, \mathrm{d}x < +\infty \right\}. \tag{1.2}$$

The space E can be endowed with the norm

$$\|u\|_E = \left(\int_\Omega h_1(x)|\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}. \tag{1.3}$$

As in [7, Lemma 2.7], it is known that E is an infinite dimensional Banach space. We say that $u \in E$ is a weak solution for problem $(\mathcal{P}_\lambda)$ if

$$\int_\Omega a(x, \nabla u)\nabla \phi \, \mathrm{d}x - \lambda \int_\Omega f(u)\phi \, \mathrm{d}x = 0$$

for all $\phi \in E$. Let

$$\Lambda(u) = \int_\Omega A(x, \nabla u) \, \mathrm{d}x, \quad F(t) = \int_0^t f(s)\mathrm{d}s, \quad G(t) = \int_0^t g(s)\mathrm{d}s,$$

$$J_{\lambda,\mu}(u) = \lambda \int_\Omega F(u)\mathrm{d}x + \mu \int_\Omega G(u)\mathrm{d}x,$$

and

$$I_{\lambda,\mu}(u) = \Lambda(u) - J_{\lambda,\mu}(u)$$

for all $u \in E$. The following remark plays an important role in our arguments.

Remark 1.1. (i) $\|u\| \leqq \|u\|_E$ for all $u \in E$ since $h_1(x) \geqq 1$.

(ii) By (A_1), A verifies the growth condition

$$|A(x,\xi)| \leqq c_0(h_0(x)|\xi| + h_1(x)|\xi|^p)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(iii) By (ii) above and (A_4), it is easy to see that

$$E = \{u \in W_0^{1,p}(\Omega) : \Lambda(u) < +\infty\} = \{u \in W_0^{1,p}(\Omega) : I_{\lambda,\mu}(u) < +\infty\}.$$

(iv) $C_0^\infty(\Omega) \subset E$ since $|\nabla u|$ is in $C_c(\Omega)$ for any $u \in C_0^\infty(\Omega)$ and $h_1 \in L^1_{\text{loc}}(\Omega)$.

Now we describe our main result.

Theorem 1.2. *Assume conditions* (A_1)–(A_5) *and* (f_1) *are fulfilled. Then problem* $(\mathcal{P}_\lambda)$ *has at least a weak solution* u *in* E *for every* λ. *If we assume further that* (f_2) *and* $f(0) \neq 0$ *hold true, then* u *is nontrivial provided* λ *is large enough.*

Theorem 1.3. *Assume conditions* (A_1)–(A_5), (f_1) *and* (g) *are fulfilled. Then for each* $\lambda \in \mathbb{R}$, *there exists* $\overline{\mu} > 0$ *such that problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least a weak solution* u *in* E *for every* $\mu \in (0, \overline{\mu})$. *If we assume further that* (f_2) *and* $g(0) \neq 0$ *hold true, then* u *is non-trivial provided* λ *is large enough.*

2. Auxiliary Results

Usually, if a functional is of class $C^1(E, \mathbb{R})$, then it possesses a global minimum value provided it is coercive and satisfies the Palais–Smale condition. Due to the presence of h_0 and h_1, the functional Λ may not belong to $C^1(E, \mathbb{R})$. This means that we cannot apply directly the Minimum Principle, see [3, Theorem 3.1]. In this situation, we need some modifications.

Definition 2.1. Let $\mathcal{F}$ be a map from a Banach space Y to $\mathbb{R}$. We say that $\mathcal{F}$ is weakly continuous differentiable on Y if and only if following two conditions are satisfied

(i) For any $u \in Y$ there exists a linear map $D\mathcal{F}(u)$ from Y to $\mathbb{R}$ such that

$$\lim_{t \to 0} \frac{\mathcal{F}(u + tv) - \mathcal{F}(u)}{t} = D\mathcal{F}(u)(v)$$

for every $v \in Y$.

(ii) For any $v \in Y$, the map $u \mapsto D\mathcal{F}(u)(v)$ is continuous on Y.

Remark 2.2. If we suppose further that $v \mapsto D\mathcal{F}(u)(v)$ is a continuous linear mapping on Y, then $\mathcal{F}$ is Gâteaux differentiable.

