SEVERAL INTERESTING INTEGRAL INEQUALITIES

WENJUN LIU, QUỐC-ANH NGÔ AND VU NHAT HUY

(*Communicated by N. Ujević*)

Abstract. In this paper, several interesting integral inequalities are presented and some open problems are proposed later on.

1. Introduction

In the paper [22], Qi proposed the following open problem, which has attracted much attention from some mathematicians (cf. [1, 2, 3, 4, 5, 8, 9, 12, 13, 14, 18, 21, 23, 24, 25]).

Open problem 1. Under what conditions does the inequality

$$\int_a^b [f(x)]^t \, dx \geqslant \left(\int_a^b f(x) \, dx \right)^{t-1} \tag{1}$$

hold for $t > 1$?

Shortly after the paper [22] was published, the authors in [25] proved that

THEOREM A. *The inequality* (1) *holds true for all $t > 1$ provided*

$$\int_a^b f(x) \, dx \geqslant (b-a)^{t-1}. \tag{2}$$

After years, Qi et al. [23] got the following extension.

THEOREM B. *Let f(x) be continuous and not identically zero on $[a,b]$, differentiable in (a,b), with $f(a) = 0$, and let α, β be positive real numbers such that $\alpha > \beta > 1$. If*

$$\left(f^{\frac{\alpha-\beta}{\beta-1}}(x) \right)' \gtreqless \frac{(\alpha-\beta)\beta^{\frac{1}{\beta-1}}}{\alpha-1}$$

for all $x \in (a,b)$ then

$$\int_a^b f^\alpha(x) \, dx \gtreqless \left(\int_a^b f(x) \, dx \right)^\beta.$$

Then, the first author of this paper [13, 14] obtained the following result.

THEOREM C. *Let $f(x)$ be continuous and not identically zero on $[a,b]$, differentiable in (a,b) with $f(a) = 0$, and let α, β be positive real numbers such that $\alpha > \beta > 1$. If*

$$0 \leqslant \left(f^{\frac{\alpha-\beta}{\alpha(\beta-1)}}(x) \right)' \begin{smallmatrix} \geqslant \\ \leqslant \end{smallmatrix} \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \qquad (3)$$

and

$$0 \leqslant \left(f^{\frac{\alpha-\beta}{\beta-1}}(x) \right)' \begin{smallmatrix} \geqslant \\ \leqslant \end{smallmatrix} \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \qquad (4)$$

for $x \in (a,b)$, then

$$\left(\int_a^b [f(x)]^\alpha \, \mathrm{d}x \right)^{\frac{1}{\beta}} \begin{smallmatrix} \geqslant \\ \leqslant \end{smallmatrix} \int_a^b f(x) \, \mathrm{d}x \begin{smallmatrix} \geqslant \\ \leqslant \end{smallmatrix} \left(\int_a^b [f(x)]^{\frac{1}{\alpha}} \, \mathrm{d}x \right)^\beta. \qquad (5)$$

Recently, in the paper [20] the second author of this paper et al. studied another very interesting integral inequality and proved the following result

THEOREM D. *Let $f(x)$ be a continous function on $[0,1]$ satisfying*

$$\int_x^1 f(t) \, dt \geqslant \int_x^1 t \, dt \quad, \quad \forall x \in [0,1]. \qquad (6)$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x) \, \mathrm{d}x \geqslant \int_0^1 x^\alpha f(x) \, \mathrm{d}x \geqslant \frac{1}{\alpha+2}$$

and

$$\int_0^1 f^{\alpha+1}(x) \, \mathrm{d}x \geqslant \int_0^1 x f^\alpha(x) \, \mathrm{d}x$$

hold for every positive real number $\alpha > 0$.

Then, they proposed the following open problem

Open problem 2. Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x) \, \mathrm{d}x \geqslant \int_0^1 x^\alpha f^\beta(x) \, \mathrm{d}x \qquad (7)$$

hold for α and β?

Theorem D is a generalization of Problem 2 (when $\alpha = 1$) of the 2nd International Mathematical Competition for University Students, Plovdiv, Bulgaria, 2–7 August, 1995 (see [11, 17]). Shortly after the paper [20] was published, the first author of this paper [15] gave an affirmative answer to the above Open problem 2 and obtained the following result (see also Bougoffa [7], Boukerrioua and Guezane-Lakoud [6]).

