An. Şt. Univ. Ovidius Constanţa Vol. 17(2), 2009, 101–114

A NEW GENERALIZATION OF OSTROWSKI TYPE INEQUALITY ON TIME SCALES

Wenjun J. Liu, Quốc-Anh Ngô and Wenbing B. Chen

Abstract

In this paper, by introducing a parameter, we first extend a generalization of Ostrowski type inequality on time scales for functions whose derivatives are bounded and then unify corresponding continuous and discrete versions. We also point out some particular integral inequalities on time scales as special cases.

1 Introduction

The following integral inequality was first established by Ostrowski in 1938.

Theorem 1 *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b)\to\mathbb{R}$ is bounded in (a,b), that is, $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x\in[a,b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty. \qquad (1)$$

The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Key Words: Ostrowski's inequality; generalization; time scales; Simpson inequality; trapezoid inequality; mid-point inequality.
Mathematics Subject Classification: 26D15, 39A10, 39A12, 39A13.
Received: May, 2009
Accepted: September, 2009

For some extensions, generalizations and similar results, please see [6, 8, 9, 14, 15, 19, 20, 21] and references therein.

The development of the theory of time scales was initiated by Hilger [10] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales. For example, we refer the reader to [1, 4, 5, 7, 11, 13]. In [5], Bohner and Matthews established the following so-called Ostrowski's inequality on time scales.

Theorem 2 *(See [5], Theorem 3.5) Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s)\Delta s \right| \leqslant \frac{M_1}{b-a}\Big(h_2(t,a) + h_2(t,b)\Big), \tag{2}$$

where $M_1 = \sup\limits_{a<t<b} |f^\Delta(t)|$. This inequality is sharp in the sense that the right-hand side of (2) cannot be replaced by a smaller one.

Liu and Ngô [16] then generalized the above Ostrowski inequality on time scales for k points $x_1, x_2, \cdots, x_k$ for functions whose derivatives are bounded. They also extended the result by considering functions whose second derivatives are bounded in [17]. They obtained:

Theorem 3 *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on (a, b) and $f^{\Delta\Delta} : (a, b) \to \mathbb{R}$ is bounded, i.e. $M_2 := \sup\limits_{a<t<b} |f^{\Delta\Delta}(x)| < \infty$. Then we have*

$$\left| \int_a^b f^\sigma(t)\Delta t - f^\sigma(x)(b-a) + (h_2(x,a) - h_2(x,b))f^\Delta(x) \right| \leqslant M_2\big(h_3(x,a) - h_3(x,b)\big).$$

The Theorem 3 may be thought of as a perturbed version of the Theorem 2. In [18], the authors derive a perturbed Ostrowski type inequality on time scales for k points $x_1, x_2, \cdots, x_k$ for functions whose second derivatives are bounded.

In the present paper, by introducing a parameter, we shall first extend another generalization of Ostrowski type inequality on time scales for functions whose derivatives are bounded and then unify corresponding continuous and discrete versions. We also point out some particular integral inequalities on time scales as special cases.

2 The generalized Ostrowski type inequality on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For a general introduction to the theory of time scale we refer the reader to Hilger [10] (see also [16, 17, 18]) and the books [2, 3, 12] .

Definition 1 *Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by*

$$h_0(t,s) = 1 \quad \textit{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau \quad \textit{for all} \quad s,t \in \mathbb{T}. \tag{3}$$

Our main result reads as follow.

Theorem 4 *Let $a,b,s,t \in \mathbb{T}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be differentiable. Then*

$$\begin{aligned} &\left|(1-\lambda)f(t) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \\ &\qquad \leqslant \frac{M}{b-a}\left(h_2\left(a, a+\lambda\frac{b-a}{2}\right) + h_2\left(t, a+\lambda\frac{b-a}{2}\right)\right. \\ &\qquad\qquad \left. + h_2\left(t, b-\lambda\frac{b-a}{2}\right) + h_2\left(b, b-\lambda\frac{b-a}{2}\right)\right) \end{aligned} \tag{4}$$

for all $\lambda \in [0,1]$ such that $a + \lambda(b-a)/2$ and $b - \lambda(b-a)/2$ are in $\mathbb{T}$ and $t \in [a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}] \cap \mathbb{T}$, where $M := \sup_{a<t<b} |f^\Delta(t)| < \infty$. This inequality is sharp provided

$$\frac{\lambda}{2}a(b-a) + \frac{\lambda^2}{4}(b-a)^2 \leqslant \int_a^{a+\lambda\frac{b-a}{2}} s\Delta s. \tag{5}$$

Remark 1 *We note that the condition (5) is trivial if $\lambda = 0$.*

To prove the Theorem 4, we need the following Generalized Montgomery Identity. This is motivated by the ideas of Dragomir et al. in [9], where the continuous version of a generalized Ostrowski integral inequality for mappings whose derivatives are bounded was proved.