Definition 2.3. We call u a generalized critical point (critical point, for short) of $\mathcal{F}$ if $D\mathcal{F}(u) = 0$. c is called a generalized critical value (critical value, for short) of $\mathcal{F}$ if $\mathcal{F}(u) = c$ for some critical point u of $\mathcal{F}$.

Denote by $C_w^1(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C_w^1(Y)$ where we denote by $C^1(Y)$ the set of all continuously Fréchet differentiable functionals on Y. Now let $\mathcal{F} \in C_w^1(Y)$. We put

$$\|D\mathcal{F}(u)\| = \sup\{|D\mathcal{F}(u)(h)| h \in Y, \|h\| = 1\}$$

for any $u \in Y$, where $\|D\mathcal{F}(u)\|$ may be $+\infty$.

Definition 2.4. We say that $\mathcal{F}$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (denoted by $(\mathrm{PS})_c$) if any sequence $\{u_n\} \subset X$ for which

$$\mathcal{F}(u_n) \to c \quad \text{and} \quad D\mathcal{F}(u_n) \to 0 \quad \text{in } X^\star$$

possesses a convergent subsequence. If this is true at every level c then we simply say that $\mathcal{F}$ satisfies the Palais–Smale condition (denoted by (PS)).

Motivated by [3, Theorem 3.1], [12, Theorem 2.3], and [13, Theorem 2], we shall obtain a similar version for weakly continuously differentiable functional which is our main ingredient in this paper.

Theorem 2.5. *Let $\mathcal{F} \in C_w^1(X)$ where X is a Banach space. Assume that*

(i) *$\mathcal{F}$ is bounded from below, $c = \inf \mathcal{F}$,*
(ii) *$\mathcal{F}$ satisfies $(PS)_c$ condition.*

Then c is a critical value of $\mathcal{F}$ (i.e. there exists a critical point $u_0 \in X$ such that $\mathcal{F}(u_0) = c$).

Proof. Let us assume, by negation, that c is not a critical value of $\mathcal{F}$. By $(\mathrm{PS})_c$ we deduce that there exists a constant $\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon]$ contains no critical value of $\mathcal{F}$. Also by $(\mathrm{PS})_c$ we deduce that there exists a constant $\delta > 0$ such that $\|D\mathcal{F}(u)\| \geqq \delta$ for all u such that $\mathcal{F}(u) \in [c - 2\varepsilon, c + 2\varepsilon]$ (see [4, Lemma 2.2]).

Next, we define

$$\begin{aligned} X_1 &:= \{u \in X : c - 2\varepsilon < \mathcal{F}(u) < c + 2\varepsilon\}, \\ X_2 &:= \{u \in X : \mathcal{F}(u) \leqq c - 2\varepsilon \text{ or } c + 2\varepsilon \leqq \mathcal{F}(u)\}, \\ X_3 &:= \{u \in X : c - \varepsilon \leqq \mathcal{F}(u) \leqq c + \varepsilon\}. \end{aligned} \tag{2.1}$$

We firstly see that X_1 is a open set, X_2 and X_3 are closed sets with $X_3 \subset X_1$, $X_2 \cap X_3 = \emptyset$ and $X_1 \cup X_2 = X$.

We now prove that there exists a vector field W on X which is locally Lipschitz continuous on X, $\|W(u)\| \leqq 1$ for all $u \in X$ and $\|W(u)\| = 0$ for each $u \in X_2$. Furthermore, W also satisfies the following inequalities

$$D\mathcal{F}(u)(W(u)) \geqq 0, \quad \text{if } u \in X, \quad D\mathcal{F}(u)(W(u)) \geqq \frac{\delta}{2}, \quad \text{if } u \in X_3. \tag{2.2}$$

Indeed, for each $u \in X$, we can find a vector $w(u) \in X$ such that $\|w(u)\| = 1$ and $D\mathcal{F}(u)(w(u)) \geqq \frac{2}{3}\|D\mathcal{F}(u)\|$. If $u \in X_1$, then we have $D\mathcal{F}(u)(w(u)) > \frac{\delta}{2}$. Hence, there exists an open neighborhood N_u of u in X_1 such that $D\mathcal{F}(v)(w(u)) > \frac{\delta}{2}$ for all $v \in N_u$ since $v \mapsto D\mathcal{F}(v)(w(u))$ is continuous on X.

