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Some mean value theorems for integrals on time scales

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ABSTRACT

In this short paper, we present time scales version of mean value theorems for integrals in the single variable case.

1. Introduction and preliminaries

The following two mean value theorems for time scales are due to M. Bohner and G. Guseinov.

Theorem A (See [1], Theorem 4.1). *Suppose that f is continuous on $[a,b]$ and has a delta derivative at each point of $[a,b)$. If $f(a)=f(b)$, then there exist points $\xi,\eta\in[a,b)$ such that*

$$f^{\Delta}(\xi)\leqq 0\leqq f^{\Delta}(\eta).$$

Theorem B (See [1], Theorem 4.2). *Suppose that f is continuous on $[a,b]$ and has a delta derivative at each point of $[a,b)$. If $f(a)=f(b)$, then there exist points $\xi,\eta\in[a,b)$ such that*

$$f^{\Delta}(\xi)(b-a)\leqq f(b)-f(a)\leqq f^{\Delta}(\eta)(b-a).$$

Motivated by Theorem A, the main aim of this paper is to present time scale version of mean value results for integrals in the single variable case. We first introduce some preliminaries on time scales (see [2,3,5] for details).

Definition 1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers.

The calculus of time scales was initiated by Stefan Hilger in his PhD thesis [4] in order to create a theory that can unify discrete and continuous analysis. Let $\mathbb{T}$ be a time scale. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

Definition 2. Let $\sigma(t)$ and $\rho(t)$ be the forward and backward jump operators in $\mathbb{T}$, respectively. For $t\in\mathbb{T}$, we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},$$

while the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}.$$

If $\sigma(t)>t$, then we say that t is right-scattered, while if $\rho(t)<t$ then we say that t is left-scattered.

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In this definition we put $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set.

Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$ and $t \neq \sup\mathbb{T}$, then t is called *right-dense*, and if $\rho(t) = t$ and $t \neq \inf\mathbb{T}$, then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called dense.

Definition 3. Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ satisfying

$$\mu(t) := \sigma(t) - t, \quad \nu(t) := t - \rho(t)$$

are called the graininess functions.

We now introduce the set $\mathbb{T}^\kappa$ which is derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t, then $\mathbb{T}^\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number, if one exists (finite), such that for all $\varepsilon > 0$ there is a neighborhood U of t such that for all $s \in U$

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leqq \varepsilon|\sigma(t) - s|.$$

We say that f is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$ ($t \neq \min\mathbb{T}$). Then we have the following

(i) If f is Δ-differentiable at t, then f is continuous at t.
(ii) If f is left continuous at t and t is right-scattered, then f is Δ-differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is Δ-differentiable at t if and only if

$$\lim_{s\to t}\frac{f(t) - f(s)}{t - s},$$

exists a finite number. In this case

$$f^\Delta(t) = \lim_{s\to t}\frac{f(t) - f(s)}{t - s}.$$

(iv) If f is Δ-differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

Proposition 1 (See [2], Theorem 1.20). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

Definition 5. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided if it satisfies

(1) f is continuous at each right-dense point.
(2) The left-sided limit $\lim_{s\to t-} f(s) = f(t-)$ exists at each left-dense point t of $\mathbb{T}$.

Remark 1. It follows from Theorem 1.74 of Bohner and Peterson [2] that every rd-continuous function has an anti-derivative.

Definition 6. A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $f^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the Δ-integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Proposition 2 (See [2], Theorem 1.77). *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^b (\alpha f(t) + \beta g(t))\Delta t = \alpha\int_a^b f(t)\Delta t + \beta\int_a^b g(t)\Delta t$,
(2) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(3) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(4) $\int_a^a f(t)\Delta t = 0$.

Definition 7. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0, \quad \forall t \in \mathbb{T}^\kappa$$

holds.

Definition 8. If a function p is regressive, then we define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad \forall s,t \in \mathbb{T}$$

where $\xi_h(z)$ is the cylinder transformation which is defined by

$$\xi_h(\tau) = \begin{cases} \frac{1}{h}\mathrm{Log}(1+\tau h), & \text{if } h > 0, \\ \tau, & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm function.

Remark 2. It is obviously to see that $e_1(t,s)$ is well-defined and $e_1(t,s) > 0$ for all $t,s \in \mathbb{T}$.

We now list here two properties of $e_p(t,s)$ which we will use in the rest of this paper.

