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A MULTIPLICITY RESULT FOR A CLASS OF EQUATIONS OF p-LAPLACIAN TYPE WITH SIGN-CHANGING NONLINEARITIES

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Abstract. Using variational arguments we study the non-existence and multiplicity of non-negative solutions for a class equations of the form

$$-\mathrm{div}(a(x, \nabla u)) = \lambda f(x, u) \text{ in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqq 3$, f is a sign-changing Carathéodory function on $\Omega \times [0, +\infty)$ and λ is a positive parameter.

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1. Introduction. This paper deals with the non-existence and multiplicity of non-negative, non-trivial solutions to the following problem,

$$-\mathrm{div}(a(x, \nabla u)) = \lambda f(x, u) \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, function a satisfies

$$|a(x, \xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, $h_0(x) \geqq 0$, $h_1(x) \geqq 1$ for a.e. $x \in \Omega$, $p \geqq 2$ and $\lambda > 0$ is a parameter. When h_0 and h_1 belong to $L^\infty(\Omega)$, the problem has been studied by many authors (see [**2, 8, 10**] for details). Here we study the situation that $h_0 \in L^{\frac{p}{p-1}}(\Omega)$ and $h_1 \in L^1_{loc}(\Omega)$. Then problem (1.1)–(1.2) now may be non-uniform in the sense that the functional associated to the problem may be infinity for some u.

We point out the fact that in [**4, 13**], D. M. Duc and N. T. Vu have studied the following Dirichlet elliptic problem,

$$-\mathrm{div}(a(x, \nabla u)) = f(x, u) \text{ in } \Omega, \tag{1.3}$$

$$u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where the nonlinear term f verifies the so-called Ambrosetti–Rabinowitz condition. The authors obtained the existence of a weak solution by using a variant of the

Mountain pass theorem introduced in [**3**]. Then, H. Q. Toan and Q.-A. Ngô [**12**] gave some multiplicity results in the case when $f(x,u) = h(x)|u|^{r-1}u + g(x)|u|^{s-1}u$. Using the Mountain pass theorem in [**3**] combined with Ekeland's variational principle in [**5**] they proved that problem (1.3)–(1.4) has at least two weak solutions.

Motivated by K. Perera [**10**] and M. Mihăilescu and V. Rădulescu [**7**], the goal of this work is to investigate the problem (1.1)–(1.2) with positive parameter λ and the sign-changing nonlinearity f. We also do not require that the nonlinear term f verifies the Ambrosetti–Rabinowitz condition as in [**4, 12**].

In order to state our main result, let us introduce the following hypotheses on problem (1.1)–(1.2).

Assume that $N \geqq 3$ and $2 \leqq p < N$. Let Ω be a bounded domain with a smooth boundary $\partial\Omega$. Consider that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x,\xi)$, is the continuous derivative with respect to ξ of the continuous function $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$, that is, $a(x,\xi) = \partial A(x,\xi)/\partial\xi$ and $A(x,0) = 0$ for a.e. $x \in \Omega$. Assume that there are positive constant c_0 and two non-negative measurable functions h_0, h_1 such that $h_0 \in L^{p/p-1}(\Omega)$, $h_1 \in L^1_{loc}(\Omega)$, $h_1(x) \geqq 1$, a.e. $x \in \Omega$. Suppose that a and A satisfy the following hypotheses.

(**A$_1$**) $|a(x,\xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(**A$_2$**) The following inequality holds

$$0 \leqq (a(x,\xi) - a(x,\psi)) \cdot (\xi - \psi)$$

for all $\xi, \psi \in \mathbb{R}^N$, a.e. $x \in \Omega$, with equality if and only if $\xi = \psi$.

(**A$_3$**) There exists a positive constant k_0 such that

$$A\left(x, \frac{\xi+\psi}{2}\right) \leqq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\psi) - k_0 h_1(x)|\xi - \psi|^p$$

for all $\xi, \psi \in \mathbb{R}^N$, a.e. $x \in \Omega$, that is, A is p-uniformly convex.

(**A$_4$**) There exists a constant $k_1 > 0$ such that the following inequalities hold true

$$k_1 h_1(x)|\xi|^p \leqq a(x,\xi)\cdot\xi \leqq pA(x,\xi)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

Example 1.