Because $\{N_u : u \in X_1\}$ is an open covering of X_1, it possesses a locally finite refinement which will be denoted by $\{N_{u_j}\}_{j\in J}$. For each $j \in J$, let $\rho_j(u)$ denote the distance from $u \in X_1$ to the complement of N_{u_j}. Then $\rho_j(\cdot)$ is Lipschitz continuous on X_1 and $\rho_j(u) = 0$ if $u \notin N_{u_j}$. Set

$$\beta_j(x) = \frac{\rho_j(x)}{\sum\limits_{k\in J} \rho_k(x)}, \quad \forall x \in X_1.$$

Since each u belongs to only finitely many sets N_{u_k}, then $\sum_{k\in J} \rho_k(u)$ is only a finite sum. Set

$$W_0(u) = \sum_{j\in J} \beta_j(x) w(u_j), \quad \forall u \in X_1.$$

Then W_0 is locally Lipschitz continuous on X_1 and $W_0(u) > \frac{\delta}{2}$ for all $u \in X_1$. Put

$$\alpha(u) = \frac{\operatorname{dist}(u, X_2)}{\operatorname{dist}(u, X_2) + \operatorname{dist}(x, X_3)}, \quad \forall u \in X.$$

Then $\alpha(u) : X \to [0, 1]$ is Lipschitz continuous on X and

$$\alpha(u) = \begin{cases} 0, & \text{on } X_2, \\ 1, & \text{on } X_3. \end{cases}$$

Set

$$W(u) = \begin{cases} \alpha(u) W_0(u), & \text{for all } u \in X_1, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $W(u)$ is the vector field on X that we need.

Consider the flow $\eta(t) = \eta(t, u)$ defined by $\frac{d\eta}{dt} = -W(\eta)$ with $\eta(0, u) = u$. It can be proved that the solution $\eta(t, u) \in C(\mathbb{R} \times X, X)$ (see [8] for detailed proof). Next, we explore the properties of the pseudo-gradient flow $\eta(t, u)$. By definition,

$$\frac{d}{dt}\mathcal{F}(\eta(t)) = D\mathcal{F}(\eta(t))(-W(\eta(t))) = -D\mathcal{F}(\eta(t))(W(\eta(t))). \tag{2.3}$$

Therefore, by (2.2) and (2.3), $\frac{d}{dt}\mathcal{F}(\eta(t)) \leqq 0$ and the strict inequality holds if $\mathcal{F}(u) \in (c - 2\varepsilon, c + 2\varepsilon)$. Thus, $\mathcal{F}(\eta(t))$ is non-increasing in t, and strictly decreasing if $\mathcal{F}(u) \in (c - 2\varepsilon, c + 2\varepsilon)$. Fixing u, we now claim that if $\mathcal{F}(u) \in [c - \varepsilon, c + \varepsilon]$ and

$\mathcal{F}(\eta(t)) \in [c-\varepsilon, c+\varepsilon]$ for all $t > 0$, then there exists a unique $t_0 > 0$ such that $\mathcal{F}(\eta(t_0)) \leqq c - \varepsilon$.

Indeed, assume that $\mathcal{F}(\eta(t)) \in [c-\varepsilon, c+\varepsilon]$ for all $t > 0$. Then for all $t > 0$, we have

$$2\varepsilon \geqq \mathcal{F}(\eta(0)) - \mathcal{F}(\eta(t)) = -\int_t^0 \mathcal{DF}(\eta(s))W(\eta(s))ds \geqq \int_0^t \frac{\delta}{2}ds = \frac{\delta t}{2}. \tag{2.4}$$

Therefore $t \leqq \frac{4\varepsilon}{\delta}$. We see that the last inequality cannot hold for large t. Hence, for each u such that $\mathcal{F}(u) \in [c-\varepsilon, c+\varepsilon]$ there exists $t_0 > 0$ such that $\mathcal{F}(\eta(t_0, u)) \leqq c - \varepsilon$. This is a contradiction since $c = \inf \mathcal{F}$. Thus c is a critical value of the functional $\mathcal{F}$. □

The following lemma concerns the smoothness of the functional Λ.

