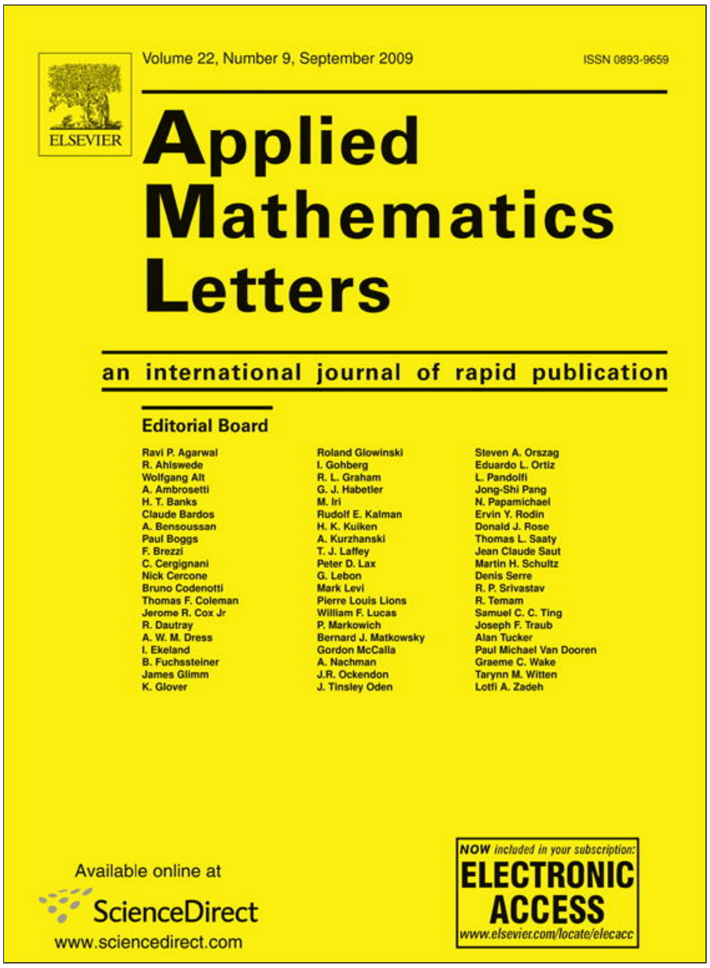

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New inequalities of Ostrowski-like type involving n knots and the L^p-norm of the m-th derivative

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ABSTRACT

On the basis of recent results due to Nenad Ujević, we obtain some new inequalities of Ostrowski-like type involving n knots and the L^p-norm of the m-th derivative where n, m, p are arbitrary numbers.

1. Introduction

In recent years, a number of authors have considered error inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of these formulas; see [1,2] and the references therein where the mid-point and trapezoid quadrature rules are considered.

This work is motivated by some results due to Nenad Ujević. Here we recall them.

Theorem 1 (*See [1]*). *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice-differentiable function such that f'' is bounded and integrable. Then we have*

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{2}\left(f\left(\frac{a+b}{2} - \left(2-\sqrt{3}\right)(b-a)\right) + f\left(\frac{a+b}{2} + \left(2-\sqrt{3}\right)(b-a)\right)\right)\right| \leqq \frac{7-4\sqrt{3}}{8}\left\|f''\right\|_\infty (b-a)^3. \tag{1}$$

Theorem 2 (*See [2]*). *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice-differentiable function such that $f'' \in L^2(a, b)$. Then we have*

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{2}\left(f\left(\frac{a+b}{2} - \frac{3-\sqrt{6}}{2}(b-a)\right)\right.\right.$$

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$$+f\left(\frac{a+b}{2}+\frac{3-\sqrt{6}}{2}(b-a)\right)\Bigg| \leqq \sqrt{\frac{49}{80}-\frac{1}{4}\sqrt{6}}\left\|f''\right\|_2 (b-a)^{\frac{5}{2}}. \quad (2)$$

In the above mentioned results, constants $\frac{7-4\sqrt{3}}{8}$ in (1) and $\sqrt{\frac{49}{80}-\frac{1}{4}\sqrt{6}}$ in (2) are sharp in the sense that these cannot be replaced by smaller ones. This leads us to strengthen (1) and (2) by enlarging the number of knots (two knots in both (1) and (2)) and replacing the norms $\|\cdot\|_\infty$ in (1) and $\|\cdot\|_2$ in (2).