1. Let

$$A(x,\xi) = \frac{h(x)}{p}|\xi|^p, \quad a(x,\xi) = h(x)|\xi|^{p-2}\xi,$$

with $p \geqq 2$ and $h \in L^1_{loc}(\Omega)$. Then we get the operator $\mathrm{div}(h(x)|\nabla u|^{p-2}\nabla u)$, and if $h(x) \equiv 1$ in Ω we conclude the well-known p-Laplacian operator

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

as in [**8, 10**].

2. Let

$$A(x,\xi) = \frac{h(x)}{p}\left((1+|\xi|^2)^{\frac{p}{2}} - 1\right)$$

with $p \geqq 2$, $h \in L^{\frac{p}{p-1}}(\Omega)$. Then

$$a(x,\xi) = h(x)(1+|\xi|^2)^{\frac{p-2}{2}}\xi.$$

We obtain the generalised mean curvature operator

$$\operatorname{div}\big(h(x)(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\big).$$

It should be observed that the above examples have not been considered in [**2, 8, 10**] yet. For more information and connection on these operators, the reader may consult either [**2**] or [**8**] and the references therein.

As in [**10**], we assume that function $f:\Omega\times[0,+\infty)\to\mathbb{R}$ is a sign-changing Carathéodory function and satisfies the following hypotheses:

(**F$_1$**) $f(x,0)=0$, $|f(x,t)| \leqq Ct^{p-1}$ for all $t\in[0.+\infty)$, a.e. $x\in\Omega$, and for some constant $C>0$.

(**F$_2$**) There exist two positive constants $t_0, t_1 > 0$ such that $F(x,t)\leqq 0$ for $0\leqq t\leqq t_0$ and $F(x,t_1)>0$.

(**F$_3$**) $\limsup\limits_{t\to\infty}\frac{F(x,t)}{t^p}\leqq 0$ uniformly in x, where $F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

We now consider the following subspace of $W_0^{1,p}(\Omega)$:

$$H := \left\{u\in W_0^{1,p}(\Omega):\int_\Omega h_1(x)|\nabla u|^p\,\mathrm{d}x < +\infty\right\}.$$

Then H is an infinite dimensional Banach space with respect to the norm (see [**4**])

$$\|u\|_H = \left(\int_\Omega h_1(x)|\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

We define the functional $\Phi_\lambda: H\to\mathbb{R}$ by

$$\Phi_\lambda(u) = \Lambda(u) - I(u), \tag{1.5}$$

where

$$\Lambda(u) = \int_\Omega A(x,\nabla u)\,\mathrm{d}x, \quad I(u) = \lambda\int_\Omega F(x,u)\,\mathrm{d}x, \quad u\in H. \tag{1.6}$$

Since $h_0\in L^{p/p-1}(\Omega)$, then the value $\Phi_\lambda(u)$ may be infinity for some $u\in W_0^{1,p}(\Omega)$, that is, the functional may not be defined throughout $W_0^{1,p}(\Omega)$. In order to overcome this difficulty, we choose the subspace H of $W_0^{1,p}(\Omega)$.

DEFINITION 1. We say that $u \in H$ is a weak solution of problem (1.1)–(1.2) if and only if

$$\int_\Omega a(x, \nabla u)\nabla\varphi \, dx - \lambda \int_\Omega f(x, u)\varphi \, dx = 0 \tag{1.7}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Then we have the following remark which plays an important role in our arguments.

REMARK 1.

(i) By (A_4) and (i) in Proposition 2, it is easy to see that

$$H = \left\{u \in W_0^{1,p}(\Omega) : \Lambda(u) < \infty\right\} = \left\{u \in W_0^{1,p}(\Omega) : \Phi_\lambda(u) < \infty\right\}.$$

(ii) Since $h_1(x) \geqq 1$, a.e. $x \in \Omega$, we have $\|u\| \leqq \|u\|_H$ for all $u \in H$. Thus, the continuous embeddings

$$H \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^i(\Omega), \quad p \leqq i \leqq p^*$$

hold true.

(iii) Since $\int_\Omega h_1(x)|\nabla u|^p \, dx < +\infty$ for any $u \in C_0^\infty(\Omega)$ and $h_1 \in L^1_{loc}(\Omega)$, we have $C_0^\infty(\Omega) \subset H$.

The main result for the existence of solutions of (1.1) can be formulated as follows.