Theorem C (See [2], Theorem 2.33). *If p is regressive, then for each $t_0 \in \mathbb{T}$ fixed, $e_p(t,s)$ is a solution of the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

on $\mathbb{T}$.

Theorem D (See [2], Theorem 2.36). *If p is regressive, then*

(1) $e_p(t,s) = \frac{1}{e_p(s,t)}$.
(2) $\left(\frac{1}{e_p(t,s)}\right)^{\Delta_t} = \frac{-p}{e_p(\sigma(t),s)}$.

Throughout this paper, we suppose that $\mathbb{T}$ is a time scale, $a,b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of real interval with the given time scale.

2. Main results

Theorem 1. *Let f be a continuous function on $[a,b]$ such that*

$$\int_a^b f(x)\Delta x = 0.$$

Then there exist $\xi, \eta \in [a,b)$ so that

$$f(\xi) \leqq \int_a^\xi f(x)\Delta x, \quad \int_a^\eta f(x)\Delta x \leqq f(\eta).$$

Proof of Theorem 1. Let

$$h(x) = e_1(a,x)\int_a^x f(t)\Delta t, \quad x \in [a,b).$$

Then

$$\begin{aligned} h^\Delta(x) &= \left(e_1(a,x)\int_a^x f(t)\Delta t\right)^\Delta = (e_1(a,x))^\Delta \int_a^x f(t)\Delta t + e_1(a,\sigma(x))\left(\int_a^x f(t)\Delta t\right)^\Delta \\ &= \left(\frac{1}{e_1(x,a)}\right)^\Delta \int_a^x f(t)\Delta t + e_1(a,\sigma(x))\left(\int_a^x f(t)\Delta t\right)^\Delta = \frac{-1}{e_1(\sigma(x),x)}\int_a^x f(t)\Delta t + e_1(a,\sigma(x))f(x) \\ &= -e_1(a,\sigma(x))\int_a^x f(t)\Delta t + e_1(a,\sigma(x))f(x). \end{aligned}$$

Since $h(a) = h(b)$ then there exists $\xi, \eta \in [a,b)$ such that

$$h^\Delta(\xi) \leqq 0 \leqq h^\Delta(\eta).$$

Hence

$$-e_1(a,\sigma(\xi))\int_a^\xi f(t)\Delta t + e_1(a,\sigma(\xi))f(\xi) \leqq 0 \leqq -e_1(a,\sigma(\eta))\int_a^\eta f(t)\Delta t + e_1(a,\sigma(\eta))f(\eta),$$

which implies that

$$f(\xi) \leqq \int_a^\xi f(x)\Delta x, \quad \int_a^\eta f(x)\Delta x \leqq f(\eta).$$

The proof is complete. □

Theorem 2. *Let f be a continuous function on* $[a,b]$ *such that*

$$\int_a^b f(x)\Delta x = 0.$$

Then there exist $\xi,\eta \in [a,b)$ *so that*

$$\frac{e_1(a,\xi)}{e_1(a,\sigma(\xi))}f(\xi) \leqq \int_a^{\sigma(\xi)} f(t)\Delta t,$$

and

$$\int_a^{\sigma(\eta)} f(t)\Delta t \leqq \frac{e_1(a,\eta)}{e_1(a,\sigma(\eta))}f(\eta).$$

Proof of Theorem 2. Let

$$h(x) = e_1(a,x)\int_a^x f(t)\Delta t.$$

Then

$$h^\Delta(x) = e_1(a,x)f(x) - e_1(a,\sigma(x))\int_a^{\sigma(x)} f(t)\Delta t.$$

Since $h(a) = h(b)$ then there exists $\xi,\eta \in [a,b)$ such that

$$g^\Delta(\xi) \leqq 0 \leqq g^\Delta(\eta).$$

Hence

$$e_1(a,\xi)f(\xi) - e_1(a,\sigma(\xi))\int_a^{\sigma(\xi)} f(t)\Delta t \leqq 0 \leqq e_1(a,\eta)f(\eta) - e_1(a,\sigma(\eta))\int_a^{\sigma(\eta)} f(t)\Delta t,$$

which implies that

$$\frac{e_1(a,\xi)}{e_1(a,\sigma(\xi))}f(\xi) \leqq \int_a^{\sigma(\xi)} f(t)\Delta t,$$
$$\int_a^{\sigma(\eta)} f(t)\Delta t \leqq \frac{e_1(a,\eta)}{e_1(a,\sigma(\eta))}f(\eta).$$

