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Some Remarks on a Class of Nonuniformly Elliptic Equations of p-Laplacian Type

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Abstract This paper deals with the existence of weak solutions in $W_0^1(\Omega)$ to a class of elliptic problems of the form

$$-\operatorname{div}(a(x,\nabla u)) = \lambda_1 |u|^{p-2} u + g(u) - h$$

in a bounded domain Ω of $\mathbb{R}^N$. Here a satisfies

$$|a(x,\xi)| \leqq c_0 \left(h_0(x) + h_1(x)|\xi|^{p-1}\right)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1 \in L^1_{loc}(\Omega)$, $h_1(x) \geqq 1$ for a.e. x in Ω; λ_1 is the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition and g, h satisfy some suitable conditions.

Keywords p-Laplacian · Nonuniform · Landesman-Laser · Elliptic · Divergence form · Landesman-Laser type

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. In the present paper we study the existence of weak solutions of the following Dirichlet problem

$$-\operatorname{div}(a(x,\nabla u)) = \lambda_1 |u|^{p-2} u + g(u) - h \tag{1}$$

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where $|a(x,\xi)| \leqq c_0(h_0(x)+h_1(x)|\xi|^{p-1})$ for any ξ in $\mathbb{R}^N$ and a.e. $x\in\Omega$, $h_0(x)\geqq 0$ and $h_1(x)\geqq 1$ for any x in Ω. λ_1 is the first eigenvalue for $-\Delta_p$ on Ω with zero Dirichlet boundary condition, that is,

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x \,\middle|\, \int_\Omega |u|^p \,\mathrm{d}x = 1 \right\}.$$

Recall that λ_1 is simple and positive. Moreover, there exists a unique positive eigenfunction ϕ_1 whose norm in $W_0^{1,p}(\Omega)$ equals to one. Regarding the functions g, we assume that g is a continuous function. We also assume that $h\in L^{p'}(\Omega)$ where we denote p' by $\frac{p}{p-1}$.

In the present paper, we study the case in which h_0 and h_1 belong to $L^{p'}(\Omega)$ and $L^1_{loc}(\Omega)$, respectively. The problem now may be non-uniform in sense that the functional associated to the problem may be infinity for some u in $W_0^{1,p}(\Omega)$. Hence, weak solutions of the problem must be found in some suitable subspace of $W_0^{1,p}(\Omega)$. To our knowledge, such equations were firstly studied by [4, 9, 10]. Our paper was motivated by the result in [2] and the generalized form of the Landesman–Lazer conditions considerred in [7, 8]. While the semilinear problem is studied in [7, 8] and the quasilinear problem is studied in [2], it turns out that a different technique allows us to use these conditions also for problem (1) and to generalize the result of [1]. In order to state our main theorem, let us introduce our hypotheses on the structure of problem (1).

Assume that $N\geqq 1$ and $p>1$. Ω be a bounded domain in $\mathbb{R}^N$ having C^2 boundary $\partial\Omega$. Consider $a:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$, $a=a(x,\xi)$, as the continuous derivative with respect to ξ of the continuous function $A:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, $A=A(x,\xi)$, that is, $a(x,\xi)=\frac{\partial A(x,\xi)}{\partial\xi}$. Assume that there are a positive real number c_0 and two nonnegative measurable functions h_0, h_1 on Ω such that $h_1\in L^1_{loc}(\Omega)$, $h_0\in L^{p'}(\Omega)$, $h_1(x)\geqq 1$ for a.e. x in Ω.

Suppose that a and A satisfy the hypotheses below

(A_1) $|a(x,\xi)|\leqq c_0(h_0(x)+h_1(x)|\xi|^{p-1})$ for all $\xi\in\mathbb{R}^N$, a.e. $x\in\Omega$.