Lemma 1 (Generalized Montgomery Identity) *Under the assumptions of Theorem 4, we have*

$$(1-\lambda)f(t)+\lambda\frac{f(a)+f(b)}{2}=\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s+\frac{1}{b-a}\int_a^b K(t,s)f^\Delta(s)\Delta s,$$

where

$$K(t,s)=\begin{cases} s-\left(a+\lambda\dfrac{b-a}{2}\right), & s\in[a,t),\\ s-\left(b-\lambda\dfrac{b-a}{2}\right), & s\in[t,b]. \end{cases} \tag{6}$$

Proof. Integrating by parts, we have

$$\begin{aligned} &\int_a^b K(t,s)f^\Delta(s)\Delta s\\ =&\int_a^t \left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)f^\Delta(s)\Delta s+\int_t^b \left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)f^\Delta(s)\Delta s\\ =&\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)f(t)+\frac{\lambda}{2}(b-a)f(a)-\int_a^t f^\sigma(s)\Delta s\\ &-\left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)f(t)+\frac{\lambda}{2}(b-a)f(b)-\int_t^b f^\sigma(s)\Delta s\\ =&(b-a)\left((1-\lambda)f(t)+\lambda\frac{f(a)+f(b)}{2}\right)-\int_a^b f^\sigma(s)\Delta s, \end{aligned}$$

from which we get the desired identity.

Corollary 1 (Continuous case) *Let* $\mathbb{T}=\mathbb{R}$. *Then*

$$(1-\lambda)f(t)+\lambda\frac{f(a)+f(b)}{2}=\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s+\frac{1}{b-a}\int_a^b K(t,s)f'(s)\,\mathrm{d}s. \tag{7}$$

This is the Montgomery identity in the continuous case, which can be found in [9].

Corollary 2 (Discrete case) *Let* $\mathbb{T}=\mathbb{Z}$, $a=0$, $b=n$, $s=j$, $t=i$ *and* $f(k)=x_k$. *Then*

$$(1-\lambda)x_i+\lambda\frac{x_0+x_n}{2}=\frac{1}{n}\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=0}^{n-1}K\left(i,j\right)\Delta x_j,$$

where

$$\begin{aligned} K(i,0) &= -\frac{n\lambda}{2}, \\ K(1,j) &= j - \left(n - \frac{n\lambda}{2}\right) \quad \textit{for} \ \ 1 \leqslant j \leqslant n-1, \\ K(n,j) &= j - \frac{n\lambda}{2} \quad \textit{for} \ \ 0 \leqslant j \leqslant n-1, \\ K(i,j) &= \begin{cases} j - \frac{n\lambda}{2}, & j \in [0,i), \\ j - \left(n - \frac{n\lambda}{2}\right), & j \in [i, n-1], \end{cases} \end{aligned}$$

as we just need $1 \leqslant i \leqslant n$ *and* $1 \leqslant j \leqslant n-1$.

Corollary 3 (Quantum calculus case) *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m, b = q^n$ *with* $m < n$. *Then*

$$\begin{aligned} &(1-\lambda)f(t) + \lambda\frac{f(q^m)+f(q^n)}{2} \\ =&\frac{\sum\limits_{k=m}^{n-1} q^k f(q^{k+1})}{\sum\limits_{k=m}^{n-1} q^k} + \frac{1}{q^n - q^m}\sum_{k=m}^{n-1}\left[f(q^{k+1}) - f(q^k)\right]K(t,q^k), \end{aligned}$$

where

$$K(t,q^k) = \begin{cases} q^k - \left(q^m + \lambda\dfrac{q^m - q^n}{2}\right), & q^k \in [q^m, t), \\ q^k - \left(q^n - \lambda\dfrac{q^m - q^n}{2}\right), & q^k \in [t, q^n]. \end{cases}$$