Lemma 2.6 (see [7, Lemma 2.4]). (i) *If $\{u_n\}$ is a sequence weakly converging to u in X, denoted by $u_n \rightharpoonup u$, then $\Lambda(u) \leqq \liminf_{n\to\infty} \Lambda(u_n)$.*

(ii) *For all $u, z \in E$*

$$\Lambda\left(\frac{u+z}{2}\right) \leqq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(z) - k_1\|u-z\|_E^p.$$

(iii) *Λ is continuous on E.*

(iv) *Λ is weakly continuously differentiable on E and*

$$D\Lambda(u)(v) = \int_\Omega a(x, \nabla u)\nabla v \, \mathrm{d}x$$

for all $u, v \in E$.

(v) *$\Lambda(u) - \Lambda(v) \geqq D\Lambda(v)(u-v)$ for all $u, v \in E$.*

The following lemma concerns the smoothness of the functional $J_{\lambda,\mu}$. The proof is standard and simple, so we omit it.

Lemma 2.7. (i) *If $u_n \rightharpoonup u$ in X, then $\lim_{n\to\infty} J_{\lambda,\mu}(u_n) = J_{\lambda,\mu}(u)$.*

(ii) *$J_{\lambda,\mu}$ is continuous on E.*

(iii) *$J_{\lambda,\mu}$ is weakly continuously differentiable on E and*

$$DJ_{\lambda,\mu}(u)(v) = \lambda \int_\Omega f(u)v \, \mathrm{d}x + \mu \int_\Omega g(u)v \, \mathrm{d}x$$

for all $u, v \in E$.

Remark 2.8. The continuity of f and g together with conditions (f_1) and (g) imply that $J_{\lambda,\mu}$ is of class C^1.

We are now in a position to prove our main results.

3. Proof of Theorem 1.2

Throughout this section, we always assume that the assumptions (A_1)–(A_5) and (f_1) are fulfilled. We remark that the critical points of the functional $I_{\lambda,0}$ correspond to the weak solutions of $(\mathcal{P}_\lambda)$.

Lemma 3.1. *For every $\lambda \in \mathbb{R}$, the functional $I_{\lambda,0}$ is coercive on E.*

Proof. First, let $\mathcal{S}$ be the best Sobolev constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, that is,

$$\mathcal{S} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}}{\left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}}.$$

Thus, we obtain

$$\mathcal{S}|v|_{L^p} \leqq \|v\|$$

for all $v \in E$. Let us fix $\lambda \in \mathbb{R}$, arbitrarily. By (f_1), there exists $\delta = \delta(\lambda)$ such that

$$|f(t)| \leqq pk_0\mathcal{S}^p \frac{1}{1+|\lambda|}|t|^{p-1}, \quad \forall |t| \geqq \delta.$$

Integrating the above inequality, we have

$$|F(t)| \leqq k_0\mathcal{S}^p \frac{1}{1+|\lambda|}|t|^p + \max_{|s|\leqq\delta}|f(s)||t|, \quad \forall t \in \mathbb{R}.$$

Thus, for every $u \in E$ we obtain

$$\begin{aligned}
I_{\lambda,0}(u) &\geqq \Lambda(u) - |J_{\lambda,0}(u)| \\
&\geqq k_0\|u\|_E^p - k_0\mathcal{S}^p \frac{|\lambda|}{1+|\lambda|}|u|_{L^p}^p - |\lambda||\Omega|^{\frac{1}{p'}}|u|_{L^p} \max_{|s|\leqq\delta}|f(s)| \\
&\geqq k_0\|u\|_E^p - k_0 \frac{|\lambda|}{1+|\lambda|}\|u\|^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\| \max_{|s|\leqq\delta}|f(s)| \\
&\geqq k_0\|u\|_E^p - k_0 \frac{|\lambda|}{1+|\lambda|}\|u\|_E^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E \max_{|s|\leqq\delta}|f(s)| \\
&= \frac{k_0}{1+|\lambda|}\|u\|_E^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E \max_{|s|\leqq\delta}|f(s)|,
\end{aligned}$$

where $p' = \frac{p}{p-1}$. Since $p > 1$, then $I_{\lambda,0}(u) \to +\infty$ whenever $\|u\|_E \to +\infty$. Hence, $I_{\lambda,0}$ is coercive on E. □