(A_2) There exists a constant $k_1>0$ such that

$$A\left(x,\frac{\xi+\psi}{2}\right)\leqq \frac{1}{2}A(x,\xi)+\frac{1}{2}A(x,\psi)-k_1h_1(x)|\xi-\psi|^p$$

for all x, ξ, ψ, that is, A is p-uniformly convex.

(A_3) A is p-subhomogeneous, that is,

$$0\leqq a(x,\xi)\xi\leqq pA(x,\xi)$$

for all $\xi\in\mathbb{R}^N$, a.e. $x\in\Omega$.

(A_4) There exists a constant $k_0\geqq\frac{1}{p}$ such that

$$A(x,\xi)\geqq k_0h_1(x)|\xi|^p$$

for all $\xi\in\mathbb{R}^N$, a.e. $x\in\Omega$.

(A_5) $A(x,0)=0$ for all $x\in\Omega$.

We refer the reader to [4–6, 9, 10] for various examples. We suppose also that

(H_1)

$$\lim_{|t|\to\infty}\frac{g(t)}{|t|^{p-1}}=0.$$

Let us define

$$F(t) = \begin{cases} \frac{p}{t} \int_0^t g(s)\,\mathrm{d}s - g(t), & t \neq 0, \\ (p-1)\,g(0), & t = 0, \end{cases} \tag{2}$$

and set

$$\overline{F(-\infty)} = \limsup_{t \to -\infty} F(t), \qquad \overline{F(+\infty)} = \limsup_{t \to +\infty} F(t), \tag{3}$$

$$\underline{F(-\infty)} = \liminf_{t \to -\infty} F(t), \qquad \underline{F(+\infty)} = \liminf_{t \to +\infty} F(t). \tag{4}$$

We suppose also that

(H_2)

$$\overline{F(+\infty)} \int_\Omega \phi_1(x)\,\mathrm{d}x < (p-1) \int_\Omega h(x)\,\phi_1(x)\,\mathrm{d}x < \underline{F(-\infty)} \int_\Omega \phi_1(x)\,\mathrm{d}x.$$

By mean of (H_2), we see that $-\infty < \underline{F(-\infty)}$ and $\overline{F(+\infty)} < +\infty$. It is known that under (H_1) and (H_2), when $A(x,\xi) = \frac{1}{p}|\xi|^p$, our problem (1) has a weak solution, see [2, Theorem 1.1]. In that paper, property $pA(x,\xi) = a(x,\xi)\cdot\xi$, which may not hold under our assumptions by (A_4), play an important role in the arguments. This leads us to study the case when $pA(x,\xi) \geqq a(x,\xi)\cdot\xi$. Our paper is also motivated by some results obtained in [2]. We shall extend some results in [2] in two directions: one is from p-Laplacian operators to general elliptic operators in divergence form and the other is to the case on non-uniform problem.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space. Next, we define $X := W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x \right)^{\frac{1}{p}}.$$

We now consider the following subspace of $W_0^{1,p}(\Omega)$

$$E = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega h_1(x)\,|\nabla u|^p\,\mathrm{d}x < +\infty \right\}. \tag{5}$$

The space E can be endowed with the norm

$$\|u\|_E = \left(\int_\Omega h_1(x)\,|\nabla u|^p\,\mathrm{d}x \right)^{\frac{1}{p}}. \tag{6}$$

As in [4, Lemma 2.7], it is known that E is an infinite dimensional Banach space. We say that $u \in E$ is a weak solution for problem (1) if

$$\int_\Omega a(x,\nabla u)\,\nabla\phi\,\mathrm{d}x - \lambda_1 \int_\Omega |u|^{p-2} u\phi\,\mathrm{d}x - \int_\Omega g(u)\,\phi\,\mathrm{d}x + \int_\Omega h\phi\,\mathrm{d}x = 0$$

for all $\phi \in E$. Let

$$\Lambda(u) = \int_\Omega A(x,\nabla u)\,\mathrm{d}x, \qquad G(t) = \int_0^t g(s)\,\mathrm{d}s,$$

$$J(u) = \frac{\lambda_1}{p} \int_\Omega |u|^p\,\mathrm{d}x + \int_\Omega G(u)\,\mathrm{d}x - \int_\Omega hu\,\mathrm{d}x,$$

and

$$I(u) = \Lambda(u) - J(u)$$

for all $u \in E$. The following remark plays an important role in our arguments.

