

A SHARP GRÜSS TYPE INEQUALITY ON TIME SCALES AND APPLICATION TO THE SHARP OSTROWSKI-GRÜSS INEQUALITY

QUỐC ANH NGÔ*

Department of Mathematics
College of Science, Việt Nam National University
Hà Nội, Việt Nam.

Department of Mathematics
National University of Singapore
2 Science Drive 2, Singapore 117543.

WENJUN LIU[†]

College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China.

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Abstract

In this paper we first derive a sharp Grüss type inequality on time scales and then apply it to the sharp Ostrowski-Grüss inequality on time scales which improves our a recent result.

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1 Introduction

In 2002, almost at the same time, by using similar somewhat complicated methods, Cheng and Sun in [7] as well as Matic in [16] have proved the following Grüss type inequality, respectively.

*E-mail address: bookworm.vn@yahoo.com

[†]E-mail address: wjliu@nuist.edu.cn

Theorem 1.1. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two integrable functions such that

$$\gamma \leq g(s) \leq \Gamma \quad (1.1)$$

for some constants γ, Γ for all $x \in [a, b]$. Then

$$\begin{aligned} & \left| \int_a^b f(x)g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx \right| \\ & \leq \frac{\Gamma - \gamma}{2} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y) \, dy \right| dx \end{aligned} \quad (1.2)$$

Moreover, Matić has proved that there exists a function g attaining the equality in (1.2), Cerone and Dragomir in [8] have proved that the constant $\frac{1}{2}$ in (1.2) is sharp. The result stated in Theorem 1.1 is of particular interest and very useful in the case when

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y) \, dy \right| dx$$

can be evaluated exactly. A great deal of sharp integral inequalities can be established by using this theorem.

The development of the theory of time scales was initiated by Hilger [10] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied certain integral inequalities on time scales ([1, 4, 5, 11, 17]). Recently, Liu and Ngô [14] proved the following Ostrowski-Grüss type inequality on time scales, which is a combination of both Grüss inequality and Ostrowski inequality on time scales due to Bohner and Matthews ([4, 5]).

Theorem 1.2. Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. If f^Δ is rd-continuous and

$$\gamma \leq f^\Delta(t) \leq \Gamma, \quad \forall t \in [a, b].$$

Then we have

$$\left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} (h_2(t, a) - h_2(t, b)) \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma),$$

for all $t \in [a, b]$ where $h_i(t, s)$ is as in Definition 2.1.

It is easy to see that the foregoing inequality is not sharp. In the first part of this paper, we shall extend Theorem 1.1 to arbitrary time scale (see Theorem 3.1 below). We also note that, inspired by Liu in [13], our proof is very simple. As an application, we shall slightly improve the above Theorem 1.2 as follow by giving a sharp bound.

Theorem 1.3. Under the assumptions of Theorem 1.2, we have

$$\begin{aligned} & \left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} (h_2(t, a) - h_2(t, b)) \right| \\ & \leq \frac{\Gamma - \gamma}{2(b-a)} \int_a^b \left| p(t, x) - \frac{h_2(t, a) - h_2(t, b)}{b-a} \right| \Delta x, \end{aligned} \quad (1.3)$$

for all $t \in [a, b]$, where $p(t, x)$ is defined as in (4.1). Moreover, the constant $\frac{1}{2}$ in (1.3) is sharp.

In fact, we do not know the relationship between the inequalities in Theorem 1.3 and Theorem 1.2. But we are sure that when $\mathbb{T} = \mathbb{R}$, Theorem 1.3 gives a better result than Theorem 1.2.

2 Time scales essentials

Now we briefly introduce the time scales theory and refer the reader to Hilger [10] and the books [2, 3, 12] for further details.