THEOREM E. *Let $f(x) \geqslant 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f^\beta(t)dt \geqslant \int_x^1 t^\beta \, dt, \quad \forall \, x \in [0,1]. \tag{8}$$

Then the inequality (7) holds for every positive real number $\alpha > 0$ and $\beta > 0$.

More recently, the first author of this paper [16] obtained an extension of the above Open problem 2.

THEOREM F. *Let $f(x) \geqslant 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b f^{\min\{1,\beta\}}(t)dt \geqslant \int_x^b (t-a)^{\min\{1,\beta\}} \, dt, \quad \forall \, x \in [a,b]. \tag{9}$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geqslant \int_a^b (x-a)^\alpha f^\beta(x)dx \tag{10}$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

It is clear that inequality (10) is a special case of the following inequality

$$\int_a^b f^{\alpha+\beta}(x) \, dx \geqslant \int_a^b f^\alpha(x) g^\beta(x) \, dx \tag{11}$$

where f, g are positive functions on $[a,b]$. In [10], the author obtained the following result

THEOREM G. *Let $f \in L^1[a,b]$, $g \in C([a,b])$ such that $f(x), g(x) \geqslant 0$ and g is nondecreasing on $[a,b]$. If*

$$\int_x^b f(t)dt \geqslant \int_x^b g(t)dt, \quad \forall \, x \in [a,b], \tag{12}$$

then inequality (11) holds for every positive real number $\alpha > 0$ and $\beta > 0$ satisfying $\alpha + \beta \geqslant 1$.

The purpose of this paper is to establish other similar integral inequalities. At last, some open problems are proposed. Our main results are from Theorem 1 to Theorem 10 which will be stated and proved in Section 2.

2. Main Results and Proofs

In the first part of this section, we shall study the similar combination between inequalities (1) and (11). For instance, we have

THEOREM 1. *Let $f(x), g(x) \geqslant 0$ be continuous functions on $[a,b]$, g is nondecreasing. Assume that λ, α, β are positive constants with $\alpha + \beta \geqslant \lambda > 1$. If*

$$\int_x^b f(t)dt \geqslant \int_x^b g(t)dt, \quad \forall x \in [a,b] \tag{13}$$

and

$$\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(t)dt \geqslant (b-a)^{\lambda-1},$$ (14)

then the following inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geqslant \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x)g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\lambda-1}$$ (15)

holds true.

Proof. By using Theorem A, we deduce that

$$\int_a^b f^{\alpha+\beta}(x)\,dx = \int_a^b \left(f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\lambda} dx \geqslant \left(\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\lambda-1}.$$

On the other hand, using Theorem G, we deduce that

$$\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x) \geqslant \int_a^b f^{\frac{\alpha}{\lambda}}(x)g^{\frac{\beta}{\lambda}}(x)\,dx.$$

Summing up the above two inequalities, we get the result.

THEOREM 2. *Let* $f(x), g(x) \geqslant 0$ *be continuous functions on* $[a,b]$, g *is nondecreasing. Assume that* $\lambda, \gamma, \alpha, \beta$ *are positive constants with* $\alpha + \beta \geqslant \lambda > \gamma > 1$. *If*

$$\int_x^b f(t)dt \geqslant \int_x^b g(t)dt, \quad \forall x \in [a,b]$$ (16)

and

$$\left(f^{\frac{\alpha+\beta}{\lambda}\cdot\frac{\lambda-\gamma}{\gamma-1}}(x)\right)' \geqslant \frac{(\lambda-\gamma)\gamma^{\frac{1}{\gamma-1}}}{\lambda-1}, \quad \forall x \in (a,b),$$ (17)

then the following inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geqslant \left(\int_a^b f^{\frac{\alpha}{\lambda}}(x)g^{\frac{\beta}{\lambda}}(x)\,dx\right)^{\gamma}$$ (18)

holds true.