Remark 1

(i) $\|u\| \leqq \|u\|_E$ for all $u \in E$ since $h_1(x) \geqq 1$.
(ii) By (A_1), A verifies the growth condition

$$|A(x,\xi)| \leqq c_0 \left(h_0(x)|\xi| + h_1(x)|\xi|^p\right)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.
(iii) By (ii) above and (A_4), it is easy to see that

$$E = \left\{u \in W_0^{1,p}(\Omega) : \Lambda(u) < +\infty\right\} = \left\{u \in W_0^{1,p}(\Omega) : I(u) < +\infty\right\}.$$

(iv) $C_0^\infty(\Omega) \subset E$ since $|\nabla u|$ is in $C_c(\Omega)$ for any $u \in C_0^\infty(\Omega)$ and $h_1 \in L^1_{loc}(\Omega)$.
(v) By (A_4) and Poincaré inequality, we see that

$$\int_\Omega A(x, \nabla u)\,\mathrm{d}x \geqq \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x \geqq \frac{\lambda_1}{p}\int_\Omega |u|^p\,\mathrm{d}x,$$

for all $u \in W_0^{1,p}(\Omega)$.

Now we describe our main result.

Theorem 1 *Assume conditions* (A_1)–(A_5) *and* (H_1)–(H_2) *are fulfilled. Then problem* (1) *has at least a weak solution in* E.

2 Auxiliary Results

Due to the presence of h_1, the functional Λ may not belong to $C^1(E,\mathbb{R})$. This means that we cannot apply the Minimum Principle directly, see [3, Theorem 3.1]. In this situation, we need some modifications.

Definition 1 Let $\mathcal{F}$ be a map from a Banach space Y to $\mathbb{R}$. We say that $\mathcal{F}$ is weakly continuous differentiable on Y if and only if following two conditions are satisfied

(i) For any $u \in Y$ there exists a linear map $D\mathcal{F}(u)$ from Y to $\mathbb{R}$ such that

$$\lim_{t\to 0}\frac{\mathcal{F}(u+tv) - \mathcal{F}(u)}{t} = \langle D\mathcal{F}(u), v\rangle$$

for every $v \in Y$.
(ii) For any $v \in Y$, the map $u \mapsto \langle D\mathcal{F}(u), v\rangle$ is continuous on Y.

Denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C^1_w(Y)$ where we denote by $C^1(Y)$ the set of all continuously Fréchet differentiable functionals on Y. Now let $\mathcal{F} \in C^1_w(Y)$, we put

$$\|D\mathcal{F}(u)\| = \sup\{|\langle D\mathcal{F}(u), h\rangle : |h \in Y, \|h\| = 1\}$$

for any $u \in Y$, where $\|D\mathcal{F}(u)\|$ may be $+\infty$.

Definition 2 We say that $\mathcal{F}$ satisfies the Palais-Smale condition if any sequence $\{u_n\} \subset Y$ for which $\mathcal{F}(u_n)$ is bounded and $\lim_{n\to\infty} \|D\mathcal{F}(u_n)\| = 0$ possesses a convergent subsequence.

The following theorem is our main ingredient.

Theorem 2 (The Minimum Principle) *Let $\mathcal{F} \in C^1_w(Y)$ where Y is a Banach space. Assume that*

(i) *$\mathcal{F}$ is bounded from below, $c = \inf \mathcal{F}$,*
(ii) *$\mathcal{F}$ satisfies Palais-Smale condition.*

Then c is a critical value of $\mathcal{F}$ (i.e., there exists a critical point $u_0 \in Y$ such that $\mathcal{F}(u_0) = c$).