By a time scale \mathbb{T} we mean any closed subset of \mathbb{R} with order and topological structure present in canonical way. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, while the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, then we say that t is *right-scattered*, while if $\rho(t) < t$ then we say that t is *left-scattered*. If $\sigma(t) = t$, the t is called *right-dense*, and if $\rho(t) = t$ then t is called *left-dense*. If \mathbb{T} has a left-scattered maximum t , then $\mathbb{T}^\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \rightarrow \mathbb{R}$, we denote the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number, if one exists, with the property that for any given $\varepsilon > 0$ there is a neighborhood U of t such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We say that f is Δ -differentiable on \mathbb{T}^κ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

A mapping $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) if: f is continuous at each right-dense point or maximal element of \mathbb{T} ; the left-sided limit $\lim_{s \rightarrow t^-} f(s) = f(t^-)$ exists at each left-dense point t of \mathbb{T} . A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a Δ -antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$.

Definition 2.1. Let $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t, s) = 1 \quad \text{for all } s, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau \quad \text{for all } s, t \in \mathbb{T}.$$

Throughout this paper, we suppose that \mathbb{T} is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of real interval with the given time scale.

3 The Grüss type inequality on time scales

We firstly state our Grüss type inequality for general time scales.

Theorem 3.1. Let $a, b, s \in \mathbb{T}$, $f, g \in C_{rd}$ and $f, g : [a, b] \rightarrow \mathbb{R}$. Then for

$$\gamma \leq g(s) \leq \Gamma, \tag{3.1}$$

we have

$$\begin{aligned} & \left| \int_a^b f(x)g(x)\Delta x - \frac{1}{b-a} \int_a^b f(x)\Delta x \int_a^b g(x)\Delta x \right| \\ & \leq \frac{\Gamma-\gamma}{2} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y \right| \Delta x. \end{aligned} \quad (3.2)$$

Moreover, the constant $\frac{1}{2}$ in (3.2) is sharp.

Proof. Since

$$\int_a^b \left(f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y \right) \Delta x = 0,$$

then it is clear that

$$\begin{aligned} & \left| \int_a^b f(x)g(x)\Delta x - \frac{1}{b-a} \int_a^b f(x)\Delta x \int_a^b g(x)\Delta x \right| \\ & = \left| \int_a^b \left(f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y \right) \left(g(x) - \frac{\gamma+\Gamma}{2} \right) \Delta x \right| \\ & \leq \sup_{a < x < b} \left| g(x) - \frac{\gamma+\Gamma}{2} \right| \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y \right| \Delta x. \end{aligned}$$

By assumption (3.1), we get

$$\left| g(x) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{\Gamma-\gamma}{2}$$

and the desired inequality (3.2) follows immediately.

To prove the sharpness of this inequality, let us define

$$g(x) = \begin{cases} \Gamma & \text{if } f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y \geq 0, \\ \gamma & \text{if } f(x) - \frac{1}{b-a} \int_a^b f(y)\Delta y < 0. \end{cases}$$

Then it is easy to verify that (3.2) is equality. \square

Similarly to Theorem 3.1, we obtain the weighted Grüss type inequality for general time scales.

Theorem 3.2. Let $a, b, s \in \mathbb{T}$, $f, g, w \in C_{rd}$ and $f, g, w : [a, b] \rightarrow \mathbb{R}$. Then for

$$\gamma \leq g(s) \leq \Gamma, \quad (3.3)$$

and

$$w(x) > 0 \quad \text{for a.e. } x \in [a, b], \quad (3.4)$$

we have

$$\begin{aligned} & \left| \int_a^b w(x)f(x)g(x)\Delta x - \frac{1}{\int_a^b w(x)\Delta x} \int_a^b w(x)f(x)\Delta x \int_a^b w(x)g(x)\Delta x \right| \\ & \leq \frac{\Gamma-\gamma}{2} \int_a^b w(x) \left| f(x) - \frac{1}{\int_a^b w(y)\Delta y} \int_a^b w(y)f(y)\Delta y \right| \Delta x. \end{aligned} \quad (3.5)$$