Lemma 3.2. *For every $\lambda \in \mathbb{R}$, the functional $I_{\lambda,0}$ satisfies the Palais–Smale condition on E.*

Proof. Let $\{u_n\}$ be a sequence in E and β be a real number such that

$$|I_{\lambda,0}(u_n)| \leqq \beta \quad \text{for all } n \tag{3.1}$$

and

$$DI_{\lambda,0}(u_n) \to 0 \quad \text{in } E^\star. \tag{3.2}$$

Since the functional $I_{\lambda,0}$ is coercive on E, then $\{u_n\}$ is bounded in E. By Remark 1.1(i), we deduce that $\{u_n\}$ is bounded in X. Since X is reflexive, then by passing to a subsequence, still denoted by $\{u_n\}$, we can assume that the sequence $\{u_n\}$ converges weakly to some u in X. We shall prove that the sequence $\{u_n\}$ converges strongly to u in E.

We observe by Remark 1.1(iii) that $u \in E$. Hence $\{\|u_n - u\|_E\}$ is bounded. Since $\{\|DI_{\lambda,0}(u_n)\|_{E^\star}\}$ converges to 0, then $DI_{\lambda,0}(u_n)(u_n - u)$ converges to 0.

We note that (f_1) implies the existence of a constant $c > 0$ such that

$$|f(t)| \leqq c(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} 0 &\leqq \int_\Omega |f(u_n)||u_n - u| \,\mathrm{d}x \\ &\leqq c\int_\Omega |u_n - u| \,\mathrm{d}x + c\int_\Omega |u_n|^{p-1}|u_n - u| \,\mathrm{d}x \\ &\leqq c(|\Omega|^{\frac{1}{p'}} + |u_n|_{L^p}^{p-1})|u_n - u|_{L^p}. \end{aligned}$$

Since $u_n \to u$ strongly in $L^p(\Omega)$, we get

$$\lim_{n\to\infty} \int_\Omega |f(u_n)||u_n - u| \,\mathrm{d}x = 0.$$

Thus

$$\lim_{n\to\infty} DJ_{\lambda,0}(u_n)(u_n - u) = 0.$$

This and the fact that

$$D\Lambda(u_n)(u_n - u) = DI_{\lambda,0}(u_n)(u_n - u) + DJ_{\lambda,0}(u_n)(u_n - u)$$

give

$$\lim_{n\to\infty} D\Lambda(u_n)(u_n - u) = 0.$$

By using (v) in Lemma 2.6, we get

$$\Lambda(u) - \lim_{n\to\infty} \Lambda(u_n) = \lim_{n\to\infty} (\Lambda(u) - \Lambda(u_n)) \geqq \lim_{n\to\infty} D\Lambda(u_n)(u - u_n) = 0.$$

This and (i) in Lemma 2.6 give

$$\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u).$$

Now if we assume by contradiction that $\|u_n - u\|_E$ does not converge to 0, then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m} - u\|_E \geqq \varepsilon$. By using relation (ii) in Lemma 2.6, we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{n_m}) - \Lambda\left(\frac{u_{n_m} + u}{2}\right) \geqq k_1\|u_{n_m} - u\|_E^p \geqq k_1\varepsilon^p.$$

Letting $m \to \infty$, we find that

$$\limsup_{m\to\infty} \quad \Lambda\left(\frac{u_{n_m} + u}{2}\right) \leqq \Lambda(u) - k_1\varepsilon^p.$$

We also have that $\frac{u_{n_m}+u}{2}$ converges weakly to u in E. Using (i) in Lemma 2.6 again, we get

$$\Lambda(u) \leqq \liminf_{m\to\infty} \quad \Lambda\left(\frac{u_{n_m} + u}{2}\right).$$