Let Y be a real Banach space, $\mathcal{F} \in C^1_w(Y)$ and c is a arbitrary real number. Before proving Theorem 2, we need the following notations.

$$\mathcal{F}^c = \{u \in Y \,|\mathcal{F}(u) \le c\},$$

$$K_c = \{u \in Y \,|\mathcal{F}(u) = c, D\mathcal{F}(u) = 0\}.$$

In order to prove Theorem 2, we need a modified Deformation Lemma which is proved in [10]. Here we recall it for completeness.

Lemma 1 (See [10], Theorem 2.2) *Let Y be a real Banach space, and $\mathcal{F} \in C^1_w(Y)$. Suppose that $\mathcal{F}$ satisfies Palais-Smale condition. Let $c \in \mathbb{R}$, $\overline{\varepsilon} > 0$ be given and let $\mathcal{O}$ be any neighborhood of K_c. Then there exists a number $\varepsilon \in (0, \overline{\varepsilon})$ and $\eta \in C((0, +\infty], Y \times Y)$ such that*

(i) *$\eta(0, u) = u$ in Y.*
(ii) *$\eta(t, u) = u$ for all $t \geqq 0$ and $u \in Y \backslash \mathcal{F}^{-1}([c - \overline{\varepsilon}, c + \overline{\varepsilon}])$.*
(iii) *$\eta(t, \cdot)$ is a homeomorphism of Y onto Y for each $t \geqq 0$.*
(iv) *$\|\eta(t, u) - u\| \leqq t$ for all $t \geqq 0$ and $u \in Y$.*
(v) *For all $u \in Y$, $\mathcal{F}(\eta(t, u))$ is non-increasing with respect to t.*
(vi) *$\eta(1, \mathcal{F}^{c+\varepsilon} \backslash \mathcal{O}) \subset \mathcal{F}^{c-\varepsilon}$.*
(vii) *If $K_c = \emptyset$ then $\eta(1, \mathcal{F}^{c+\varepsilon}) \subset \mathcal{F}^{c-\varepsilon}$.*
(viii) *If $\mathcal{F}$ is even on Y then $\eta(t, \cdot)$ is odd in Y.*

Proof of Theorem 2 Let us assume, by negation, that c is not a critical value of $\mathcal{F}$. Then, Lemma 1 implies the existence of $\varepsilon > 0$ and $\eta \in C([0, +\infty), Y \times Y)$ satisfying $\eta(1, \mathcal{F}^{c+\varepsilon}) \subset \mathcal{F}^{c-\varepsilon}$. This is a contradiction since $\mathcal{F}^{c-\varepsilon} = \emptyset$ due to the fact that $c = \inf \mathcal{F}$. □

For simplicity of notation, we shall denote $D\mathcal{F}(u)$ by $\mathcal{F}'(u)$. The following lemma concerns the smoothness of the functional Λ.

Lemma 2 (See [4], Lemma 2.4)

(i) *If* $\{u_n\}$ *is a sequence weakly converging to* u *in* X, *denoted by* $u_n \rightharpoonup u$, *then* $\Lambda(u) \leqq \liminf_{n\to\infty} \Lambda(u_n)$.
(ii) *For all* $u, z \in E$

$$\Lambda\left(\frac{u+z}{2}\right) \leqq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(z) - k_1 \|u-z\|_E^p.$$

(iii) Λ *is continuous on* E.
(iv) Λ *is weakly continuously differentiable on* E *and*

$$\left\langle \Lambda'(u), v\right\rangle = \int_\Omega a(x, \nabla u)\nabla v \mathrm{d}x$$

for all $u, v \in E$.
(v) $\Lambda(u) - \Lambda(v) \geqq \langle \Lambda'(v), u-v\rangle$ *for all* $u, v \in E$.

The following lemma concerns the smoothness of the functional J. The proof is standard and simple, so we omit it.