THEOREM 1. *Under hypotheses* (A_1)–(A_4) *and* (F_1)*, there exists a positive constant* $\underline{\lambda}$ *such that for all* $\lambda \in (0, \underline{\lambda})$*, problem* (1.1)–(1.2) *has no weak solution.*

THEOREM 2. *Under hypotheses* (A_1)–(A_4) *and* (F_1)–(F_3)*, there exists a positive constant* $\overline{\lambda}$ *such that for all* $\lambda \geqq \overline{\lambda}$*, problem* (1.1)–(1.2) *has at least two distinct non-negative, non-trivial weak solutions.*

To prove Theorem 2, we first prove that the functional associated to the problem (1.1)–(1.2) is bounded from below and coercive, and thus the first weak solution is obtained due to a variant of the minimum principle which we will prove in the next section (see Theorem 4). To obtain the second solution to the problem (1.1)–(1.2), we shall use a variant of the mountain pass theorem due to Duc (see Proposition 1).

2. Auxiliary results. Due to the presence of h_1, functional Λ may not be continuously Fréchet differentiable functionals on H. This means that we cannot apply the classical Mountain pass theorem by Ambrosetti–Rabinowitz (see [**1**] for details). To overcome this difficulty, we shall use a weak version of the Mountain pass theorem introduced by Duc [**3**]. Now we introduce the following concept of weakly continuously differentiability due to Duc.

DEFINITION 2. Let $\mathcal{F}$ be a map from a Banach space X to $\mathbb{R}$. We say that $\mathcal{F}$ is weakly continuously differentiable on X if and only if the following two conditions are satisfied:

(i) For any $u \in X$ there exists a linear map $D\mathcal{F}(u)$ from X to $\mathbb{R}$ such that

$$\lim_{t\to 0} \frac{\mathcal{F}(u + tv) - \mathcal{F}(u)}{t} = D\mathcal{F}(u)(v)$$

for every $v \in X$.

(ii) For any $v \in X$, the map $u \mapsto D\mathcal{F}(u)(v)$ is continuous on X.

REMARK 2. If we suppose further that $v \mapsto D\mathcal{F}(u)(v)$ is continuous linear mapping on X, then $\mathcal{F}$ is Gâteaux differentiable.

DEFINITION 3. We call u a generalised critical point (critical point, for short) of $\mathcal{F}$ if $D\mathcal{F}(u) = 0$. c is called a generalised critical value (critical value, for short) of $\mathcal{F}$ if $\mathcal{F}(u) = c$ for some critical point u of $\mathcal{F}$.

Denote by $C^1_w(X)$ the set of weakly continuously differentiable functionals on X. It is clear that $C^1(X) \subset C^1_w(X)$ where we denote by $C^1(X)$ the set of all continuously Fréchet differentiable functionals on X. Now let $\mathcal{F} \in C^1_w(X)$; we put

$$\|D\mathcal{F}(u)\| = \sup\{|D\mathcal{F}(u)(h)| h \in Y, \|h\| = 1\}$$

for any $u \in X$, where $\|D\mathcal{F}(u)\|$ may be $+\infty$.

DEFINITION 4. We say that $\mathcal{F}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (denoted by $(\mathrm{PS})_c$) if any sequence $\{u_n\} \subset X$ for which

$$\mathcal{F}(u_n) \to c \quad \text{and} \quad D\mathcal{F}(u_n) \to 0 \text{ in } X^\star$$

possesses a convergent subsequence. If this is true at every level c then we simply say that $\mathcal{F}$ satisfies the Palais-Smale condition (denoted by (PS)).

DEFINITION 5. We say that $\mathcal{F}$ satisfies the Cerami condition at level $c \in \mathbb{R}$ (denoted by $(\mathrm{C})_c$) if any sequence $\{u_n\} \subset X$ for which

$$\mathcal{F}(u_n) \to c \quad \text{and} \quad (1 + \|x_n\|)D\mathcal{F}(u_n) \to 0 \text{ in } X^\star$$

possesses a convergent subsequence. If this is true at every level c, then we simply say that $\mathcal{F}$ satisfies the Cerami condition (denoted by (C)).

In the proof of our main theorems, we shall use the following results which is proved in [**9**]. We will recall its proof for completeness.