Proof. (Proof of the Theorem 4) By applying Lemma 1, we get

$$\begin{aligned} &\left|(1-\lambda)f(t) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \\ \leqslant&\frac{1}{b-a}\left|\int_a^b K(t,s) f^\Delta(s)\Delta s\right| \\ \leqslant&\frac{M}{b-a}\left(\int_a^t |K(t,s)|\,\Delta s + \int_t^b |K(t,s)|\,\Delta s\right) \\ =&\frac{M}{b-a}\left(\int_a^t \left|s - \left(a + \lambda\frac{b-a}{2}\right)\right|\Delta s + \int_t^b \left|s - \left(b - \lambda\frac{b-a}{2}\right)\right|\Delta s\right) \end{aligned}$$

$$
\begin{aligned}
=&\frac{M}{b-a}\left(\int_a^{a+\lambda\frac{b-a}{2}}\left|s-\left(a+\lambda\frac{b-a}{2}\right)\right|\Delta s+\int_{a+\lambda\frac{b-a}{2}}^{t}\left|s-\left(a+\lambda\frac{b-a}{2}\right)\right|\Delta s\right.\\
&\left.+\int_t^{b-\lambda\frac{b-a}{2}}\left|s-\left(b-\lambda\frac{b-a}{2}\right)\right|\Delta s+\int_{b-\lambda\frac{b-a}{2}}^{b}\left|s-\left(b-\lambda\frac{b-a}{2}\right)\right|\Delta s\right)\\
=&\frac{M}{b-a}\left(\int_{a+\lambda\frac{b-a}{2}}^{a}\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)\Delta s+\int_{a+\lambda\frac{b-a}{2}}^{t}\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)\Delta s\right.\\
&\left.+\int_{b-\lambda\frac{b-a}{2}}^{t}\left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)\Delta s+\int_{b-\lambda\frac{b-a}{2}}^{b}\left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)\Delta s\right)\\
=&\frac{M}{b-a}\left(h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(t,a+\lambda\frac{b-a}{2}\right)\right.\\
&\left.+h_2\left(t,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right),
\end{aligned}
$$

which completes the first part of our proof.

To prove the sharpness of this inequality, let $f(t)=t$, $t=b-\lambda\frac{b-a}{2}$. It follows that $M=1$. Starting with the righ-hand side of (4), we have

$$
\begin{aligned}
&\frac{M}{b-a}\left(h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(t,a+\lambda\frac{b-a}{2}\right)\right.\\
&\qquad\left.+h_2\left(t,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right)\\
=&\frac{1}{b-a}\left(h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(b-\lambda\frac{b-a}{2},a+\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
h_2\left(a,a+\lambda\frac{b-a}{2}\right)&=\int_{a+\lambda\frac{b-a}{2}}^{a}\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)\Delta s\\
&=\int_{a+\lambda\frac{b-a}{2}}^{a}s\Delta s-\left(a+\lambda\frac{b-a}{2}\right)\left(a-\left(a+\lambda\frac{b-a}{2}\right)\right)\\
&=\int_{a+\lambda\frac{b-a}{2}}^{a}s\Delta s+\left(a+\lambda\frac{b-a}{2}\right)\lambda\frac{b-a}{2}.
\end{aligned}
$$

$$\begin{aligned}h_2\left(b-\lambda\frac{b-a}{2},a+\lambda\frac{b-a}{2}\right)&=\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}\left(s-\left(a+\lambda\frac{b-a}{2}\right)\right)\Delta s\\&=\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}s\Delta s-\left(a+\lambda\frac{b-a}{2}\right)\left(b-\lambda\frac{b-a}{2}-\left(a+\lambda\frac{b-a}{2}\right)\right)\\&=\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}s\Delta s-\left(a+\lambda\frac{b-a}{2}\right)(b-a)(1-\lambda).\end{aligned}$$

$$\begin{aligned}h_2\left(b,b-\lambda\frac{b-a}{2}\right)&=\int_{b-\lambda\frac{b-a}{2}}^{b}\left(s-\left(b-\lambda\frac{b-a}{2}\right)\right)\Delta s\\&=\int_{b-\lambda\frac{b-a}{2}}^{b}s\Delta s-\left(b-\lambda\frac{b-a}{2}\right)\left(b-\left(b-\lambda\frac{b-a}{2}\right)\right)\\&=\int_{b-\lambda\frac{b-a}{2}}^{b}s\Delta s-\left(b-\lambda\frac{b-a}{2}\right)\lambda\frac{b-a}{2}.\end{aligned}$$

