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Multiplicity of weak solutions for a class of nonuniformly elliptic equations of p-Laplacian type

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Abstract

This paper deals with the multiplicity of weak solutions in $W_0^1(\Omega)$ to a class of nonuniformly elliptic equations of the form

$$-\text{div}(a(x,\nabla u)) = h(x)|u|^{r-1}u + g(x)|u|^{s-1}u$$

in a bounded domain Ω of $\mathbb{R}^N$. Here a satisfies $|a(x,\xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1 \in L^1_{\text{loc}}(\Omega)$, $h_1(x) \geqq 1$ for a.e. x in Ω, $1 < r < p-1 < s < (Np - N + p)/(N - p)$.

Keywords: p-Laplacian; Nonuniform elliptic equations; Multiplicity

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. In the present paper we study the multiplicity of nontrivial solutions of the following Dirichlet elliptic problem:

$$-\,\text{div}(a(x,\nabla u)) = h(x)|u|^{r-1}u + g(x)|u|^{s-1}u \tag{1}$$

where $|a(x,\xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for any ξ in $\mathbb{R}^N$ and a.e. $x \in \Omega$, $h_0(x) \geqq 0$ and $h_1(x) \geqq 1$ for any x in Ω. For h_0 and h_1 belonging to L^∞, the problem has been studied. Here we study the case in which h_0 and h_1 belong to $L^{\frac{p}{p-1}}(\Omega)$ and $L^1_{\text{loc}}(\Omega)$ respectively. The equation now may be nonuniformly elliptic. To our knowledge, such equations were first studied by Duc et al. and Vu [3,6]. In both papers, the authors studied the following problem:

$$-\,\text{div}(a(x,\nabla u)) = f(x,u) \tag{2}$$

where the nonlinear term f verifies the Ambrosetti–Rabinowitz type condition, see [1]. They then obtained the existence of a weak solution by using a variation of the Mountain-Pass Theorem introduced in [2]. We also point out the fact that for the case when $h_1 \equiv 1$, our problem (1) was studied in [4]. Our goal is to extend the results of [4] (for the "nonuniform case") and of [3,6] (the existence of at least two weak solutions).

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In order to state our main theorem, let us introduce our hypotheses on the structure of problem (1).

Assume that $N \geqq 3$ and $2 \leqq p < N$. Let Ω be a bounded domain in $\mathbb{R}^N$ having C^2 boundary $\partial\Omega$. Consider $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $a = a(x, \xi)$, as the continuous derivative with respect to ξ of the continuous function $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x, \xi)$, that is, $a(x, \xi) = \frac{\partial A(x,\xi)}{\partial \xi}$. Assume that there are a positive real number c_0 and two nonnegative measurable functions h_0, h_1 on Ω such that $h_1 \in L^1_{\mathrm{loc}}(\Omega)$, $h_0 \in L^{\frac{p}{p-1}}(\Omega)$, $h_1(x) \geqq 1$ for a.e. x in Ω.

Suppose that a and A satisfy the hypotheses below:

(A$_1$) $|a(x, \xi)| \leqq c_0(h_0(x) + h_1(x)|\xi|^{p-1})$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_2$) There exists a constant $k_1 > 0$ such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leqq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k_1 h_1(x)|\xi - \psi|^p$$

for all x, ξ, ψ, that is, A is p-uniformly convex.

(A$_3$) A is p-subhomogeneous, that is,

$$0 \leqq a(x, \xi)\xi \leqq pA(x, \xi)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_4$) There exists a constant $k_0 > 0$ such that

$$A(x, \xi) \geqq k_0 h_1(x)|\xi|^p$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

(A$_5$) $A(x, 0) = 0$ for all $x \in \Omega$.

(A$_6$) $A(x, -\xi) = A(x, \xi)$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

Example 1. **(i)** $A(x, \xi) = \frac{1}{p}|\xi|^p$, $a(x, \xi) = |\xi|^{p-2}\xi$ with $p \geqq 2$. We get the p-Laplacian operator.

(ii) $A(x, \xi) = \frac{1}{p}|\xi|^p + \theta(x)(\sqrt{1 + |\xi|^2} - 1)$, $a(x, \xi) = |\xi|^{p-2}\xi + \theta(x)\frac{\xi}{\sqrt{1+|\xi|^2}}$ with $p \geqq 2$ and θ a suitable function. We get the operator

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \operatorname{div}\left(\theta(x)\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)$$

which can be regarded as the sum of the p-Laplacian operator and a degenerate form of the mean curvature operator.