THEOREM 3 (see [**9**]). *Let $\mathcal{F} \in C^1_w(X)$ where X is a Banach space. Assume that*
(i) *$\mathcal{F}$ is bounded from below, $c = \inf \mathcal{F}$,*
(ii) *$\mathcal{F}$ satisfies the (PS) condition.*
Then c is a critical value of $\mathcal{F}$ (i.e. there exists a critical point $u_0 \in X$ such that $\mathcal{F}(u_0) = c$)

Proof of Theorem 3. Let c be an arbitrary real number. Before proving the theorem, we need the following notation:

$$\mathcal{F}^c = \{u \in X | \mathcal{F}(u) \leqq c\}.$$

Let us assume, by negation, that c is not a critical value of $\mathcal{F}$. Then, Theorem 2.2 in [**13**] implies the existence of $\varepsilon > 0$ and $\eta \in C([0, +\infty) \times X, X)$ satisfying $\eta(1, \mathcal{F}^{c+\varepsilon}) \subset \mathcal{F}^{c-\varepsilon}$. This is a contradiction since $\mathcal{F}^{c-\varepsilon} = \emptyset$ due to the fact that $c = \inf \mathcal{F}$. □

REMARK 3. By Corollary 2.1.1 in [**6**], if $\mathcal{F} : X \to \mathbb{R}$ is a locally Lipschitz, bounded from below function and it satisfies the (C) condition, then $\mathcal{F}$ is coercive. This leads us to state the following lemma.

LEMMA 1. *If $\mathcal{F}: X \to \mathbb{R}$ is a locally Lipschitz, bounded from below function and it satisfies the (C) condition then it satisfies the (PS) condition.*

Proof. Let $\{u_n\}_n \subset X$ be a sequence such that $\mathcal{F}(u_n)$ is bounded and $D\mathcal{F}(u_n) \to 0$ in $X^\star$. By Remark 3, $\mathcal{F}$ is coercive, and this helps us to deduce that $\{u_n\}_n$ is bounded in X. Hence also $(1 + \|u_n\|)D\mathcal{F}(u_n) \to 0$ in $X^\star$, and because $\mathcal{F}$ satisfies the (C) condition, it follows that $\{u_n\}_n$ has a strongly convergent subsequence. This completes the proof. □

Similar to Theorem 3, we have the following new result.

THEOREM 4. *Let $\mathcal{F}$ be continuous on X and be of class $C^1_w(X)$ where X is a Banach space. Assume that*

(i) *$\mathcal{F}$ is bounded from below, $c = \inf \mathcal{F}$,*
(ii) *$\mathcal{F}$ satisfies the (C) condition.*

Then c is a critical value of $\mathcal{F}$ (i.e. there exists a critical point $u_0 \in X$ such that $\mathcal{F}(u_0) = c$).

The proof of Theorem 4 follows from Lemma 1, so we omit it. Next we provide a variant Mountain pass theorem due to Duc [**3**].

PROPOSITION 1 (see [**3**]). *Let $\mathcal{F} \in C^1_w(X)$ where X is a Banach space and satisfies (PS) condition. Assume that $\mathcal{F}(0) = 0$ and there exist a positive constant ρ and $z_0 \in X$ such that*

(i) *$\|z_0\|_X > \rho$ and $\mathcal{F}(z_0) \leqq 0$.*
(ii) *$\alpha = \inf\{\mathcal{F}(u) : u \in X, \|u\|_X = \rho\} > 0$.*

Assume that the set

$$G = \{\varphi \in C([0, 1], X) : \varphi(0) = 0, \varphi(1) = z_0\}$$

is not empty. Put

$$\beta := \inf\{\max \mathcal{F}(\varphi([0, 1])) : \varphi \in G\}.$$

Then $\beta \geqq \alpha$ and β is a critical value of $\mathcal{F}$.

For the use of Proposition 1, we refer the reader to [**3, 12, 13**]. We end this section by studying some certain properties of the functional Φ_λ given by (1.5) but we first recall some results which will be used throughout this work.

PROPOSITION 2 (see [**4**]).

(i) *A verifies the growth condition*

$$|A(x, \xi)| \leqq c_0(h_0(x)|\xi| + h_1(x)|\xi|^p)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(ii) *$A(x, \xi)$ is convex with respect to ξ. Moreover, by (A_3) for all $u, v \in H$ we have*

$$\Lambda\left(\frac{u + v}{2}\right) \leqq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k_0 \|u - v\|^p_H. \tag{2.1}$$

Using the method as in [**4**] with some simple computations we obtain the following proposition which concerns the smoothness of the functional Φ_λ.

