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An Ostrowski–Grüss type inequality on time scales

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ABSTRACT

We derive a new inequality of Ostrowski–Grüss type on time scales by using the Grüss inequality on time scales and thus unify corresponding continuous and discrete versions. We also apply our result to the quantum calculus case.

1. Introduction

In 1997, Dragomir and Wang [1] proved the following Ostrowski–Grüss type integral inequality.

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \le f'(x) \le \Gamma$, $x \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma), \tag{1}$$

for all $x \in [a, b]$.

This inequality is a connection between the Ostrowski inequality [2] and the Grüss inequality [3]. It can be applied to bound some special mean and some numerical quadrature rules. For other related results on similar integral inequalities please see the papers [4–7] and the references therein.

The aim of this paper is to extend a generalization of the Ostrowski–Grüss type integral inequality to an arbitrary time scale.

2. Time scale essentials

A time scale is an arbitrary nonempty closed subset of the real numbers. The development of the theory of time scales was initiated by Hilger [8] in 1988 as a theory capable of containing difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales. For example, we refer the reader to [9–13].

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For a general introduction to the theory of time scales we refer the reader to the books [14–16]. Throughout this paper, we suppose that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of a real interval with the given time scale.

The present paper is motivated by the following results: the Grüss inequality on time scales and the Ostrowski inequality on time scales due to Bohner and Matthews. To be precise, the following so-called Grüss inequality on time scales was established in [10].

Theorem 2 *(See [10, Theorem 3.1]). Suppose $a, b, s \in \mathbb{T}, f, g \in C_{rd}$ and $f, g : [a, b] \to \mathbb{R}$. Then for*

$$m_1 \le f(s) \le M_1, \qquad m_2 \le g(s) \le M_2, \tag{2}$$

we have

$$\left|\frac{1}{b-a}\int_a^b f^\sigma(s)g^\sigma(s)\Delta s - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\,\frac{1}{b-a}\int_a^b g^\sigma(s)\Delta s\right| \le \frac{1}{4}(M_1-m_1)(M_2-m_2). \tag{3}$$

The same authors also proved the following so-called Ostrowski inequality on time scales in [11].

Theorem 3 *(See [11, Theorem 3.5]). Suppose $a, b, s, t \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ is differentiable. Then*

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right| \le \frac{M}{b-a}(h_2(t,a)+h_2(t,b)), \tag{4}$$

where $M = \sup_{a<t<b}|f^\Delta(t)|$. This inequality is sharp in the sense that the right-hand side of (4) cannot be replaced by a smaller one.

In the present paper we shall first derive a new inequality of Ostrowski–Grüss type on time scales by using Theorem 2 and then unify corresponding continuous and discrete versions. We also apply our result to the quantum calculus case.

3. The Ostrowski–Grüss type inequality on time scales

By adapting the techniques presented in [1] and using Theorem 2, the Ostrowski–Grüss type inequality can be shown for general time scales.

Theorem 4. *Suppose $a, b, s, t \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ is differentiable. Suppose f^Δ is rd-continuous and*

$$\gamma \le f^\Delta(t) \le \Gamma, \quad \forall\, t \in [a,b].$$

Then we have

$$\left|f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\,(h_2(t,a)-h_2(t,b))\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{5}$$

for all $t \in [a, b]$.

To prove Theorem 4, we need the following Montgomery identity.

Lemma 1 *(See [11, Lemma 3.1]). Suppose $a, b, s, t \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ is differentiable. Then*

$$f(t) = \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s + \frac{1}{b-a}\int_a^b p(t,s)f^\Delta(s)\Delta s, \tag{6}$$

where

$$p(t,s) = \begin{cases} s-a, & a \le s < t,\\ s-b, & t \le s \le b.\end{cases}$$

Proof (*Proof of* Theorem 4). By applying Lemma 1, we get

$$f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s = \frac{1}{b-a}\int_a^b p(t,s)f^\Delta(s)\Delta s, \tag{7}$$

for all $t \in [a, b]$. It is clear that for all $t \in [a, b]$ and $s \in [a, b]$ we have

$$t - b \le p(t,s) \le t - a.$$

Applying Theorem 2 to the mapping $p(t,\cdot)$ and $f^{\Delta}(\cdot)$, we get

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b p(t,s)f^{\Delta}(s)\Delta s-\frac{1}{b-a}\int_a^b p(t,s)\Delta s\,\frac{1}{b-a}\int_a^b f^{\Delta}(s)\Delta s\right| &\le \frac{1}{4}[(t-a)-(t-b)](\Gamma-\gamma)\\ &\le \frac{1}{4}(b-a)(\Gamma-\gamma).\end{aligned} \tag{8}$$

A simple calculation gives that

$$\begin{aligned}\int_a^b p(t,s)\Delta s &= \int_a^t (s-a)\Delta s+\int_t^b (s-b)\Delta s\\ &= \int_a^t (s-a)\Delta s-\int_b^t (s-b)\Delta s\\ &= h_2(t,a)-h_2(t,b)\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f^{\Delta}(s)\Delta s=\frac{f(b)-f(a)}{b-a}.$$

By combining (7) and (8) and the above two equalities, we obtain (5). □

If we apply the Ostrowski–Grüss type inequality to different time scales, we will get some well-known and some new results.