Thus, in this situation, the right-hand side of (4) equals to

$$\begin{aligned}&\frac{1}{b-a}\left(-\int_a^{a+\lambda\frac{b-a}{2}}s\Delta s+\int_{a+\lambda\frac{b-a}{2}}^{b-\lambda\frac{b-a}{2}}s\Delta s+\int_{b-\lambda\frac{b-a}{2}}^{b}s\Delta s\right)\\&+\left(a+\lambda\frac{b-a}{2}\right)\frac{\lambda}{2}-\left(a+\lambda\frac{b-a}{2}\right)(1-\lambda)-\left(b-\lambda\frac{b-a}{2}\right)\frac{\lambda}{2}\\=&\frac{1}{b-a}\left(-2\int_a^{a+\lambda\frac{b-a}{2}}s\Delta s+\int_a^b s\Delta s\right)-(a+\lambda(b-a))(1-\lambda).\end{aligned}$$

Starting with the left-hand side of (4), we have

$$\begin{aligned}&\left|(1-\lambda)f(t)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|\\&\qquad=\left|(1-\lambda)\left(b-\lambda\frac{b-a}{2}\right)+\lambda\frac{a+b}{2}-\frac{1}{b-a}\int_a^b\sigma(s)\Delta s\right|\\&\qquad=\left|(1-\lambda)\left(b-\lambda\frac{b-a}{2}\right)+\lambda\frac{a+b}{2}+\frac{1}{b-a}\int_a^b s\Delta s-b-a\right|\\&\qquad=\left|-\lambda\left(1-\frac{\lambda}{2}\right)(b-a)-a+\frac{1}{b-a}\int_a^b s\Delta s\right|,\end{aligned}$$

where we have used

$$\int_a^b \sigma(s)\,\Delta s = \int_a^b \big(\sigma(s)+s\big)\Delta s - \int_a^b s\Delta s = \int_a^b (s^2)^{\Delta}\Delta s - \int_a^b s\Delta s = b^2 - a^2 - \int_a^b s\Delta s.$$

So, if

$$\frac{\lambda}{2}a(b-a) + \frac{\lambda^2}{4}(b-a)^2 \leqslant \int_a^{a+\lambda\frac{b-a}{2}} s\Delta s$$

holds true, then

$$\begin{aligned} &\left| -\lambda\left(1-\frac{\lambda}{2}\right)(b-a) - a + \frac{1}{b-a}\int_a^b s\Delta s\right| \\ &\qquad \geqslant -\lambda\left(1-\frac{\lambda}{2}\right)(b-a) - a + \frac{1}{b-a}\int_a^b s\Delta s \\ &\qquad \geqslant \frac{1}{b-a}\left(-2\int_a^{a+\lambda\frac{b-a}{2}} s\Delta s + \int_a^b s\Delta s\right) - \big(a+\lambda(b-a)\big)(1-\lambda), \end{aligned}$$

which helps us to complete our proof.

If we apply the the inequality (4) to different time scales, we will get some well-known and some new results.

Corollary 4 (Continuous case) *Let $\mathbb{T}=\mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Hence,*

$$h_2(t,s) = \frac{(t-s)^2}{2}, \quad \text{for all} \quad t,s\in\mathbb{R}.$$

This leads us to state the following inequality

$$\begin{aligned} &\left|(1-\lambda)f(t) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \\ &\qquad \leqslant M\left(\frac{1}{4}(b-a)\big((1-\lambda)^2+\lambda^2\big) + \frac{1}{b-a}\left(x-\frac{a+b}{2}\right)^2\right) \end{aligned}$$

for all $\lambda\in[0,1]$ and $a+\lambda\frac{b-a}{2} \leqslant t \leqslant b-\lambda\frac{b-a}{2}$, where $M = \sup_{x\in(a,b)}|f'(x)| < \infty$, which is exactly the generalized Ostrowski type inequality shown in Theorem 2 of [9].

Corollary 5 (Discrete case) *Let $\mathbb{T}=\mathbb{Z}$, $a=0$, $b=n$, $s=j$, $t=i$ and $f(k)=x_k$. Thus, we have*

$$\left|(1-\lambda)x_i+\lambda\frac{x_0+x_n}{2}-\frac{1}{n}\sum_{j=1}^{n}x_j\right|\leqslant\frac{M}{n}\left(\left|i-\frac{n+1}{2}\right|^2+\frac{(2\lambda^2-2\lambda+1)n^2-1}{4}\right)$$

for all $\lambda\in[0,1]$ such that $\frac{\lambda n}{2}$ and $n-\frac{\lambda n}{2}$ are in $\mathbb{Z}$ and $i\in\left[\frac{\lambda n}{2},n-\frac{\lambda n}{2}\right]\cap\mathbb{T}$, where $M=\max\limits_{1\leqslant i\leqslant n-1}|\Delta x_i|<\infty$.