PROPOSITION 3.
(i) *If* $\{u_m\}$ *is a sequence weakly converging to* u *in* $W_0^{1,p}(\Omega)$*, then*

$$\Lambda(u) \leqq \liminf_{m\to\infty} \Lambda(u_m)$$

and

$$\lim_{m\to\infty} I(u_m) = I(u).$$

(ii) *The functionals* Λ *and* I *are continuous on* H.
(iii) *Functional* Φ_λ *is weakly continuously differentiable on* H *and we have*

$$D\Phi_\lambda(u)(\varphi) = \int_\Omega a(x, \nabla u)\nabla\varphi \,\mathrm{d}x - \lambda \int_\Omega f(x, u)\varphi \,\mathrm{d}x$$

for all $u, \varphi \in H$.

3. Proofs of the theorems.

Proof of Theorem 1. Let us denote by S the best constant in the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, i.e.

$$S = \inf_{W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}}{\left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}}. \tag{3.1}$$

Then, if u is a weak solution of problem (1.1)–(1.2), multiplying (1.1) by u and integrating by parts combined with conditions (A_4) and (F_1) gives

$$\begin{aligned} k_1 \int_\Omega |\nabla u|^p \,\mathrm{d}x &\leqq k_1 \int_\Omega h_1(x)|\nabla u|^p \,\mathrm{d}x \\ &\leqq \int_\Omega a(x, \nabla u)\nabla u \,\mathrm{d}x = \lambda \int_\Omega f(x, u)u \,\mathrm{d}x \leqq C\lambda \int_\Omega |u|^p \,\mathrm{d}x\,. \end{aligned} \tag{3.2}$$

Hence, choosing $\underline{\lambda} = k_1 S/C$, where S is given by (3.1), we conclude the proof. □

We will prove Theorem 2 by using critical point theory. Set $f(x, t) = 0$ for all $t < 0$ and consider the energy functional $\Phi_\lambda : H \to \mathbb{R}$ which is given by (1.5).

LEMMA 2. *If* u *is a critical point of* Φ_λ *then* u *is non-negative in* Ω.

Proof. Observe that if u is a critical point of Φ_λ, denoting by u^- the negative part of u, i.e. $u^-(x) = \min\{u(x), 0\}$ we have

$$\begin{aligned} 0 = D\Phi_\lambda(u)(u^-) &= \int_\Omega a(x, \nabla u)\nabla u^- \,\mathrm{d}x - \lambda \int_\Omega f(x, u)u^- \,\mathrm{d}x \\ &\geqq k_1 \int_\Omega h_1(x)|\nabla u^-|^p \,\mathrm{d}x = k_1\|u^-\|_H^p, \end{aligned} \tag{3.3}$$

which yields that $u \geqq 0$ for a.e. x in Ω. Thus, non-trivial critical points of the functional Φ_λ are non-negative, non-trivial solutions of problem (1.1)–(1.2). □

The following lemma shows that the functional Φ_λ satisfies all of the assumptions of Theorem 3. Then problem (1.1)–(1.2) admits a weak solution $u_1 \in H$ as a global minimiser and $u_1 \geqq 0$.

LEMMA 3. *The functional Φ_λ is bounded from below and satisfies the (PS) condition on H.*

Proof. By conditions (F_1) and (F_3), there exists a constant $C_\lambda = C(\lambda) > 0$ such that

$$\lambda F(x,t) \leqq \frac{k_1 S}{2p}|t|^p + C_\lambda \tag{3.4}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Hence,

$$\begin{aligned} \Phi_\lambda(u) &= \int_\Omega A(x, \nabla u)\,\mathrm{d}x - \lambda \int_\Omega F(x,u)\,\mathrm{d}x \\ &\geqq \frac{k_1}{p}\int_\Omega h_1(x)|\nabla u|^p\,\mathrm{d}x - \int_\Omega \left(\frac{k_1 S}{2p}|u|^p + C_\lambda\right)\mathrm{d}x \\ &\geqq \frac{k_1}{2p}\|u\|_H^p - C_\lambda|\Omega|, \end{aligned} \tag{3.5}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbb{R}^N$. Thus, the functional Φ_λ is coercive and hence bounded from below on H.