Lemma 3

(i) *If* $u_n \rightharpoonup u$ *in* X, *then* $\lim_{n\to\infty} J(u_n) = J(u)$.
(ii) J *is continuous on* E.
(iii) J *is weakly continuously differentiable on* E *and*

$$\left\langle J'(u), v\right\rangle = \lambda_1 \int_\Omega |u|^{p-2} uv\mathrm{d}x + \int_\Omega g(u)v\mathrm{d}x - \int_\Omega hv\mathrm{d}x$$

for all $u, v \in E$.

3 Proofs

We remark that the critical points of the functional I correspond to the weak solutions of (1). Throughout this paper, we sometimes denote by "const" a positive constant. We are now in position to prove our main result.

Lemma 4 *I satisfies the Palais-Smale condition on E provided* (H_2) *holds true.*

Proof Let $\{u_n\}$ be a sequence in E and β be a real number such that

$$|I(u_n)| \leqq \beta \quad \text{for all } n \tag{7}$$

and

$$I'(u_n) \to 0 \quad \text{in } E^\star. \tag{8}$$

We prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\|_E \to \infty$ as $n \to \infty$. Letting $v_n = \frac{u_n}{\|u_n\|_E}$ for every n. Thus $\{v_n\}$ is bounded in E. By Remark 1(i), we deduce that $\{v_n\}$ is bounded in X. Since X is reflexive, then by passing to a subsequence, still denotes by $\{v_n\}$, we can assume that the sequence $\{v_n\}$ converges weakly to some v in X. Since the embedding $X \hookrightarrow L^p(\Omega)$ is compact then $\{v_n\}$ converges strongly to v in $L^p(\Omega)$.

Dividing (7) by $\|u_n\|_E^p$ together with Remark 1(v), we deduce that

$$\limsup_{n\to+\infty}\left(\frac{1}{p}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x - \frac{\lambda_1}{p}\int_\Omega |v_n|^p\,\mathrm{d}x - \int_\Omega \frac{G(u_n)}{\|u_n\|_E^p}\mathrm{d}x + \int_\Omega h\frac{u_n}{\|u_n\|_E^p}\mathrm{d}x\right) \leqq 0.$$

Since, by the hypotheses on p, g, h and $\{u_n\}$,

$$\limsup_{n\to+\infty}\left(\int_\Omega \frac{G(u_n)}{\|u_n\|_E^p}\mathrm{d}x + \int_\Omega h\frac{u_n}{\|u_n\|_E^p}\mathrm{d}x\right) = 0,$$

while

$$\limsup_{n\to+\infty}\int_\Omega |v_n|^p\,\mathrm{d}x = \int_\Omega |v|^p\,\mathrm{d}x,$$

we have

$$\limsup_{n\to+\infty}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x \leqq \lambda_1\int_\Omega |v|^p\,\mathrm{d}x.$$

Using the weak lower semi-continuity of norm and Poincaré inequality, we get

$$\begin{aligned}\lambda_1\int_\Omega |v|^p\,\mathrm{d}x \leqq \int_\Omega |\nabla v|^p\,\mathrm{d}x \leqq \liminf_{n\to+\infty}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x\\ \leqq \limsup_{n\to+\infty}\int_\Omega |\nabla v_n|^p\,\mathrm{d}x \leqq \lambda_1\int_\Omega |v|^p\,\mathrm{d}x.\end{aligned}$$

Thus, the inequalities are indeed equalities. Beside, $\{v_n\}$ converges strongly to v in X and $\int_\Omega |\nabla v|^p\mathrm{d}x = \lambda_1\int_\Omega |v|^p\mathrm{d}x$. This implies, by the definition of ϕ_1, that $v = \pm\phi_1$. Let us assume that $v = \phi_1 > 0$ in Ω (the other case is treated similarly). By mean of (7), we deduce that

$$-\beta p \leqq p\int_\Omega A(x,\nabla u_n)\,\mathrm{d}x - \lambda_1\int_\Omega |u_n|^p\,\mathrm{d}x - p\int_\Omega G(u_n)\,\mathrm{d}x + p\int_\Omega hu_n\mathrm{d}x \leqq \beta p. \quad (9)$$