The proof is complete. □

Corollary 1. *Let* $\mathbb{T} = \mathbb{R}$, *from Theorems 1 and 2 together with the continuity of f we deduce that the existence of* $\gamma \in [a,b]$ *such that*

$$f(\gamma) = \int_a^\gamma f(x)\mathrm{d}x$$

provided

$$\int_a^b f(x)\mathrm{d}x = 0.$$

Theorem 3. *Let f be a continuous function on* $[a,b]$ *such that*

$$\int_a^b f(x)\Delta x = 0.$$

Then for each $\mathbb{T} \ni c < a$, *there exist* $\xi,\eta \in [a,b)$ *so that*

$$f(\xi)(\xi - c) \leqq \int_a^{\xi} f(t)\Delta t, \quad \int_a^{\eta} f(t)\Delta t \leqq f(\eta)(\eta - c).$$

Proof of Theorem 3. Let

$$h(x) = \frac{1}{x-c}\int_a^x f(t)\Delta t, \quad x \in [a,b),\ \mathbb{T} \ni c < a.$$

Therefore

$$h^{\Delta}(x) = \frac{-1}{(\sigma(x)-c)(x-c)}\int_a^x f(t)\Delta t + \frac{1}{\sigma(x)-c}f(x).$$

Since $h(a) = h(b)$ then there exists $\xi, \eta \in [a,b)$ such that

$$h^{\Delta}(\xi) \leqq 0 \leqq h^{\Delta}(\eta).$$

Hence

$$\frac{-1}{(\sigma(\xi)-c)(\xi-c)}\int_a^{\xi} f(t)\Delta t + \frac{1}{\sigma(\xi)-c}f(\xi) \leqq 0 \leqq \frac{-1}{(\sigma(\eta)-c)(\eta-c)}\int_a^{\eta} f(t)\Delta t + \frac{1}{\sigma(\eta)-c}f(\eta),$$

which implies

$$\frac{f(\xi)}{\sigma(\xi)-c} \leqq \frac{\int_a^{\xi} f(t)\Delta t}{(\sigma(\xi)-c)(\xi-c)},$$
$$\frac{\int_a^{\eta} f(t)\Delta t}{(\sigma(\eta)-c)(\eta-c)} \leqq \frac{f(\eta)}{\sigma(\eta)-c}.$$

Thus

$$f(\xi)(\xi - c) \leqq \int_a^{\xi} f(t)\Delta t, \quad \int_a^{\eta} f(t)\Delta t \leqq f(\eta)(\eta - c).$$

The proof is complete. □

Corollary 2. *Let $\mathbb{T} = \mathbb{R}$, from Theorem 3 together with the continuity of f we deduce the existence of $\gamma \in [a,b]$ such that*

$$f(\gamma)(\xi - c) = \int_a^{\gamma} f(x)\mathrm{d}x,$$

for each $c < a$ provided

$$\int_a^b f(x)\Delta x = 0.$$

Theorem 4. *Let f, g be a continuous function on $[a,b]$. Then there exist $\xi, \eta \in [a,b)$ so that*

$$f(\xi)\left(\int_{\xi}^b g(t)\Delta t\right) \leqq \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi),$$

and

$$f(\eta)\left(\int_{\eta}^b g(t)\Delta t\right) \geqq \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta).$$

Proof of Theorem 4

$$h(x) = -\left(\int_a^x f(t)\Delta t\right)\left(\int_b^x g(t)\Delta t\right), \quad x \in [a,b).$$

Then

$$h^{\Delta}(x) = -f(x)\left(\int_b^x g(t)\Delta t\right) - \left(\int_a^{\sigma(x)} f(t)\Delta t\right)g(x).$$

Since $h(a) = h(b)$ then there exist $\xi, \eta \in [a,b)$ such that

$$h^{\Delta}(\xi) \leqq 0 \leqq h^{\Delta}(\eta).$$

Hence

$$-f(\xi)\left(\int_b^{\xi} g(t)\Delta t\right) - \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi) \leqq 0 \leqq -f(\eta)\left(\int_b^{\eta} g(t)\Delta t\right) - \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta),$$

which implies

$$0 \leqq f(\xi)\left(\int_b^{\xi} g(t)\Delta t\right) + \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi),$$
$$0 \geqq f(\eta)\left(\int_b^{\eta} g(t)\Delta t\right) + \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta),$$

or equivalently

$$f(\xi)\left(\int_{\xi}^{b} g(t)\Delta t\right) \leqq \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi),$$
$$f(\eta)\left(\int_{\eta}^{b} g(t)\Delta t\right) \geqq \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta).$$