Proof. Since

$$\int_a^b \left(w(x)f(x) - \frac{w(x)}{\int_a^b w(y)\Delta y} \int_a^b w(y)f(y)\Delta y \right) \Delta x = 0,$$

then

$$\begin{aligned} & \left| \int_a^b w(x)f(x)g(x)\Delta x - \frac{1}{\int_a^b w(x)\Delta x} \int_a^b w(x)f(x)\Delta x \int_a^b w(x)g(x)\Delta x \right| \\ &= \left| \int_a^b \left(w(x)f(x) - \frac{w(x)}{\int_a^b w(x)\Delta x} \int_a^b w(y)f(y)\Delta y \right) \left(g(x) - \frac{\gamma + \Gamma}{2} \right) \Delta x \right| \\ &\leq \sup_{a < x < b} \left| g(x) - \frac{\gamma + \Gamma}{2} \right| \int_a^b w(x) \left| f(x) - \frac{1}{\int_a^b w(y)\Delta y} \int_a^b w(y)f(y)\Delta y \right| \Delta x. \end{aligned}$$

By assumption (3.3), we get

$$\left| g(x) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{\Gamma - \gamma}{2}$$

and the desired inequality (3.5) follows immediately.

To prove the sharpness of this inequality, let us define

$$g(x) = \begin{cases} \Gamma & \text{if } f(x) - \frac{1}{\int_a^b w(y)\Delta y} \int_a^b w(y)f(y)\Delta y \geq 0, \\ \gamma & \text{if } f(x) - \frac{1}{\int_a^b w(y)\Delta y} \int_a^b w(y)f(y)\Delta y < 0. \end{cases}$$

Then it is easy to verify that (3.5) is equality. \square

4 The sharp Ostrowki-Grüss inequality on time scales

In this section, by applying Theorem 3.1, we shall prove Theorem 1.3.

Proof of Theorem 1.3. Let

$$p(t, s) = \begin{cases} s - a, & a \leq s < t, \\ s - b, & t \leq s \leq b. \end{cases} \quad (4.1)$$

Applying Theorem 3.1 for the choices $f(x) := p(t, x)$ and $g(x) := f^\Delta(x)$ to get

$$\begin{aligned} & \left| \int_a^b p(t, x)f^\Delta(x)\Delta x - \frac{1}{b-a} \int_a^b p(t, x)\Delta x \int_a^b f^\Delta(x)\Delta x \right| \\ &\leq \frac{\Gamma - \gamma}{2} \int_a^b \left| p(t, x) - \frac{1}{b-a} \int_a^b p(t, y)\Delta y \right| \Delta x. \end{aligned} \quad (4.2)$$

We obviously have

$$\frac{1}{b-a} \int_a^b f^\Delta(x)\Delta x = \frac{f(b) - f(a)}{b-a}$$

and

$$\frac{1}{b-a} \int_a^b p(t,x) \Delta x = \frac{h_2(t,a) - h_2(t,b)}{b-a}.$$

Hence,

$$\int_a^b \left| p(t,x) - \frac{1}{b-a} \int_a^b p(t,y) \Delta y \right| \Delta x = \int_a^b \left| p(t,x) - \frac{h_2(t,a) - h_2(t,b)}{b-a} \right| \Delta x. \quad (4.3)$$

By (4.2) and (4.3), we deduce that

$$\begin{aligned} & \left| \int_a^b p(t,x) f^\Delta(x) \Delta x - \frac{1}{b-a} \int_a^b p(t,x) \Delta x \int_a^b f^\Delta(x) \Delta x \right| \\ & \leq \frac{\Gamma - \gamma}{2} \int_a^b \left| p(t,x) - \frac{h_2(t,a) - h_2(t,b)}{b-a} \right| \Delta x. \end{aligned} \quad (4.4)$$

By using Montgomery Identity, see [5], we deduce that

$$f(t) = \frac{1}{b-a} \int_a^b f^\sigma(x) \Delta x + \frac{1}{b-a} \int_a^b p(t,x) f^\Delta(x) \Delta x,$$

which helps us to deduce that

$$\begin{aligned} & \left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} (h_2(t,a) - h_2(t,b)) \right| \\ & = \left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{1}{b-a} \int_a^b p(t,x) \Delta x \frac{1}{b-a} \int_a^b f^\Delta(x) \Delta x \right| \\ & = \frac{1}{b-a} \left| \int_a^b p(t,x) f^\Delta(x) \Delta x - \frac{1}{b-a} \int_a^b p(t,x) \Delta x \int_a^b f^\Delta(x) \Delta x \right|. \end{aligned}$$