Corollary 1. *(Continuous case). Let $\mathbb{T}=\mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Hence,*

$$h_2(t,s)=\frac{(t-s)^2}{2},\quad \text{for all } t,s\in\mathbb{R}.$$

This leads us to state the following inequality:

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{f(b)-f(a)}{b-a}\left(t-\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma), \tag{9}$$

for all $t\in[a,b]$, where $\gamma\le f'(t)\le\Gamma$, which is exactly the Ostrowski–Grüss type inequality shown in Theorem 1.

Corollary 2 (*Discrete Case*). *Let $\mathbb{T}=\mathbb{Z}$, $a=0$, $b=n$, $s=j$, $t=i$ and $f(k)=x_k$. With these, it is known that*

$$h_k(t,s)=\binom{t-s}{k},\quad \text{for all } t,s\in\mathbb{Z}.$$

Therefore,

$$h_2(t,0)=\binom{t}{2}=\frac{t(t-1)}{2},\qquad h_2(t,n)=\binom{t-n}{2}=\frac{(t-n)(t-n-1)}{2}.$$

Thus, we have

$$\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j-\frac{x_n}{n}\left(i-\frac{n+1}{2}\right)\right|\le\frac{1}{4}n(\Gamma-\gamma), \tag{10}$$

for all $i=\overline{1,n}$, where $\gamma\le\Delta x_i\le\Gamma$.

Corollary 3 (*Quantum Calculus Case*). *Let $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $a=q^m$, $b=q^n$ with $m<n$. In this situation, one has*

$$h_k(t,s)=\prod_{\nu=0}^{k-1}\frac{t-q^{\nu}s}{\sum\limits_{\mu=0}^{\nu}q^{\mu}},\quad \text{for all } t,s\in\mathbb{T}.$$

Therefore,

$$h_2\left(t,q^m\right)=\frac{(t-q^m)\left(t-q^{m+1}\right)}{1+q},\qquad h_2\left(t,q^n\right)=\frac{(t-q^n)\left(t-q^{n+1}\right)}{1+q}.$$

Then

$$\left|f(t)-\frac{1}{q^n-q^m}\int_{q^m}^{q^n} f^{\sigma}(s)\Delta s-\frac{f(q^n)-f(q^m)}{q^n-q^m}\left(t-\frac{q^{2n+1}-q^{2m+1}}{q+1}\right)\right|\le\frac{1}{4}(q^n-q^m)(\Gamma-\gamma), \tag{11}$$

where

$$\gamma\le\frac{f(qt)-f(t)}{(q-1)(t)}\le\Gamma,\quad \forall t\in[a,b].$$

If f^{Δ} is bounded on $[a,b]$ then we have the following corollary.

Corollary 4. *With the assumptions of Theorem 4, if $|f^{\Delta}(t)|\le M$ for all $t\in[a,b]$ and some positive constant M, then we have*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s-\frac{f(b)-f(a)}{(b-a)^2}\left(h_2(t,a)-h_2(t,b)\right)\right|\le\frac{1}{2}(b-a)M, \tag{12}$$

for all $t\in[a,b]$.

Furthermore, choosing $t=(a+b)/2$ and $t=b$, respectively, in (5), we have the following corollary.

Corollary 5. *With the assumptions in Theorem 4, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2\left(\frac{a+b}{2},a\right)-h_2\left(\frac{a+b}{2},b\right)\right]-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s\right|$$
$$\le\frac{1}{4}(b-a)(\Gamma-\gamma)\quad\left(\text{if } \frac{a+b}{2}\in\mathbb{T}\right) \tag{13}$$

and

$$\left|f(b)-\frac{f(b)-f(a)}{(b-a)^2}h_2(b,a)-\frac{1}{b-a}\int_a^b f^{\sigma}(s)\Delta s\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma). \tag{14}$$

Remark 1. We note that the inequality (5) in Theorem 4 is not sharp. In our further work [17], we will obtain a sharp result by deriving and using a sharp Grüss type inequality on time scales.

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