Let $\{u_m\}$ be a Palais-Smale sequence in H, i.e.

$$|\Phi_\lambda(u_m)| \leqq c \text{ for all } m, \quad \Phi'_\lambda(u_m) \to 0 \text{ in } H^\star. \tag{3.6}$$

Since Φ_λ is coercive on H, $\{u_m\}$ is bounded in H. By Remark 1 (ii), $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. It follows that there exists $u \in W_0^{1,p}(\Omega)$ such that, passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in $W_0^{1,p}(\Omega)$. We shall prove that $\{u_m\}$ converges strongly to u in H.

Indeed, we observe by Remark 1(i), Proposition 3(i) and (3.6) that $u \in H$. Hence, $\{\|u_m - u\|_H\}$ is bounded. This and (3.6) imply that $D\Phi_\lambda(u_m)(u_m - u)$ converges to 0 as $m \to \infty$.

Using condition (F_1) combined with Hölder's inequality we deduce that

$$\begin{aligned} 0 \leqq \int_\Omega |f(x,u_m)||u_m - u|\,\mathrm{d}x &\leqq C\int_\Omega |u_m|^{p-1}|u_m - u|\,\mathrm{d}x \\ &\leqq C\,\|u_m\|_{L^p(\Omega)}^{p-1}\,\|u_m - u\|_{L^p(\Omega)}. \end{aligned} \tag{3.7}$$

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, $\{u_m\}$ converges strongly to u in $L^p(\Omega)$. Therefore, relation (3.7) implies that

$$\lim_{m\to\infty} DI(u_m)(u_m - u) = 0. \tag{3.8}$$

Combining relations (3.6) and (3.8) with the fact that

$$D\Lambda(u_m)(u_m - u) = D\Phi_\lambda(u_m)(u_m - u) + DI(u_m)(u_m - u),$$

we conclude that

$$\lim_{m\to\infty} D\Lambda(u_m)(u_m - u) = 0. \tag{3.9}$$

On the other hand, the convex property of functional Λ (see Proposition 2(ii)) implies that

$$\Lambda(u) - \lim_{m\to\infty} \Lambda(u_m) = \lim_{m\to\infty} (\Lambda(u) - \Lambda(u_m)) \geqq \lim_{m\to\infty} D\Lambda(u_m)(u - u_m) = 0. \tag{3.10}$$

Combining this with Proposition 3(i), we have

$$\lim_{m\to\infty} \Lambda(u_m) = \Lambda(u). \tag{3.11}$$

We now assume by contradiction that $\{u_m\}$ does not converge strongly to u in H, and then there exist a constant $\epsilon > 0$ and a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ such that $\|u_{m_k} - u\|_H \geqq \epsilon$. Using Proposition 2(ii) we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{m_k}) - \Lambda\left(\frac{u_{m_k} + u}{2}\right) \geqq k_0\|u_{m_k} - u\|_H^p \geqq k_0\epsilon^p. \tag{3.12}$$

Letting $k \to \infty$, relation (3.12) gives

$$\limsup_{k\to\infty} \Lambda\left(\frac{u_{m_k} + u}{2}\right) \leqq \Lambda(u) - k_0\epsilon^p. \tag{3.13}$$

We remark that sequence $\{\frac{u_{m_k}+u}{2}\}$ also converges weakly to u in $W_0^{1,p}(\Omega)$. So, using Proposition 3(i) again we get

$$\Lambda(u) \leqq \liminf_{k\to\infty} \Lambda\left(\frac{u_{m_k} + u}{2}\right), \tag{3.14}$$

which contradicts (3.13). Therefore, $\{u_m\}$ converges strongly to u in H. □

LEMMA 4. *There exists a positive constant $\overline{\lambda}$ such that for all $\lambda \geqq \overline{\lambda}$,* $\inf_H \Phi_\lambda < 0$, *and hence $u_1 \not\equiv 0$, i.e. the solution u_1 is not trivial.*

Proof. Let $\Omega_0 \subset \Omega$ be a compact subset large enough and a function $\varphi_0 \in C_0^\infty(\Omega)$ such that $\varphi_0(x) = t_1$ in Ω_0 and $0 \leqq \varphi_0(x) \leqq t_1$ in $\Omega\backslash\Omega_0$, where t_1 is chosen as in assumption (F_2): then we have

$$\int_\Omega F(x, \varphi_0)\,dx \geqq \int_{\Omega_0} F(x, \varphi_0)\,dx - Ct_1^p|\Omega\backslash\Omega_0| > 0. \tag{3.15}$$