In view of (8),

$$\begin{aligned}-\varepsilon_n\|u_n\|_E \leqq -\int_\Omega a(x,\nabla u_n)\nabla u_n\mathrm{d}x + \lambda_1\int_\Omega |u_n|^p\,\mathrm{d}x\\ + \int_\Omega g(u_n)u_n\mathrm{d}x - \int_\Omega hu_n\mathrm{d}x \leqq \varepsilon_n\|u_n\|_E.\end{aligned} \quad (10)$$

By summing up (9) and (10), we get

$$\begin{aligned}-\beta p - \varepsilon_n\|u_n\|_E \leqq &\int_\Omega (pA(x,\nabla u_n) - a(x,\nabla u_n)\nabla u_n)\,\mathrm{d}x\\ &- \int_\Omega (pG(u_n) - g(u_n)u_n)\,\mathrm{d}x + (p-1)\int_\Omega hu_n\mathrm{d}x\\ &\leqq \beta p + \varepsilon_n\|u_n\|_E,\end{aligned}$$

which gives

$$-\int_\Omega (pG(u_n) - g(u_n)u_n)\,\mathrm{d}x + (p-1)\int_\Omega hu_n \mathrm{d}x \leqq \beta p + \varepsilon_n \|u_n\|_E,$$

and after dividing by $\|u_n\|_E$, we obtain

$$-\int_\Omega \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|_E}\mathrm{d}x + (p-1)\int_\Omega hv_n \mathrm{d}x \leqq \frac{\beta p}{\|u_n\|_E} + \varepsilon_n.$$

Taking lim sup to both sides, we then deduce

$$(p-1)\int_\Omega h\phi_1(x)\,\mathrm{d}x \leqq \limsup_{n\to+\infty}\int_\Omega \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|_E}\mathrm{d}x$$

which gives

$$(p-1)\int_\Omega h\phi_1(x)\,\mathrm{d}x \leqq \limsup_{n\to+\infty}\int_\Omega F(u_n)\frac{u_n}{\|u_n\|_E}\mathrm{d}x = \limsup_{n\to+\infty}\int_\Omega F(u_n)v_n \mathrm{d}x.$$

For $\varepsilon > 0$, let

$$c_\varepsilon = \begin{cases} \overline{F(+\infty)} + \varepsilon, & \text{if } \overline{F(+\infty)} > -\infty, \\ -\frac{1}{\varepsilon}, & \text{if } \overline{F(+\infty)} = -\infty, \end{cases} \tag{11}$$

and

$$d_\varepsilon = \begin{cases} \underline{F(-\infty)} - \varepsilon, & \text{if } \underline{F(-\infty)} > -\infty, \\ \frac{1}{\varepsilon}, & \text{if } \underline{F(-\infty)} = +\infty. \end{cases} \tag{12}$$

Then there exists $M > 0$ such that $c_\varepsilon t \geqq F(t)t$ for all $t > M$ and $d_\varepsilon t \geqq F(t)t$ for all $t < -M$. Moreover, the continuity of F on $\mathbb{R}$ implies that for any $K > 0$ there exists $c(K) > 0$ such that $|F(t)| \leqq c(K)$ for all $t \in [-K, K]$. We now set

$$\int_\Omega F(u_n)v_n \mathrm{d}x = \underbrace{\int_{|u_n(x)|\leqq K} F(u_n)v_n \mathrm{d}x}_{A_{K,n}} + \underbrace{\int_{u_n(x)<-K} F(u_n)v_n \mathrm{d}x}_{C_{K,n}} + \underbrace{\int_{u_n(x)>K} F(u_n)v_n \mathrm{d}x}_{B_{K,n}}.$$