Proof. In this situation, it is known that

$$h_k(t,s)=\begin{pmatrix}t-s\\k\end{pmatrix},\qquad\text{for all}\quad t,s\in\mathbb{Z}.$$

Therefore,

$$h_2\left(a,a+\lambda\frac{b-a}{2}\right)=\begin{pmatrix}-\frac{n\lambda}{2}\\2\end{pmatrix}=\frac{\frac{n\lambda}{2}\left(\frac{n\lambda}{2}+1\right)}{2},$$

$$h_2\left(t,a+\lambda\frac{b-a}{2}\right)=\begin{pmatrix}i-\frac{n\lambda}{2}\\2\end{pmatrix}=\frac{\left(i-\frac{n\lambda}{2}\right)\left(i-\frac{n\lambda}{2}-1\right)}{2},$$

$$h_2\left(t,b-\lambda\frac{b-a}{2}\right)=\begin{pmatrix}i-n+\frac{n\lambda}{2}\\2\end{pmatrix}=\frac{\left(i-n+\frac{n\lambda}{2}\right)\left(i-n+\frac{n\lambda}{2}-1\right)}{2},$$

and

$$h_2\left(t,b-\lambda\frac{b-a}{2}\right)=\begin{pmatrix}\frac{n\lambda}{2}\\2\end{pmatrix}=\frac{\frac{n\lambda}{2}\left(\frac{n\lambda}{2}-1\right)}{2},$$

Thus, we get the desired result.

Corollary 6 (Quantum calculus case). *Let $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $a=q^m,b=q^n$ with $m<n$. Then*

$$\left|(1-\lambda)f(t)+\lambda\frac{f(q^m)+f(q^n)}{2}-\frac{1}{q^n-q^m}\int_{q^m}^{q^n}f^\sigma(s)\Delta s\right|$$
$$\leqslant\frac{M}{(1+q)(q^n-q^m)}\Bigg(2t^2-(1+q)(q^m+q^n)t$$
$$+\left(\left(2\lambda^2-\frac{3}{2}\lambda+1\right)(q^{2m+1}+q^{2n+1})-\lambda(3-2\lambda)q^{m+n+1}+\frac{\lambda}{2}(q^m-q^n)^2\right)\Bigg)$$

for all $\lambda \in [0,1]$ *such that* $q^m + \lambda \frac{q^n - q^m}{2}$ *and* $q^n - \lambda \frac{q^n - q^m}{2}$ *are in* $\mathbb{T}$ *and*

$$t \in \left[q^m + \lambda \frac{q^n - q^m}{2}, q^n - \lambda \frac{q^n - q^m}{2} \right] \cap \mathbb{T}$$

where

$$M = \sup_{t \in (q^m, q^n)} \left| \frac{f(qt) - f(t)}{(q-1)t} \right|.$$

Proof. In this situation, one has

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum\limits_{\mu=0}^{\nu} q^\mu}, \quad \text{for all} \quad t, s \in \mathbb{T}$$

and

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

Therefore,

$$h_2\left(q^m, q^m + \lambda \frac{q^n - q^m}{2}\right) = \frac{\frac{\lambda}{2}(q^m - q^n)\left[q^m - \left(1 - \frac{\lambda}{2}\right)q^{m+1} - \frac{\lambda}{2}q^{n+1}\right]}{1+q},$$

$$h_2\left(t, q^m + \lambda \frac{q^n - q^m}{2}\right) = \frac{\left[t - \left(1 - \frac{\lambda}{2}\right)q^m - \frac{\lambda}{2}q^n\right]\left[t - \left(1 - \frac{\lambda}{2}\right)q^{m+1} - \frac{\lambda}{2}q^{n+1}\right]}{1+q},$$

$$h_2\left(t, q^n - \lambda \frac{q^n - q^m}{2}\right) = \frac{\left[t - \left(1 - \frac{\lambda}{2}\right)q^n - \frac{\lambda}{2}q^m\right]\left[t - \left(1 - \frac{\lambda}{2}\right)q^{n+1} - \frac{\lambda}{2}q^{m+1}\right]}{1+q},$$

and

$$h_2\left(q^m, q^n - \lambda \frac{q^n - q^m}{2}\right) = \frac{\frac{\lambda}{2}(q^n - q^m)\left[q^n - \left(1 - \frac{\lambda}{2}\right)q^{n+1} - \frac{\lambda}{2}q^{m+1}\right]}{1+q}.$$

Thus, we get the result.