Thus, $\Phi_\lambda(\varphi_0) < 0$ for $\lambda \geqq \overline{\lambda}$ with $\overline{\lambda}$ large enough. This implies that $\inf_H \Phi_\lambda < 0$ and then $\Phi_\lambda(u_1) < 0$ for $\lambda \geqq \overline{\lambda}$, i.e. $u_1 \not\equiv 0$. □

In the next part of this paper, we shall show the existence of the second solution $u_2 \neq u_1$ by using the Mountain pass theorem introduced in [**3**]. To this purpose, we first fix $\lambda \geqq \overline{\lambda}$ and set

$$\widehat{f}(x,t)=\begin{cases}0, & \text{for } t<0,\\ f(x,t) & \text{for } 0\leqq t\leqq u_1(x),\\ f(x,u_1(x)) & \text{for } t>u_1(x),\end{cases} \tag{3.16}$$

and $\widehat{F}(x,t)=\int_0^t \widehat{f}(x,s)\,\mathrm{d}s$. Define the functional $\widehat{\Phi}_\lambda : H \to \mathbb{R}$ by

$$\widehat{\Phi}_\lambda(u)=\int_\Omega A(x,\nabla u)\,\mathrm{d}x-\lambda\int_\Omega \widehat{F}(x,u)\,\mathrm{d}x\,. \tag{3.17}$$

With the same arguments as those used for the functional Φ_λ, we can show that $\widehat{\Phi}_\lambda$ is weakly continuously differentiable on H and

$$D\widehat{\Phi}_\lambda(u)(\varphi)=\int_\Omega a(x,\nabla u)\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega \widehat{f}(x,u)\varphi\,\mathrm{d}x$$

for all $u,\varphi\in H$.

LEMMA 5. *If $u\in H$ is a critical point of $\widehat{\Phi}_\lambda$ then $u\leqq u_1$. So, u is a solution of problem* (1.1)–(1.2) *in the order interval* $[0,u_1]$.

Proof. By the definitions of Φ_λ and $\widehat{\Phi}_\lambda$, if u is a critical point of $\widehat{\Phi}_\lambda$ then $u\geqq 0$ as before and by condition (A_2) we have

$$\begin{aligned}0&=(D\widehat{\Phi}'_\lambda(u)-D\Phi_\lambda(u_1))((u-u_1)^+)\\ &=\int_\Omega (a(x,\nabla u)-a(x,\nabla u_1))\cdot\nabla(u-u_1)^+\,\mathrm{d}x\\ &\quad-\lambda\int_\Omega(\widehat{f}(x,u)-f(x,u_1))(u-u_1)^+\,\mathrm{d}x\\ &=\int_{\{u>u_1\}}(a(x,\nabla u)-a(x,\nabla u_1))\cdot(\nabla u-\nabla u_1)\,\mathrm{d}x\geqq 0.\end{aligned} \tag{3.18}$$

According to (3.18) and condition (A_2), the equality holds if and only if $\nabla u=\nabla u_1$. It follows that $\nabla u(x)=\nabla u_1(x)$ for all $x\in\Omega_1:=\{y\in\Omega: u(y)>u_1(y)\}$. Hence,

$$\int_{\Omega_1}|\nabla(u-u_1)|^p\,\mathrm{d}x=0 \text{ and thus } \int_\Omega|\nabla(u-u_1)^+|^p\,\mathrm{d}x=0.$$

Combining this with Remark 1(ii), we conclude that $\|(u-u_1)^+\|_{L^p(\Omega)}=0$ and then $(u-u_1)^+=0$ in Ω, that is, $u\leqq u_1$ in Ω. □

LEMMA 6. *There exist a constant $\rho\in(0,\|u_1\|_H)$ and a constant $\alpha>0$ such that $\widehat{\Phi}_\lambda(u)\geqq\alpha$ for all $u\in H$ with $\|u\|_H=\rho$.*