Thanks to Lemma 2.1 in [2], we have

$$\lim_{n\to\infty} \text{meas}\left\{x \in \Omega \,\middle|\, u_n(x) \leqq K\right\} = 0.$$

We are now ready to estimate $A_{K,n}$, $B_{K,n}$ and $C_{K,n}$.

$$\left|A_{K,n}\right| \leqq \int_{|u_n(x)|\leq K} |F(u_n)|\frac{|u_n|}{\|u_n\|}\mathrm{d}x \leqq \frac{c(K)\,K\,\text{meas}(\Omega)}{\|u_n\|} \to 0,$$

$$B_{K,n} \leqq c_\varepsilon \int_{u_n(x)>K} v_n \mathrm{d}x = c_\varepsilon\left(\int_\Omega v_n \mathrm{d}x - \int_{u_n(x)\leqq K} v_n \mathrm{d}x\right) \to c_\varepsilon \int_\Omega \phi_1 \mathrm{d}x,$$

$$C_{K,n} \leqq d_\varepsilon \int_{u_n(x)<-K} v_n \mathrm{d}x \to 0.$$

Summing up we deduce that

$$\limsup_{n\to+\infty}\int_\Omega F(u_n)\frac{u_n}{\|u_n\|_E}\mathrm{d}x \leqq c_\varepsilon\int_\Omega \phi_1(x)\,\mathrm{d}x$$

for any $\varepsilon \searrow 0$ which yields

$$(p-1)\int_\Omega h\phi_1(x)\,\mathrm{d}x \leqq \overline{F(+\infty)}\int_\Omega \phi_1(x)\,\mathrm{d}x$$

which contradicts (H_2).

Hence $\{u_n\}$ is bounded in E. By Remark 1(i), we deduce that $\{u_n\}$ is bounded in X. Since X is reflexible, then by passing to a subsequence, still denoted by $\{u_n\}$, we can assume that the sequence $\{u_n\}$ converges weakly to some u in X. We shall prove that the sequence $\{u_n\}$ converges strongly to u in E.

We observe by Remark 1(iii) that $u \in E$. Hence $\{\|u_n-u\|_E\}$ is bounded. Since $\{\|I'(u_n-u)\|_{E^\star}\}$ converges to 0, then $\langle I'(u_n-u), u_n-u\rangle$ converges to 0.

By the hypotheses on g and h, we easily deduce that

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x = 0,$$

$$\lim_{n\to+\infty}\int_\Omega g(u_n)(u_n-u)\,\mathrm{d}x = 0, \qquad \lim_{n\to+\infty}\int_\Omega h(u_n-u)\,\mathrm{d}x = 0.$$

On the other hand,

$$\left\langle J'(u_n), u_n-u\right\rangle\rangle = \lambda_1\int_\Omega |u_n|^{p-2}u_n(u_n-u)\mathrm{d}x + \int_\Omega g(u_n)(u_n-u)\mathrm{d}x + \int_\Omega h(u_n-u)\mathrm{d}x.$$

Thus

$$\lim_{n\to\infty}\left\langle J'(u_n), u_n-u\right\rangle = 0.$$

This and the fact that

$$\left\langle \Lambda'(u_n), u_n-u\right\rangle = \left\langle I'(u_n), u_n-u\right\rangle + \left\langle J'(u_n), u_n-u\right\rangle$$

give

$$\lim_{n\to\infty}\left\langle \Lambda'(u_n), u_n-u\right\rangle = 0.$$

By using (v) in Lemma 2, we get

$$\Lambda(u) - \limsup_{n\to\infty}\Lambda(u_n) = \liminf_{n\to\infty}\big(\Lambda(u)-\Lambda(u_n)\big) \geqq \lim_{n\to\infty}\left\langle \Lambda'(u_n), u-u_n\right\rangle = 0.$$