3 Some particular integral inequalities on time scales

In this section we point out some particular integral inequalities on time scales as special cases, such as: *rectangle inequality* on time scales, *trapezoid inequality* on time scales, *mid-point inequality* on time scales, *Simpson inequality* on time scales, *averaged mid-point-trapezoid inequality* on time scales and others.

Throughout this section, we always assume $\mathbb{T}$ is a time scale; $a, b \in \mathbb{T}$ with $a < b$; $f : [a,b] \to \mathbb{R}$ is differentiable. We denote

$$M = \sup_{a<x<b} |f^\Delta(x)|.$$

Corollary 7 *Under the assumptions of Theorem 4 with* $\lambda = 1$ *and* $t = \frac{a+b}{2} \in \mathbb{T}$*. Then we have the trapezoid inequality on time scales*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \leqslant \frac{M}{b-a}\left(h_2\left(a, \frac{a+b}{2}\right) + h_2\left(b, \frac{a+b}{2}\right)\right). \tag{8}$$

Remark 2 *If we take* $\lambda = 0$ *in Theorem 4, then Theorem 2 is recaptured. Therefore, Theorem 4 may be regarded as a generalization of Theorem 2.*

Corollary 8 *Under the assumptions of Theorem 4 with* $\lambda = \frac{1}{3}$*. Then we have the following integral inequality on time scales*

$$\begin{aligned}\left|\frac{1}{6}\left(f(a)+f(b)+4f(t)\right) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \\ \leqslant \frac{M}{b-a}\Bigg(h_2\left(a, \frac{5a+b}{6}\right) + h_2\left(t, \frac{5a+b}{6}\right) \\ + h_2\left(t, \frac{a+5b}{6}\right) + h_2\left(b, \frac{a+5b}{6}\right)\Bigg)\end{aligned} \tag{9}$$

for all $t \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right] \cap \mathbb{T}$.

Remark 3 *If we choose* $t = \frac{a+b}{2}$ *in (9), we get the Simpson inequality on time scales*

$$\begin{aligned}\left|\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \\ \leqslant \frac{M}{b-a}\Bigg(h_2\left(a, \frac{5a+b}{6}\right) + h_2\left(\frac{a+b}{2}, \frac{5a+b}{6}\right) \\ + h_2\left(\frac{a+b}{2}, \frac{a+5b}{6}\right) + h_2\left(b, \frac{a+5b}{6}\right)\Bigg).\end{aligned}$$

Corollary 9 *Under the assumptions of Theorem 4 with* $\lambda = \frac{1}{2}$*. Then we have* *the following integral inequality on time scales*

$$\left|\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f(t)\right)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|$$
$$\leqslant \frac{M}{b-a}\left(h_2\left(a,\frac{3a+b}{4}\right)+h_2\left(t,\frac{3a+b}{4}\right)\right. \tag{10}$$
$$\left.+h_2\left(t,\frac{a+3b}{4}\right)+h_2\left(b,\frac{a+3b}{4}\right)\right)$$

for all $t\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\cap\mathbb{T}$.

Remark 4 *If we choose* $t=\frac{a+b}{2}$ *in (10), we get the averaged mid-point-trapezoid inequality on time scales*

$$\left|\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|$$
$$\leqslant \frac{M}{b-a}\left(h_2\left(a,\frac{3a+b}{4}\right)+h_2\left(\frac{a+b}{2},\frac{3a+b}{4}\right)\right.$$
$$\left.+h_2\left(\frac{a+b}{2},\frac{a+3b}{4}\right)+h_2\left(b,\frac{a+3b}{4}\right)\right).$$

Corollary 10 *Under the assumptions of Theorem 4 with* $t=\frac{a+b}{2}\in\mathbb{T}$. *Then we have the following integral inequality on time scales*

$$\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right|$$
$$\leqslant \frac{M}{b-a}\left(h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(\frac{a+b}{2},a+\lambda\frac{b-a}{2}\right)\right. \tag{11}$$
$$\left.+h_2\left(\frac{a+b}{2},b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right)$$

for all $\lambda\in[0,1]$ *such that* $a+\lambda(b-a)/2$ *and* $b-\lambda(b-a)/2$ *are in* $\mathbb{T}$.