Before stating our main result, let us introduce the following notation.

$$I(f)=\int_a^b f(x)\mathrm{d}x.$$

Let $1 \leqq m, n < \infty$ and $1 \leqq p \leqq \infty$. For each $i=\overline{1,n}$, we assume $0 < x_i < 1$ such that

$$\begin{cases} x_1+x_2+\cdots+x_n=\dfrac{n}{2},\\ \cdots\\ x_1^j+x_2^j+\cdots+x_n^j=\dfrac{n}{j+1},\\ \cdots\\ x_1^{m-1}+x_2^{m-1}+\cdots+x_n^{m-1}=\dfrac{n}{m}.\end{cases}$$

Put

$$Q\left(f,n,m,x_1,\ldots,x_n\right)=\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+x_i\left(b-a\right)\right).$$

Remark 3. With the above notation, (1) reads as follows:

$$\left|I(f)-Q\left(f,2,2,\frac{1}{2}-\left(2-\sqrt{3}\right),\frac{1}{2}+\left(2-\sqrt{3}\right)\right)\right| \leqq \frac{7-4\sqrt{3}}{8}\left\|f''\right\|_\infty (b-a)^3 \quad (3)$$

while (2) reads as follows:

$$\left|I(f)-Q\left(f,2,2,\frac{1}{2}-\frac{3-\sqrt{6}}{2},\frac{1}{2}+\frac{3-\sqrt{6}}{2}\right)\right| \leqq \sqrt{\frac{49}{80}-\frac{1}{4}\sqrt{6}}\left\|f''\right\|_2 (b-a)^{\frac{5}{2}}. \quad (4)$$

We are in a position to state our main result.

Theorem 4. *Let $I \subset \mathbb{R}$ be an open interval such that $[a,b] \subset I$ and let $f: I \to \mathbb{R}$ be an m-times-differentiable function such that $f^{(m)} \in L^p(a,b)$. Then we have*

$$\left|I(f)-Q\left(f,n,m,x_1,\ldots,x_n\right)\right| \leqq \frac{1}{m!}\left(\left(\frac{1}{mq+1}\right)^{\frac{1}{q}}+\left(\frac{1}{(m-1)q+1}\right)^{\frac{1}{q}}\right)\left\|f^{(m)}\right\|_p (b-a)^{m+\frac{1}{q}} \quad (5)$$

where

$$\frac{1}{p}+\frac{1}{q}=1$$

and

$$\|f\|_r := \begin{cases}\left(\displaystyle\int_a^b |f(x)|^r\,\mathrm{d}x\right)^{\frac{1}{r}}, & \text{when } 1 \leq r < \infty,\\ \operatorname*{ess\,sup}_{[a,b]} |f|, & \text{when } r=\infty.\end{cases}$$

Remark 5. It is worth noticing that the right hand side of (5) does not involve $x_i, i=\overline{1,n}$, and that m can be chosen arbitrarily. This means that our inequality (5) is better in some sense. However, the inequality (5) is not sharp due to the restriction of the technique that we use. We hope that we will soon see some responses on this problem.

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 6 (*See [3]*). *Let $f : [a, b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a, b]$, $x_0 \in (a, b)$, then for all $x \in (a, b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x)$$

where $T_{r-1}(f, x_0, \cdot)$ is Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!} \mathrm{d}t. \tag{6}$$

By a simple calculation, the remainder in (6) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x - x_0 - t)^{r-1} f^{(r)}(x_0 + t)}{(r-1)!} \mathrm{d}t$$

which helps us to deduce a similar representation of f as follows:

$$f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t)\, \mathrm{d}t. \tag{7}$$

Proof of Theorem 4. Define

$$F(x) = \int_a^x f(x) \mathrm{d}x.$$

By the Fundamental Theorem of Calculus

$$I(f) = F(b) - F(a).$$

Applying Lemma 6 to $F(x)$ with $x = a$ and $u = b - a$, we get

$$F(b) = F(a) + \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)\, \mathrm{d}t$$

which yields

$$I(f) = \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)\, \mathrm{d}t.$$

Equivalently,

$$I(f) = \sum_{k=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)\, \mathrm{d}t.$$

For each $1 \leqq i \leqq n$, applying Lemma 6 to $f(x)$ with $x = a$ and $u = x_i(b-a)$, we get

$$\begin{aligned} f(a + x_i(b-a)) &= \sum_{k=0}^{m-1} \frac{x_i^k (b-a)^k}{k!} f^{(k)}(a) + \int_0^{x_i(b-a)} \frac{(x_i(b-a) - t)^{m-1}}{(m-1)!} f^{(m)}(a+t)\, \mathrm{d}t \\ &= \sum_{k=0}^{m-1} \frac{x_i^k (b-a)^k}{k!} f^{(k)}(a) + \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\, \mathrm{d}u. \end{aligned} \tag{8}$$