Proof. By using Theorem B, we deduce that

$$\int_a^b f^{\alpha+\beta}(x)\,dx = \int_a^b \left(f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\lambda} dx \geqslant \left(\int_a^b f^{\frac{\alpha+\beta}{\lambda}}(x)\right)^{\gamma}.$$

Then we use the same proof as above to get the result.

In the sequel, we shall study some inequalities which is similar to (10) or (11). We firstly obtain the following theorem.

THEOREM 3. *Let $f(x) > 0$ be a continuous and decreasing function on $[a,b]$. Then the inequality*

$$\frac{\int_a^b f^\beta(x)\,dx}{\int_a^b f^\gamma(x)\,dx} \geqslant \frac{\int_a^b (x-a)^\alpha f^\beta(x)\,dx}{\int_a^b (x-a)^\alpha f^\gamma(x)\,dx} \tag{19}$$

holds for every positive real number $\alpha > 0$ and $\beta \geqslant \gamma > 0$. If $f(x)$ is increasing, then the inequality in (19) reverses.

Proof. We only prove the case "$\geqslant$". For this purpose, we only need to prove

$$\int_a^b f^\beta(x)\,dx \int_a^b (x-a)^\alpha f^\gamma(x)\,dx \geqslant \int_a^b f^\gamma(x)\,dx \int_a^b (x-a)^\alpha f^\beta(x)\,dx,$$

which is equivalent to

$$\int_a^b f^\beta(x)\,dx \int_a^b (y-a)^\alpha f^\gamma(y)\,dy \geqslant \int_a^b f^\gamma(x)\,dx \int_a^b (y-a)^\alpha f^\beta(y)\,dy.$$

That is to prove

$$\int_a^b \int_a^b f^\gamma(x)f^\gamma(y)(y-a)^\alpha \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right)\,dx\,dy \geqslant 0.$$

We denote

$$G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)(y-a)^\alpha \left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right)\,dx\,dy.$$

Then

$$G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)(x-a)^\alpha \left(f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)\right)\,dx\,dy.$$

Therefore

$$2G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)\left((y-a)^\alpha - (x-a)^\alpha\right)\left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right)\,dx\,dy.$$

Since $f(x) \geqslant 0$ is a continuous and decreasing function on $[a,b]$, we have

$$\left((y-a)^\alpha - (x-a)^\alpha\right)\left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) > 0$$

for all $x,y \in [a,b]$. Therefore $2G \geqslant 0$. We complete the proof.

REMARK 1. The following special case of the Theorem 3 (for $f(x) \geqslant 0$ be a continuous and decreasing function on $[0,1]$)

$$\frac{\int_0^1 f^2(x)\,dx}{\int_0^1 f(x)\,dx} \geqslant \frac{\int_0^1 xf^2(x)\,dx}{\int_0^1 xf(x)\,dx} \tag{20}$$

gives an answer to the test question 5 in the Mathematics Contest of Tsinghua University, China in 1985.

Similar to the method used in the proof of the previous theorem, we obtain the following three theorems.

THEOREM 4. *Let $f(x), g(x) \geqslant 0$ be continuous functions on $[a,b]$ such that $f(x)$ is decreasing and $g(x)$ is increasing. Then the inequality*

$$\frac{\int_a^b f^\beta(x)\,dx}{\int_a^b f^\gamma(x)\,dx} \geqslant \frac{\int_a^b g^\alpha(x)f^\beta(x)\,dx}{\int_a^b g^\alpha(x)f^\gamma(x)\,dx} \tag{21}$$

holds for every positive real number $\alpha > 0$ and $\beta \geqslant \gamma > 0$. If $f(x)$ is increasing, then (21) *reverses.*

Proof. We only prove the case "$\geqslant$". For this purpose, we only need to prove

$$\int_a^b f^\beta(x)\,dx \int_a^b g^\alpha(x)f^\gamma(x)\,dx \geqslant \int_a^b f^\gamma(x)\,dx \int_a^b g^\alpha(x)f^\beta(x)\,dx,$$

which is equivalent to

$$\int_a^b f^\beta(x)\,dx \int_a^b g^\alpha(y)f^\gamma(y)\,dy \geqslant \int_a^b f^\gamma(x)\,dx \int_a^b g^\alpha(y)f^\beta(y)\,dy.$$