Remark 5 *If we choose* $\lambda=0$ *in (11), we get the mid-point inequality on time scales*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f^\sigma(t)\Delta t\right|\leqslant\frac{M}{b-a}\left(h_2\left(\frac{a+b}{2},a\right)+h_2\left(\frac{a+b}{2},b\right)\right).$$

Acknowledgements

The work was supported by the Science Research Foundation of Nanjing University of Information Science and Technology (Grant No. 20080295) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 09KJB110005).

References

[1] R. Agarwal, M. Bohner, A. Peterson, *Inequalities on time scales: A survey*, Math. Inequal. Appl., **4** (2001), 535–557.

[2] M. Bohner, A. Peterson, *Dynamic equations on time scales*, Birkhäuser, Boston, 2001.

[3] M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[4] M. Bohner, T. Matthews, *The Grüss inequality on time scales*, *Commun. Math. Anal.* **3** (2007), 1-8.

[5] M. Bohner, T. Matthews, *Ostrowski inequalities on time scales*, JIPAM. J. Inequal. Pure Appl. Math., **9** (2008), Art. 6, 8 pp.

[6] P. Cerone, S. S. Dragomir, J. Roumeliotis, *An inequality of Ostrowski-Grüss type for twice differentiable mappings and applications in numerical integration*, Kyungpook Math. J., **39** (1999), 331-341.

[7] C. Dinu, *A weighted Hermite Hadamard inequality for Steffensen-Popoviciu and Hermite-Hadamard weights on time scales*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **17** (2009), 77-90.

[8] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl., **5** (2000), 533-579.

[9] S. S. Dragomir, P. Cerone, J. Roumelitis, *A new generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means*, Appl. Math. Lett., **13** (2000), 19-25.

[10] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität. Würzburg, 1988.

[11] R. Hilscher, *A time scales version of a Wirtinger-type inequality and applications, Dynamic equations on time scales*, J. Comput. Appl. Math., **141** (2002), 219-226.

[12] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, *Dynamic systems on measure chains*, Kluwer Academic Publishers, 1996.

[13] W. N. Li, W. H. Sheng, *Some nonlinear dynamic inequalities on time scales*, Proc. Indian Acad. Sci. (Math. Sci.), **117** (2007), 545-554.

[14] W. J. Liu, *Several error inequalities for a quadrature formula with a parameter and applications*, Comput. Math. Appl., **56** (2008), 1766-1772.

[15] W. J. Liu, Y. Huang, X. X. Pan, *New weighted Ostrowski-Grüss-Cebysev type inequalities*, Bull. Korean Math. Soc., **45** (2008), 477-483.

[16] W. J. Liu, Q. A. Ngô, *A generalization of Ostrowski inequality on time scales for k points*, Appl. Math. Comput., **203** (2008), 754-760.

[17] W. J. Liu, Q. A. Ngô, *An Ostrowski type inequality on time scales for functions whose second derivatives are bounded*, Inequality Theory and Applications, **6**, 2009, to appear.

[18] W. J. Liu, Q. A. Ngô, W. B. Chen, *A perturbed Ostrowski type inequality on time scales for k points for functions whose second derivatives are bounded*, J. Inequal. Appl., **2008**, Art. ID 597241, 12 pp.

[19] W. J. Liu, Q. L. Xue, S. F. Wang, *Several new perturbed Ostrowski-like type inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **8** (2007), Art.110, 6 pp.

[20] D. S. Mitrinović, J. Pecarić, A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1994.

[21] D. S. Mitrinović, J. Pecarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

(Wenjun J. Liu) College of Mathematics and Physics,
Nanjing University of Information Science and Technology,
Nanjing 210044, China,
E-mail: wjliu@nuist.edu.cn

(Quốc-Anh Ngô) Department of Mathematics,
College of Science,
Việt Nam National University,
Hà Nội, Việt Nam,
E-mail: bookworm_vn@yahoo.com

(Wenbing B. Chen) College of Mathematics and Physics,
Nanjing University of Information Science and Technology,
Nanjing 210044, China,
E-mail: chenwb@nuist.edu.cn