Proof. We set $\Omega_u=\{x\in\Omega: u(x)>\min\{u_1(x),t_0\}\}$, where t_0 is given as in (F_2). Then, by (3.16) and condition (F_1), we have $\widehat{F}(x,u(x))\leqq 0$ on $\Omega\setminus\Omega_u$. Hence,

$$\widehat{\Phi}_\lambda(u)\geqq k_1\,\|u\|_H^p-\lambda\int_{\Omega_u}\widehat{F}(x,u)\,\mathrm{d}x\,. \tag{3.19}$$

Using (F_1), Hölder's inequality and Remark 1(ii), we get

$$\int_{\Omega_u} \widehat{F}(x,u)\,\mathrm{d}x \leqq C \int_{\Omega_u} |u|^p\,\mathrm{d}x \leqq C|\Omega_u|^{1-\frac{p}{q}} \|u\|_H^p, \tag{3.20}$$

where $q = \frac{Np}{N-p}$. Therefore,

$$\widehat{\Phi}_\lambda(u) \geqq \|u\|_H^p \left(k_1 - \lambda C|\Omega_u|^{1-\frac{p}{q}}\right). \tag{3.21}$$

In order to prove Lemma 6, it is enough to show that $|\Omega_u| \to 0$ as $\|u\|_H \to 0$. Indeed, let $\epsilon > 0$ be arbitrary; we choose $\Omega_\epsilon \subset \Omega$ a compact subset, large enough such that $|\Omega \backslash \Omega_\epsilon| < \epsilon$, and denote by $\Omega_{u,\epsilon} := \Omega_u \cap \Omega_\epsilon$. Then it is clear that

$$\|u\|_H^p \geqq \|u\|^p \geqq \int_\Omega |u|^p\,\mathrm{d}x \geqq \int_{\Omega_{u,\epsilon}} u^p\,\mathrm{d}x \geqq l^p|\Omega_{u,\epsilon}|, \tag{3.22}$$

where $l = \min\{\min u_1(\Omega_\epsilon), t_0\}$. Letting $\|u\|_H \to 0$ we deduce that $|\Omega_{u,\epsilon}| \to 0$. Since $\Omega_u \subset \Omega_{u,\epsilon} \cup \Omega\backslash\Omega_\epsilon$ we have $|\Omega_u| \leqq |\Omega_{u,\epsilon}| + \epsilon$ with $\epsilon > 0$ as arbitrary. Thus, $|\Omega_u| \to 0$ as $\|u\|_H \to 0$. □

LEMMA 7. *Functional $\widehat{\Phi}_\lambda$ satisfies the (PS) condition on H.*

Proof. We observe by (3.21) that $\widehat{\Phi}_\lambda$ is coercive. Therefore, every Palais-Smale sequence of $\widehat{\Phi}_\lambda$ is bounded in H. We follow the method as those used in the proof of Lemma 3. It can be shown that the functional $\widehat{\Phi}_\lambda$ satisfies the (PS) condition on H. □

Proof of Theorem 2. By Lemmas 2–4, using Theorem 3, problem (1.1)–(1.2) admits a non-negative, non-trivial weak solution u_1. Setting

$$c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \widehat{\Phi}_\lambda(u), \tag{3.23}$$

where $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = u_1\}$, Lemmas 6–7 show that all of the assumptions of Proposition 1 are fulfilled, $\widehat{\Phi}_\lambda(u_1) = \Phi_\lambda(u_1) < 0$ and $\|u_1\|_H > \rho$. Then, $c > 0$ is a critical value of $\widehat{\Phi}_\lambda$, i.e. there exists $u_2 \in H$ such that $D\widehat{\Phi}_\lambda(u_2)(\varphi) = 0$ for all $\varphi \in H$ and $\widehat{\Phi}_\lambda(u_2) = c > 0$. By Lemma 5, $0 \leqq u_2 \leqq u_1$ in Ω. Therefore, using (3.16) some simple computations give us

$$\widehat{\Phi}_\lambda(u_2) = \Phi_\lambda(u_2), \quad D\widehat{\Phi}_\lambda(u_2)(\varphi) = D\Phi_\lambda(u_2)(\varphi) \text{ for all } \varphi \in H.$$

By Remark 1(iii), we conclude that u_2 is a weak solution of problem (1.1)–(1.2). Furthermore, $\Phi_\lambda(u_2) = c > 0 > \Phi_\lambda(u_1)$. Thus, u_2 is not trivial and $u_2 \neq u_1$. The proof of Theorem 2 is now complete. □

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