That is to prove

$$\int_a^b \int_a^b f^\gamma(x)f^\gamma(y)g^\alpha(y)[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)]\,dx\,dy \geqslant 0.$$

We denote

$$G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)g^\alpha(y)[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)]\,dx\,dy.$$

Then

$$G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)g^\alpha(x)[f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)]\,dx\,dy.$$

Therefore

$$2G = \int_a^b \int_a^b f^\gamma(x)f^\gamma(y)\left(g^\alpha(y) - g^\alpha(x)\right)\left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right)\,dx\,dy.$$

Since $f(x) \geqslant 0$ is a continuous and decreasing function on $[a,b]$, we have

$$\left(g^\alpha(y) - g^\alpha(x)\right)\left(f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)\right) > 0$$

for all $x, y \in [a,b]$. Therefore $2G \geqslant 0$. We complete the proof.

THEOREM 5. *Let $f(x) \geqslant 0$ be a continuous function on $[a,b]$ and satisfies*

$$[(y-a)^\alpha f^\alpha(x) - (x-a)^\alpha f^\alpha(y)][f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)] \geqslant 0, \quad \forall\, x,y \in [a,b]. \tag{22}$$

Then the inequality

$$\frac{\displaystyle\int_a^b f^{\alpha+\beta}(x)\,dx}{\displaystyle\int_a^b f^{\alpha+\gamma}(x)\,dx} \geqslant \frac{\displaystyle\int_a^b (x-a)^\alpha f^\beta(x)\,dx}{\displaystyle\int_a^b (x-a)^\alpha f^\gamma(x)\,dx} \tag{23}$$

holds for every positive real number $\alpha > 0$ and $\beta \geqslant \gamma > 0$. If (22) reverses, then (23) reverses.

Proof. We only prove the case "$\geqslant$". For this purpose, we only need to prove

$$H := \int_a^b \int_a^b f^{\alpha+\gamma}(x) f^\gamma(y)(y-a)^\alpha [f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)]\,dx\,dy \geqslant 0.$$

Since

$$H = \int_a^b \int_a^b f^\gamma(x) f^{\alpha+\gamma}(y)(x-a)^\alpha [f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)]\,dx\,dy,$$

we have

$$2H = \int_a^b \int_a^b f^\gamma(x) f^\gamma(y)[(y-a)^\alpha f^\alpha(x) - (x-a)^\alpha f^\alpha(y)][f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)]\,dx\,dy.$$

The condition (22) implies that $H \geqslant 0$.

THEOREM 6. *Let $f(x), g(x) \geqslant 0$ be continuous functions on $[a,b]$ and satisfy*

$$[g^\alpha(y) f^\alpha(x) - g^\alpha(x) f^\alpha(y)][f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)] \geqslant 0, \quad \forall\, x,y \in [a,b]. \tag{24}$$

Then the inequality

$$\frac{\displaystyle\int_a^b f^{\alpha+\beta}(x)\,dx}{\displaystyle\int_a^b f^{\alpha+\gamma}(x)\,dx} \geqslant \frac{\displaystyle\int_a^b g^\alpha(x) f^\beta(x)\,dx}{\displaystyle\int_a^b g^\alpha(x) f^\gamma(x)\,dx} \tag{25}$$

holds for every positive real number $\alpha > 0$ and $\beta \geqslant \gamma > 0$. If (24) reverses, then (25) reverses.

Proof. We only prove the case "$\geqslant$". For this purpose, we only need to prove

$$H := \int_a^b \int_a^b f^{\alpha+\gamma}(x) f^\gamma(y) g^\alpha(y)[f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)]\,dx\,dy \geqslant 0.$$

Since

$$H = \int_a^b \int_a^b f^\gamma(x) f^{\alpha+\gamma}(y) g^\alpha(x)[f^{\beta-\gamma}(y) - f^{\beta-\gamma}(x)]\,dx\,dy,$$

we have

$$2H = \int_a^b \int_a^b f^\gamma(x) f^\gamma(y) [g^\alpha(y) f^\alpha(x) - g^\alpha(x) f^\alpha(y)] [f^{\beta-\gamma}(x) - f^{\beta-\gamma}(y)] \, dx \, dy.$$

The condition (24) implies that $H \geqslant 0$.