That is a contradiction. Therefore $\{u_n\}$ converges strongly to u in E. □

Proof of Theorem 1.2. The coerciveness and the Palais–Smale condition are enough to prove that $I_{\lambda,0}$ attains its proper infimum in Banach space E (see Theorem 2.5), so that $(\mathcal{P}_\lambda)$ has at least a solution u in E. We show that u is not trivial for λ large enough. Indeed, let s_0 be a real number as in (f_2) and let $\Omega_1 \subset \Omega$ be an open subset with $|\Omega_1| > 0$. Then, we deduce that there exists $u_1 \in C_0^\infty(\Omega) \subset E$ such that $u_1(x) \equiv s_0$ on $\overline{\Omega}_1$ and $0 \leqq u_1(x) \leqq s_0$ in $\Omega\backslash\Omega_1$. We have

$$\begin{aligned} I_{\lambda,0}(u_1) &= \int_\Omega A(x, \nabla u_1)\mathrm{d}x - \lambda \int_\Omega F(u_1)\mathrm{d}x \\ &\leqq \int_\Omega A(x, \nabla u_1)\mathrm{d}x - \lambda \int_{\Omega_1} F(u_1)\mathrm{d}x \\ &= C - \lambda|\Omega_1|F(s_0), \end{aligned}$$

where C is a positive constant. Thus for λ large enough, we get $I_{\lambda,0}(u_1) < 0$. Hence, the solution u is not trivial. The proof is complete. □

4. Proof of Theorem 1.3

Throughout this section, we always assume that the assumptions (A_1)–(A_5), (f_1) and (g) are fulfilled. The proof of Theorem 1.3 is almost similar to the proof of Theorem 1.2. Let us fix $\lambda \in \mathbb{R}$, arbitrarily.

Lemma 4.1. *For each $\lambda \in \mathbb{R}$, there exists a constant $\overline{\mu} > 0$, dependent of λ, such that for every $\mu \in (0, \overline{\mu})$, the functional $I_{\lambda,\mu}$ is coercive on E.*

Proof. Since g is asymptotically $(p-1)$-linear at infinity, then after integrating there exists a constant $m > 0$ such that

$$|g(t)| \leqq mp\mathcal{S}^p|t|^{p-1} + m, \quad \forall t \in \mathbb{R}.$$

This inequality yields that

$$|G(t)| \leqq m\mathcal{S}^p|t|^p + m|t|, \quad \forall t \in \mathbb{R}.$$

Thus, for every $u \in E$ we obtain

$$\begin{aligned} I_{\lambda,\mu}(u) &\geqq \Lambda(u) - |J_{\lambda,\mu}(u)| \\ &\geqq k_0\|u\|_E^p - k_0\frac{|\lambda|}{1+|\lambda|}\|u\|^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\| \max_{|s|\leqq\delta}|f(s)| \\ &\quad - |\mu|m\|u\|^p - m\frac{|\mu|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\| \\ &\geqq k_0\|u\|_E^p - k_0\frac{|\lambda|}{1+|\lambda|}\|u\|_E^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E \max_{|s|\leqq\delta}|f(s)| \\ &\quad - |\mu|m\|u\|_E^p - m\frac{|\mu|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E \\ &= \left(\frac{k_0}{1+|\lambda|} - |\mu|m\right)\|u\|_E^p - \frac{|\lambda|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E \max_{|s|\leqq\delta}|f(s)| - m\frac{|\mu|}{\mathcal{S}}|\Omega|^{\frac{1}{p'}}\|u\|_E, \end{aligned}$$

where $p' = \frac{p}{p-1}$. Let $\overline{\mu} = \frac{k_0}{m(1+|\lambda|)}$ and fix $\mu \in (0, \overline{m})$. Since $p > 1$, then $I_{\lambda,\mu}(u) \to +\infty$ whenever $\|u\|_E \to +\infty$. Hence, $I_{\lambda,\mu}$ is coercive on E. □

Lemma 4.2. *Let λ and $\overline{\mu}$ be chosen as in the previous lemma. Then, the functional $I_{\lambda,\mu}$ satisfies the Palais–Smale condition on E for every $\mu \in (0, \overline{\mu})$.*