The proof is complete. □

Corollary 3. *Let $\mathbb{T} = \mathbb{R}$, from Theorem 4 together with the continuity of f and g we deduce the existence of $\gamma \in [a, b]$ such that*

$$f(\gamma)\left(\int_{\gamma}^{b} g(x)\mathrm{d}x\right) = \left(\int_a^{\gamma} f(x)\mathrm{d}x\right)g(\gamma).$$

Theorem 5. *Let f, g be continuous functions on $[a, b]$. Then there exist $\xi, \eta \in [a, b)$ so that*

$$\left(\int_a^{\xi} f(t)\Delta t\right)\left(\int_b^{\xi} g(t)\Delta t\right) \leqq f(\xi)\left(\int_b^{\xi} g(t)\Delta t\right) + \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi)$$

and

$$\left(\int_a^{\eta} f(t)\Delta t\right)\left(\int_b^{\eta} g(t)\Delta t\right) \geqq f(\eta)\left(\int_b^{\eta} g(t)\Delta t\right) + \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta).$$

Proof of Theorem 5. Let

$$h(x) = -e_1(a, x)\left(\int_a^{x} f(t)\Delta t\right)\left(\int_b^{x} g(t)\Delta t\right).$$

Then

$$\begin{aligned} h^{\Delta}(x) &= e_1(a, \sigma(x))\left(\int_a^{x} f(t)\Delta t\right)\left(\int_b^{x} g(t)\Delta t\right) - e_1(a, \sigma(x))\left(\left(\int_a^{x} f(t)\Delta t\right)\left(\int_b^{x} g(t)\Delta t\right)\right)^{\Delta} \\ &= e_1(a, \sigma(x))\left(\int_a^{x} f(t)\Delta t\right)\left(\int_b^{x} g(t)\Delta t\right) - e_1(a, \sigma(x))\left(f(x)\left(\int_b^{x} g(t)\Delta t\right) + \left(\int_a^{\sigma(x)} f(t)\Delta t\right)g(x)\right). \end{aligned}$$

Since $h(a) = h(b)$ then there exists $\xi, \eta \in [a, b)$ such that

$$h^{\Delta}(\xi) \leqq 0 \leqq h^{\Delta}(\eta).$$

Hence

$$e_1(a, \sigma(\xi))\left(\int_a^{\xi} f(t)\Delta t\right)\left(\int_b^{\xi} g(t)\Delta t\right) - e_1(a, \sigma(\xi))\left(f(\xi)\left(\int_b^{\xi} g(t)\Delta t\right) + \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi)\right) \leqq 0,$$

and

$$e_1(a, \sigma(\eta))\left(\int_a^{\eta} f(t)\Delta t\right)\left(\int_b^{\eta} g(t)\Delta t\right) - e_1(a, \sigma(\eta))\left(f(\eta)\left(\int_b^{\eta} g(t)\Delta t\right) + \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta)\right) \geqq 0.$$

Thus

$$\left(\int_a^{\xi} f(t)\Delta t\right)\left(\int_b^{\xi} g(t)\Delta t\right) \leqq f(\xi)\left(\int_b^{\xi} g(t)\Delta t\right) + \left(\int_a^{\sigma(\xi)} f(t)\Delta t\right)g(\xi)$$

and

$$\left(\int_a^\eta f(t)\Delta t\right)\left(\int_b^\eta g(t)\Delta t\right) \geqq f(\eta)\left(\int_b^\eta g(t)\Delta t\right) + \left(\int_a^{\sigma(\eta)} f(t)\Delta t\right)g(\eta).$$

The proof is complete. □

Corollary 4. *Let $\mathbb{T} = \mathbb{R}$, from Theorem 4 together with the continuity of f and g we deduce the existence of $\gamma \in [a, b]$ such that*

$$\left(\int_a^\gamma f(x)\mathrm{d}x\right)\left(\int_b^\gamma g(x)\mathrm{d}x\right) = f(\gamma)\left(\int_b^\gamma g(x)\mathrm{d}x\right) + \left(\int_a^\gamma f(x)\mathrm{d}x\right)g(\gamma).$$

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