Our next result is the following theorem which will be used frequently in the rest of this section.

THEOREM 7. *Let* $f(x), g(x), h(x) \geqslant 0$ *be continuous functions on* $[a,b]$ *such that*

$$\left(g(x) - g(y) \right) \left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)} \right) \geqslant 0. \tag{26}$$

Then the inequality

$$\frac{\int_a^b f(x) \, dx}{\int_a^b h(x) \, dx} \geqslant \frac{\int_a^b f(x) g(x) \, dx}{\int_a^b h(x) g(x) \, dx} \tag{27}$$

holds. If (26) *reverses, then* (27) *reverses.*

Proof. We need to prove that

$$\int_a^b f(x) \, dx \int_a^b h(x) g(x) \, dx \geqslant \int_a^b h(x) \, dx \int_a^b f(x) g(x) \, dx,$$

which is equivalent to

$$\int_a^b f(x) \, dx \int_a^b h(y) g(y) \, dy \geqslant \int_a^b h(x) \, dx \int_a^b f(y) g(y) \, dy.$$

That is to prove

$$\int_a^b \int_a^b g(y) \Big(f(x) h(y) - f(y) h(x) \Big) \, dx \, dy \geqslant 0.$$

We denote

$$G = \int_a^b \int_a^b g(y) \Big(f(x) h(y) - f(y) h(x) \Big) \, dx \, dy.$$

Then

$$G = \int_a^b \int_a^b g(x) \Big(f(y) h(x) - f(x) h(y) \Big) \, dx \, dy.$$

Therefore

$$2G = \int_a^b \int_a^b \Big(g(x) - g(y) \Big) \Big(f(y) h(x) - f(x) h(y) \Big) \, dx \, dy$$

$$= \int_a^b \int_a^b h(x) h(y) \Big(g(x) - g(y) \Big) \left(\frac{f(y)}{h(y)} - \frac{f(x)}{h(x)} \right) \, dx \, dy.$$

From the hypotheses, it is easy to see that $2G \geqslant 0$. We complete the proof.

REMARK 2. It is obviously to see that (26) holds true for f, g, h such that either

- g is increasing and $\frac{f}{h}$ is decreasing

or

- g is decreasing and $\frac{f}{h}$ is increasing.

By applying Theorem 7, we have

THEOREM 8. *Let $f(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x)$ is increasing. Then the inequality*

$$\frac{\displaystyle\int_a^b f(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)\,\mathrm{d}x} \geqslant \frac{\displaystyle\int_a^b f^p(x)\,\mathrm{d}x}{\displaystyle\int_a^b h^p(x)\,\mathrm{d}x} \tag{28}$$

holds for all $p \geqslant 1$. If $\frac{f(x)}{h(x)}$ is increasing, then (28) reverses.

Proof. Since $p \geqslant 1$ and f is increasing, then $g(x) := f^{p-1}(x)$ is also increasing. By applying the foregoing Theorem, we deduce that

$$\frac{\displaystyle\int_a^b f(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)\,\mathrm{d}x} \geqslant \frac{\displaystyle\int_a^b f(x)f^{p-1}(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)f^{p-1}(x)\,\mathrm{d}x}. \tag{29}$$

Moreover, since $f(x) \geqslant h(x)$ for all x, then

$$\frac{\displaystyle\int_a^b f(x)f^{p-1}(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)f^{p-1}(x)\,\mathrm{d}x} \geqslant \frac{\displaystyle\int_a^b f^p(x)\,\mathrm{d}x}{\displaystyle\int_a^b h^p(x)\,\mathrm{d}x}. \tag{30}$$

Summing up (29) and (30), we get the result.

Involving convex functions, we have

THEOREM 9. *Let $f(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x)$ is increasing. Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$. Then the inequality*

$$\frac{\displaystyle\int_a^b f(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)\,\mathrm{d}x} \geqslant \frac{\displaystyle\int_a^b \varphi(f(x))\,\mathrm{d}x}{\displaystyle\int_a^b \varphi(h(x))\,\mathrm{d}x} \tag{31}$$

holds.