Hence

$$\begin{aligned} & \left| f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} (h_2(t,a) - h_2(t,b)) \right| \\ & \leq \frac{\Gamma - \gamma}{2(b-a)} \int_a^b \left| p(t,x) - \frac{h_2(t,a) - h_2(t,b)}{b-a} \right| \Delta x, \end{aligned}$$

for all $t \in [a, b]$. It is obviously to see that the foregoing inequality is sharp. \square

If we apply the Theorem 1.3 to different time scales, we will get some well-known and some new results.

Corollary 4.1 (Continuous case). *Let $\mathbb{T} = \mathbb{R}$, then inequality (1.3) becomes*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds - \frac{f(b) - f(a)}{b-a} \left(t - \frac{a+b}{2} \right) \right| \leq \frac{1}{8} (b-a) (\Gamma - \gamma) \quad (4.5)$$

for all $t \in [a, b]$, where $\gamma \leq f'(t) \leq \Gamma$. Actually, inequality (4.5) is sharp, see [8, Theorem 3].

Corollary 4.2 (Discrete case). Let $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = n$, $s = j$, $t = i$ and $f(k) = x_k$. With these, it is known that

$$h_k(t, s) = \binom{t-s}{k}, \quad \text{for all } t, s \in \mathbb{Z}.$$

Therefore,

$$h_2(t, 0) = \binom{t}{2} = \frac{t(t-1)}{2}, \quad h_2(t, n) = \binom{t-n}{2} = \frac{(t-n)(t-n-1)}{2}.$$

Thus, we have

$$\left| x_i - \frac{1}{n} \sum_{j=1}^n x_j - \frac{x_n - x_0}{n} \left(i - \frac{n+1}{2} \right) \right| \leq \frac{\Gamma - \gamma}{2n} \sum_{j=0}^{n-1} \left| p(i, j) - \left(i - \frac{n+1}{2} \right) \right| \quad (4.6)$$

for all $i = \overline{1, n}$, where $\gamma \leq \Delta x_i \leq \Gamma$ and

$$\begin{aligned} p(i, 0) &= 0, \\ p(1, j) &= j - n, \quad 1 \leq j \leq n-1, \\ p(n, j) &= j, \quad 0 \leq j \leq n-1, \\ p(i, j) &= \begin{cases} j, & 0 \leq j < i, \\ j - n, & i \leq j \leq n-1. \end{cases} \end{aligned}$$

Moreover, the constant $\frac{1}{2}$ in (4.6) is sharp.

Corollary 4.3 (Quantum calculus case). Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$, $b = q^n$ with $m < n$. In this situation, one has

$$h_k(t, s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}, \quad \text{for all } t, s \in \mathbb{T}.$$

Therefore,

$$h_2(t, q^m) = \frac{(t - q^m)(t - q^{m+1})}{1 + q}, \quad h_2(t, q^n) = \frac{(t - q^n)(t - q^{n+1})}{1 + q}.$$

Then

$$\begin{aligned} & \left| f(t) - \frac{1}{q^n - q^m} \int_{q^m}^{q^n} f^\sigma(s) \Delta s - \frac{f(q^n) - f(q^m)}{q^n - q^m} \left(t - \frac{q^{2n+1} - q^{2m+1}}{q+1} \right) \right| \\ & \leq \frac{\Gamma - \gamma}{2(q^n - q^m)} \sum_{k=m}^{n-1} \left| p(t, q^k) - \left(t - \frac{q^{2n+1} - q^{2m+1}}{q+1} \right) \right|, \end{aligned} \quad (4.7)$$

where

$$\gamma \leq \frac{f(qt) - f(t)}{(q-1)(t)} \leq \Gamma, \quad \forall t \in [a, b]$$

and

$$p(t, q^k) = \begin{cases} q^k - q^m, & q^m \leq q^k < t, \\ q^k - q^n, & t \leq q^k \leq q^n. \end{cases}$$

Moreover, the constant $\frac{1}{2}$ in (4.7) is sharp.

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