By applying (8) to $i = \overline{1, n}$ and then summing, we deduce that

$$\begin{aligned}
\sum_{i=1}^{n} f\left(a + x_i (b-a)\right) &= \sum_{i=1}^{n}\sum_{k=0}^{m-1} \frac{x_i^k (b-a)^k}{k!} f^{(k)}(a) + \sum_{i=1}^{n} \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u \\
&= \sum_{k=0}^{m-1} \frac{\sum_{i=1}^{n} x_i^k (b-a)^k}{k!} f^{(k)}(a) + \sum_{i=1}^{n} \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u \\
&= \sum_{k=0}^{m-1} \frac{n (b-a)^k}{(k+1)!} f^{(k)}(a) + \sum_{i=1}^{n} \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u. \qquad (9)
\end{aligned}$$

Thus,

$$Q(f, n, m, x_1, \dots, x_n) = \sum_{k=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \frac{b-a}{n} \sum_{i=1}^{n} \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u.$$

Therefore,

$$\begin{aligned}
&|I(f) - Q(f, n, m, x_1, \dots, x_n)| \\
&= \left| \int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)\,\mathrm{d}t - \frac{b-a}{n} \sum_{i=1}^{n} \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u \right| \\
&\leqq \left| \int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)\,\mathrm{d}t \right| + \frac{b-a}{n} \sum_{i=1}^{n} \left| \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\,\mathrm{d}u \right| \\
&= \left| \int_a^{b} \frac{(b-x)^m}{m!} f^{(m)}(x)\mathrm{d}x \right| + \frac{b-a}{n} \sum_{i=1}^{n} \left| \int_a^{b} \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}\left((1-x_i)a + x_i x\right)\mathrm{d}x \right| \\
&\leqq \left\| \frac{(b-\cdot)^m}{m!} f^{(m)} \right\|_1 + \frac{b-a}{n} \sum_{i=1}^{n} \left\| \frac{x_i^m (b-\cdot)^{m-1}}{(m-1)!} f^{(m)}\left((1-x_i)a + x_i \cdot\right) \right\|_1 .
\end{aligned}$$

Using the Hölder inequality, we get

$$\left\| \frac{(b-\cdot)^m}{m!} f^{(m)} \right\|_1 = \frac{1}{m!} \left\| (b-\cdot)^m f^{(m)} \right\|_1 \leqq \frac{1}{m!} \left\| f^{(m)} \right\|_p \left\| (b-\cdot)^m \right\|_q = \frac{1}{m!} \left(\frac{(b-a)^{mq+1}}{mq+1} \right)^{\frac{1}{q}} \left\| f^{(m)} \right\|_p . \qquad (10)$$

We see that $\left\| f^{(m)}\left((1-x_i)a + x_i \cdot\right) \right\|_\infty \leqq \left\| f^{(m)} \right\|_\infty$ while, for $1 \leqq p < \infty$, we have

$$\begin{aligned}
\left\| f^{(m)}\left((1-x_i)a + x_i \cdot\right) \right\|_p &= \left(\int_a^b \left| f^{(m)}\left((1-x_i)a + x_i x\right) \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\
&= \frac{1}{x_i^{\frac{1}{p}}} \left(\int_a^b \left| f^{(m)}\left((1-x_i)a + x_i x\right) \right|^p \mathrm{d}\left((1-x_i)a + x_i x\right) \right)^{\frac{1}{p}} \\
&= \frac{1}{x_i^{\frac{1}{p}}} \left(\int_a^{(1-x_i)a + x_i b} \left| f^{(m)}(t) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&\leqq \frac{1}{x_i^{\frac{1}{p}}} \left(\int_a^{b} \left| f^{(m)}(t) \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\
&= \frac{1}{x_i^{\frac{1}{p}}} \left\| f^{(m)} \right\|_p \\
&\leqq \frac{1}{x_i} \left\| f^{(m)} \right\|_p .
\end{aligned}$$