Proof. Since φ is convex with $\varphi(0) = 0$, then $\frac{\varphi(x)}{x}$ is increasing. This and the fact that $f(x) \leqslant h(x)$ yield

$$\frac{\varphi(f(x))}{f(x)} \leqslant \frac{\varphi(h(x))}{h(x)}.$$

Moreover, f and $\frac{\varphi(x)}{x}$ are increasing, then the following function

$$g(x) = \frac{\varphi(f(x))}{f(x)}$$

is also increasing. This helps us to deduce that

$$\frac{\int_a^b \varphi(f(x))\,dx}{\int_a^b \varphi(h(x))\,dx} = \frac{\int_a^b f(x)\frac{\varphi(f(x))}{f(x)}\,dx}{\int_a^b h(x)\frac{\varphi(h(x))}{h(x)}\,dx} \leqslant \frac{\int_a^b f(x)\frac{\varphi(f(x))}{f(x)}\,dx}{\int_a^b h(x)\frac{\varphi(f(x))}{f(x)}\,dx} \leqslant \frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx}.$$

The proof is complete.

The following theorem slightly improves Theorem 9.

THEOREM 10. *Let $f(x), g(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x), g(x)$ are increasing. Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx} \geqslant \frac{\int_a^b \varphi(f(x))g(x)\,dx}{\int_a^b \varphi(h(x))g(x)\,dx} \tag{32}$$

holds.

Proof. Similar to the proof of the foregoing theorem, we have

$$\frac{\int_a^b \phi(f(x))g(x)\,dx}{\int_a^b \phi(h(x))g(x)\,dx} = \frac{\int_a^b \frac{\phi(f(x))}{f(x)}g(x)f(x)\,dx}{\int_a^b \frac{\phi(h(x))}{h(x)}g(x)h(x)\,dx}$$

$$\leqslant \frac{\int_a^b \frac{\phi(f(x))}{f(x)}g(x)f(x)\,dx}{\int_a^b \frac{\phi(f(x))}{f(x)}g(x)h(x)\,dx} \leqslant \frac{\int_a^b f(x)\,dx}{\int_a^b h(x)\,dx}.$$

The proof is complete.

Lastly, we propose the following open problems.

Open Problem 3. Can the condition (22) in the Theorem 5 (or (24) in the Theorem 6) be improved?

Open Problem 4. Under what conditions does the inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geqslant \left(\int_a^b (x-a)^\alpha f^\beta(x)\,dx\right)^\lambda \tag{33}$$

hold for α, β and λ?

Open Problem 5. Under what conditions does the inequality

$$\frac{\displaystyle\int_a^b f^{\alpha+\beta}(x)\,dx}{\displaystyle\int_a^b f^{\alpha+\gamma}(x)\,dx} \geqslant \frac{\left(\displaystyle\int_a^b (x-a)^\alpha f^\beta(x)\,dx\right)^\delta}{\left(\displaystyle\int_a^b (x-a)^\alpha f^\gamma(x)\,dx\right)^\lambda} \tag{34}$$

hold for $\alpha, \beta, \gamma, \delta$ and λ?

Open Problem 6. Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$. Under what conditions does the inequality

$$\frac{\displaystyle\int_a^b f(x)\,dx}{\displaystyle\int_a^b h(x)\,dx} \geqslant \frac{\left(\displaystyle\int_a^b \varphi(f(x))g(x)\,dx\right)^\delta}{\left(\displaystyle\int_a^b \varphi(h(x))g(x)\,dx\right)^\lambda} \tag{35}$$

hold for δ and λ?

REFERENCES

[1] M. AKKOUCHI, *Some integral inequalities*, Divulgaciones Math., **11**, 2 (2003), 121–125.

[2] M. AKKOUCHI, *On an integral inequality of Feng Qi*, Divulgaciones Math., **13**, 1 (2005), 11–19.

[3] K. BRAHIM, *On some q-integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **9**, 4 (2008), Art. 106.

[4] L. BOUGOFFA, *Notes on Qi type integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **4**, 4 (2003), Art. 77.

[5] L. BOUGOFFA, *An integral inequality similar to Qi's inequality*, JIPAM. J. Inequal. Pure Appl. Math., **6**, 1 (2005), Art. 27.