Proof. Let $\{u_n\}$ be a sequence in E and β be a real number such that

$$|I_{\lambda,\mu}(u_n)| \leqq \beta \quad \text{for all } n \tag{4.1}$$

and

$$DI_{\lambda,\mu}(u_n) \to 0 \quad \text{in } E^\star. \tag{4.2}$$

Similar to the proof of Lemma 3.2, $\{u_n\}$ is bounded in E and then is bounded in X. Therefore, there exists $u \in X$ such that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$. We observe by Remark 1.1(iii) that $u \in E$. Hence $\{\|u_n - u\|_E\}$ is bounded. Since $\{\|DI_{\lambda,0}(u_n - u)\|_{E^\star}\}$ converges to 0, then $DI_{\lambda,0}(u_n - u)(u_n - u)$ converges to 0.

We note that (g) implies the existence of a constant $c > 0$ such that

$$|g(t)| \leqq c(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} 0 &\leqq \int_\Omega |g(u_n)||u_n - u|\mathrm{d}x \leqq c\int_\Omega |u_n - u|\,\mathrm{d}x + c\int_\Omega |u_n|^{p-1}|u_n - u|\mathrm{d}x \\ &\leqq c(|\Omega|^{\frac{1}{p'}} + \|u\|_n^{p-1})\|u_n - u\|_p. \end{aligned}$$

Since $u_n \to u$ strongly in $L^p(\Omega)$, we get

$$\lim_{n\to\infty}\int_\Omega |g(u_n)||u_n - u|\,\mathrm{d}x = 0.$$

Thus

$$\lim_{n\to\infty} DJ_{\lambda,\mu}(u_n)(u_n - u) = 0.$$

This and the fact that

$$D\Lambda(u_n)(u_n - u) = DI_{\lambda,\mu}(u_n)(u_n - u) + DJ_{\lambda,\mu}(u_n)(u_n - u)$$

give

$$\lim_{n\to\infty} D\Lambda(u_n)(u_n - u) = 0.$$

Similar to the last part of the proof of Lemma 3.2, the last equality yields $u_n \to u$ in E. This completes our proof. □

Proof of Theorem 1.3. The coerciveness and the Palais–Smale condition are enough to prove that $I_{\lambda,\mu}$ attains its proper infimum in Banach space E (see Theorem 2.5), so that $(\mathcal{P}_{\lambda,\mu})$ has at least a solution u in E. We show that u is not trivial for λ large enough. Indeed, let s_0 be a real number as in (f_2) and let $\Omega_1 \subset \Omega$ be an open subset with $|\Omega_1| > 0$. Then, we deduce that there exists $u_1 \in C_0^\infty(\Omega) \subset E$ such that $u_1(x) \equiv s_0$ on $\overline{\Omega}_1$ and $0 \leqq u_1(x) \leqq s_0$ in $\Omega \backslash \Omega_1$. We have

$$\begin{aligned} I_{\lambda,\mu}(u_1) &= \int_\Omega A(x, \nabla u_1)\mathrm{d}x - \lambda \int_\Omega F(u_1)\mathrm{d}x - \mu \int_\Omega G(u_1)\mathrm{d}x \\ &\leqq \int_\Omega A(x, \nabla u_1)\mathrm{d}x - \lambda \int_{\Omega_1} F(u_1)\mathrm{d}x - \mu \int_\Omega G(u_1)\mathrm{d}x \\ &= \int_\Omega A(x, \nabla u_1)\mathrm{d}x - \mu \int_\Omega G(u_1)\mathrm{d}x - \lambda \int_{\Omega_1} F(u_1)\mathrm{d}x \\ &= C - \lambda|\Omega_1|F(s_0), \end{aligned}$$

where C is a positive constant (it is important to notice that the constant C actually depends on the parameter μ). Thus, for λ large enough, we get $I_{\lambda,\mu}(u_1) < 0$. Hence, the solution u is not trivial. The proof is complete. □

Acknowledgments

The author wishes to express gratitude to the anonymous referees for a number of valuable comments and suggestions which helped to improve the presentation of the present paper from line to line. The author also would like to thank Professor Hoàng Quoc Toan, for his interest, encouragement, fruitful discussions and helpful comments.

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