This helps us to deduce that

$$\begin{aligned}
&\frac{b-a}{n}\sum_{i=1}^{n}\left\|\frac{x_i^m\,(b-\cdot)^{m-1}}{(m-1)!}f^{(m)}\left((1-x_i)a+x_i\cdot\right)\right\|_1\\
&\quad=\frac{b-a}{n}\sum_{i=1}^{n}\frac{x_i^m}{(m-1)!}\left\|(b-\cdot)^{m-1}f^{(m)}\left((1-x_i)\,a+x_i\cdot\right)\right\|_1\\
&\quad\leqq\frac{b-a}{n}\sum_{i=1}^{n}\frac{x_i^m}{(m-1)!}\left\|f^{(m)}\left((1-x_i)\,a+x_i\cdot\right)\right\|_p\left\|(b-\cdot)^{(m-1)}\right\|_q\\
&\quad\leqq\frac{b-a}{n}\sum_{i=1}^{n}\frac{x_i^m}{(m-1)!}\frac{\left\|f^{(m)}\right\|_p}{x_i}\left(\frac{(b-a)^{(m-1)q+1}}{(m-1)\,q+1}\right)^{\frac{1}{q}}\\
&\quad=\frac{b-a}{n}\sum_{i=1}^{n}x_i^{m-1}\frac{\left\|f^{(m)}\right\|_p}{(m-1)!}\left(\frac{(b-a)^{(m-1)q+1}}{(m-1)\,q+1}\right)^{\frac{1}{q}}\\
&\quad=\frac{\left\|f^{(m)}\right\|_p}{m!}\left(\frac{(b-a)^{mq+1}}{(m-1)\,q+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

It follows that

$$|I(f)-Q\left(f,n,m,x_1,\ldots,x_n\right)|\leqq\frac{1}{m!}\left(\frac{(b-a)^{mq+1}}{mq+1}\right)^{\frac{1}{q}}\left\|f^{(m)}\right\|_p+\frac{\left\|f^{(m)}\right\|_p}{m!}\left(\frac{(b-a)^{mq+1}}{(m-1)\,q+1}\right)^{\frac{1}{q}}$$

which completes our proof. □

3. Examples

In this section, by applying our main theorem, we will obtain some new inequalities which cannot be easily obtained from [1,2].

Example 7. *When $n=3$, $m=3$ and*

$$0<\underbrace{\frac{1}{4}-\frac{\sqrt{21}}{12}}_{x_1}<\underbrace{\frac{1}{2}}_{x_2}<\underbrace{\frac{1}{4}+\frac{\sqrt{21}}{12}}_{x_3}<1 \tag{11}$$

we deduce that

$$\begin{aligned}
&\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{3}\left(f\left(a+\left(\frac{1}{4}-\frac{\sqrt{21}}{12}\right)(b-a)\right)+f\left(a+\frac{1}{2}\,(b-a)\right)\right.\right.\\
&\qquad\left.\left.+f\left(a+\left(\frac{1}{4}+\frac{\sqrt{21}}{12}\right)(b-a)\right)\right)\right|\leqq\frac{\sqrt{7}+\sqrt{5}}{6\sqrt{35}}\left\|f'''\right\|_2(b-a)^{\frac{7}{2}}.
\end{aligned}\tag{12}$$

Example 8. *When $n=2$, $m=3$ and*

$$0<\underbrace{\frac{1}{2}-\frac{1}{2\sqrt{3}}}_{x_1}<\underbrace{\frac{1}{2}-\frac{1}{2\sqrt{3}}}_{x_2}<1 \tag{13}$$

we deduce that

$$\left|\int_a^b f(x)\mathrm{d}x-\frac{b-a}{2}\left(f\left(\frac{a+b}{2}-\frac{1}{2\sqrt{3}}\,(b-a)\right)+f\left(\frac{a+b}{2}+\frac{1}{2\sqrt{3}}\,(b-a)\right)\right)\right|\leqq\frac{7}{72}\left\|f'''\right\|_\infty(b-a)^4. \tag{14}$$

Example 9. *When* $m = 2$ *and* $0 < x_i < 1$ *such that* $\sum_{i=1}^{n} x_i = \frac{n}{2}$ *then we have*

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a)) \right| \leqq \frac{5}{12} \left\| f'' \right\|_\infty (b-a)^3, \tag{15}$$

and

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a)) \right| \leqq \frac{\sqrt{5}+\sqrt{3}}{2\sqrt{15}} \|f''\|_2 (b-a)^{\frac{5}{2}}. \tag{16}$$

Example 10. *When* $m = 3$ *and* $0 < x_i < 1$ *such that* $\sum_{i=1}^{n} x_i = \frac{n}{2}$ *and* $\sum_{i=1}^{n} x_i^2 = \frac{n}{3}$ *then we have*

$$\left| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a)) \right| \leqq \frac{7}{72} \|f'''\|_\infty (b-a)^4. \tag{17}$$

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