[6] K. BOUKERRIOUA AND A. GUEZANE-LAKOUD, *On an open question regarding an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., **8**, 3 (2007), Art. 77.

[7] L. BOUGOFFA, *Note on an open problem*, JIPAM. J. Inequal. Pure Appl. Math., **8**, 2 (2007), Art. 58.

[8] Y. CHEN AND J. KIMBALL, *Note on an open problem of Feng Qi*, JIPAM. J. Inequal. Pure Appl. Math., **7**, 1 (2005), Art. 4.

[9] B. GAVREA, *On some integral inequalities*, International Series of Numerical Mathematics **157**, Birkhäuser Basel (2009), pp. 91–95.

[10] N.S. HOANG, *Notes on an inequality*, JIPAM. J. Inequal. Pure Appl. Math., **9**, 2 (2008), Art. 42.

[11] *The 2nd International Mathematical Competition for University Students, Plovdiv, Bulgaria, 2–7 August, 1995, Solutions for problems on Day 1*, Available online at
URL: http://www.imc-math.org.uk/index.php?year=1995 or
URL: http://www.ucl.ac.uk/~ucahjej/imc/imc1995/prob_sol.pdf.

[12] J.-CH. KUANG, *Applied inequalities*, 3nd edition, Shandong Science and Technology Press, Jinan, China, 2004. (Chinese).

[13] W.J. LIU, C.C. LI AND J.W. DONG, *Note on Qi's inequality and Bougoffa's inequality*, JIPAM. J. Inequal. Pure Appl. Math., **7**, 4 (2006), Art. 129.

[14] W.J. LIU, C.C. LI AND J.W. DONG, *Consolidations of extended Qi's inequality and Bougoffa's inequality*, J. Math. Inequal., **2**, 1 (2008), 9–15.

[15] W.J. LIU, C.C. LI AND J.W. DONG, *On an open problem concerning an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., **8**, 3 (2007), Art. 74.

[16] W.J. LIU, G.S. CHENG AND C.C. LI, *Further development of an open problem concerning an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., **9**, 1 (2008), Art.14.

[17] *Mathematics today* **10**, A.Ya. Dorogovtsev (Ed.), Nauchnoe Izdatel'stvo "TViMS", Kiev, 1995, pp. 102.

[18] Y. MIAO AND J. LI, *Further development of an open problem*, JIPAM. J. Inequal. Pure Appl. Math., **9**, 4 (2008), Art. 108.

[19] S. MAZOUZI AND F. QI, *On an open problem regarding an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., **4**, 2 (2003), Art. 31.

[20] Q.A. NGÔ, D.D. THANG, T. T. DAT AND D.A. TUAN, *Note on an integral inequality*, JIPAM. J. Inequal. Pure Appl. Math., **7**, 4 (2006), Art. 120.

[21] T.K. POGANY, *On an open problem of F. Qi*, JIPAM. J. Inequal. Pure Appl. Math., **3**, 4 (2002), Art. 54.

[22] F. QI, *Several integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **1**, 2 (2000), Art. 19.

[23] F. QI, A.J. LI, W.Z. ZHAO, D.W. NIU, AND J. CAO, *Extensions of several integral inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **7**, 3 (2006), Art. 107.

[24] W. T. SULAIMAN, *On an integral inequality*, Commun. Math. Anal., **5**, 1 (2008), 57–64.

[25] K.W. YU AND F. QI, *A short note on an integral inequality*, RGMIA Res. Rep. Coll., **4**, 1 (2001), Art. 4, 23–25.

(Received November 13, 2008)

Wenjun Liu
College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
e-mail: wjliu@nuist.edu.cn

Quốc-Anh Ngô
Department of Mathematics
College of Science
Việt Nam National University
Hà Nội, Việt Nam

Department of Mathematics
National University of Singapore
2 Science Drive 2
Singapore 117543
e-mail: bookworm_vn@yahoo.com

Vu Nhat Huy
Department of Mathematics
College of Science
Việt Nam National University
Hà Nội, Việt Nam
e-mail: nhat_huy85@yahoo.com

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com