This and (i) in Lemma 2 give

$$\lim_{n\to\infty}\Lambda(u_n) = \Lambda(u).$$

Now if we assume by contradiction that $\|u_n-u\|_E$ does not converge to 0 then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that $\|u_{n_m}-u\|_E \geqq \varepsilon$. By using relation (ii) in Lemma 2, we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda\left(u_{n_m}\right) - \Lambda\left(\frac{u_{n_m}+u}{2}\right) \geqq k_1\left\|u_{n_m}-u\right\|_E^p \geqq k_1\varepsilon^p.$$

Letting $m \to \infty$ we find that

$$\limsup_{m\to\infty} \Lambda\left(\frac{u_{n_m}+u}{2}\right) \leqq \Lambda(u) - k_1\varepsilon^p.$$

We also have $\frac{u_{n_m}+u}{2}$ converges weakly to u in E. Using (i) in Lemma 2 again, we get

$$\Lambda(u) \leqq \liminf_{m\to\infty} \Lambda\left(\frac{u_{n_m}+u}{2}\right).$$

That is a contradiction. Therefore $\{u_n\}$ converges strongly to u in E. □

Lemma 5 *I is coercive on E provided* (H$_2$) *holds true.*

Proof We firstly note that, in the proof of the Palais-Smale condition, we have proved that if $I(u_n)$ is a sequence bounded from above with $\|u_n\|_E \to \infty$, then (up to a subsequence), $v_n = \frac{u_n}{\|u_n\|_E} \to \pm\phi_1$ in X. Using this fact, we will prove that I is coercive provided (H$_2$) holds true.

Indeed, if I is not coercive, it is possible to choose a sequence $\{u_n\} \subset E$ such that $\|u_n\|_E \to \infty$, $I(u_n) \leq \text{const}$ and $v_n = \frac{u_n}{\|u_n\|_E} \to \pm\phi_1$ in X. We can assume without loss of generality that $v_n \to \phi_1$ in X. By Remark 1(v),

$$-\int_\Omega G(u_n)\,\mathrm{d}x + \int_\Omega hu_n\mathrm{d}x \leqq I(u_n). \tag{13}$$

The rest of the proof follows the proof of Lemma 2.3 in [2]. We include it in brief for completeness. Dividing (13) by $\|u_n\|_E$ and then letting $n \to +\infty$ we get

$$\limsup_{n\to+\infty}\left(-\int_\Omega \frac{G(u_n)}{\|u_n\|_E}\mathrm{d}x + \int_\Omega h\frac{u_n}{\|u_n\|_E}\mathrm{d}x\right) \leqq \limsup_{n\to+\infty}\frac{I(u_n)}{\|u_n\|_E} \leqq \limsup_{n\to+\infty}\frac{\text{const}}{\|u_n\|_E} = 0,$$

which gives

$$\int_\Omega h\phi_1\mathrm{d}x \leqq \liminf_{n\to+\infty}\int_\Omega \frac{G(u_n)}{\|u_n\|_E}\mathrm{d}x \leqq \limsup_{n\to+\infty}\int_\Omega \frac{G(u_n)}{\|u_n\|_E}\mathrm{d}x.$$

Again, thanks to Lemma 2.3 in [2], we have

$$\limsup_{n\to+\infty}\int_\Omega \frac{G(u_n)}{\|u_n\|_E}\mathrm{d}x \leqq \frac{c_\varepsilon}{p-1}\int_\Omega \phi_1\mathrm{d}x,$$

where c_ε is as (11). Summing up we deduce that

$$\int_\Omega h\phi_1\mathrm{d}x \leqq \frac{1}{p-1}\overline{F(+\infty)}\int_\Omega \phi_1\mathrm{d}x,$$

which contradicts (H$_2$). The proof is complete. □

Proof of Theorem 1 The coerciveness and the Palais-Smale condition are enough to prove that I attains its proper infimum in Banach space E (see Theorem 2), so that (1) has at least a solution in E. The proof is complete. □

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