(iii) $A(x, \xi) = \frac{1}{p}((\theta^{\frac{2}{p-1}}(x) + |\xi|^2)^{\frac{p}{2}} - \theta^{\frac{p}{p-1}}(x))$, $a(x, \xi) = (\theta^{\frac{2}{p-1}}(x) + |\xi|^2)^{\frac{p-2}{2}}\xi$ with $p \geqq 2$ and θ a suitable function. Now we get the operator

$$\operatorname{div}\left(\left(\theta^{\frac{2}{p-1}}(x) + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u\right)$$

which is a variant of the generalized mean curvature operator

$$\operatorname{div}\left((1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right).$$

Regarding the functions h and g, we assume that

(H) $h(x) \geqq 0$ for all $x \in \Omega$ and $h \in L^{r_0}(\Omega) \cap L^\infty(\Omega)$, where

$$\frac{1}{r_0} + \frac{(r + 1)}{p^*} = 1,$$

that is $r_0 = \frac{Np}{Np-(r+1)(N-p)}$.

(G) $g(x) > 0$ a.e. $x \in \Omega$ and $g \in L^{s_0}(\Omega) \cap L^\infty(\Omega)$, where

$$\frac{1}{s_0} + \frac{(s + 1)}{p^*} = 1,$$

that is $s_0 = \frac{Np}{Np-(s+1)(N-p)}$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space. Next, we define $X := W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{3}$$

We now consider the following subspace of $W_0^{1,p}(\Omega)$:

$$E = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega h_1(x)|\nabla u|^p \mathrm{d}x < +\infty\right\}.$$

The space E can be endowed with the norm

$$\|u\|_E = \left(\int_\Omega h_1(x)|\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

As in [3], it is known that E is an infinite dimensional Banach space. We say that $u \in E$ is a weak solution for problem (1) if

$$\int_\Omega a(x,\nabla u)\nabla\varphi \mathrm{d}x - \int_\Omega h(x)|u|^{r-1}u\varphi \mathrm{d}x - \int_\Omega g(x)|u|^{s-1}u\varphi \mathrm{d}x = 0$$

for all $\varphi \in E$.

Let

$$J(u) = \frac{1}{r+1}\int_\Omega h(x)|u|^{r+1}\mathrm{d}x + \frac{1}{s+1}\int_\Omega g(x)|u|^{s+1}\mathrm{d}x,$$
$$\Lambda(u) = \int_\Omega A\,(x,\nabla u)\,\mathrm{d}x,$$

and

$$I(u) = \Lambda(u) - J(u)$$

for all $u \in E$.

The following remark plays an important role in our arguments.

Remark 2. **(i)** $\|u\| \leqq \|u\|_E$ for all $u \in E$ since $h_1(x) \geqq 1$. Thus the continuous embeddings

$$E \hookrightarrow X \hookrightarrow L^i(\Omega), \quad p \leqq i \leqq p^\star$$

hold true.

(ii) By $(\mathbf{A}_4)$ and (i) in Lemma 5, it is easy to see that

$$E = \{u \in W_0^{1,p}(\Omega) : \Lambda(u) < +\infty\} = \{u \in W_0^{1,p}(\Omega) : I(u) < +\infty\}.$$

(iii) $C_0^\infty(\Omega) \subset E$ since $|\nabla u|$ is in $C_c(\Omega)$ for any $u \in C_0^\infty(\Omega)$ and $h_1 \in L^1_{\mathrm{loc}}(\Omega)$.

Our main results are included in a couple of theorems below.

Theorem 3. *Assume* $1 < r < p-1 < s < (Np - N + p)/(N-p)$ *and conditions* $(\mathbf{A}_1)$– $(\mathbf{A}_5)$, $(\mathbf{H})$, and $(\mathbf{G})$ *are fulfilled. Then problem* (1) *has at least two nontrivial weak solutions in* E *provided that the product* $\|h\|_{L^{r_0}(\Omega)}^{\frac{s+1-p}{s-r}}\|g\|_{L^{s_0}(\Omega)}^{\frac{p-r-1}{s-r}}$ *is small enough.*

Theorem 4. *Assume* $1 < r < p-1 < s < (Np - N + p)/(N-p)$ *and conditions* $(\mathbf{A}_1)$– $(\mathbf{A}_6)$, $(\mathbf{H})$, and $(\mathbf{G})$ *are fulfilled. Then problem* (1) *has infinitely many nontrivial generalized solutions in* E.

To see the power of the theorems, we compare our assumptions to those considered in [5,3,4,6]. Our problem (1) covers the following cases which have been considered in the literature:

(i) $A(x, \xi) = \frac{1}{p}|\xi|^p$ with $p \geqq 2$.

(ii) $A(x, \xi) = \frac{1}{p}((1 + |\xi|^2)^{\frac{p}{2}} - 1)$ with $p \geqq 2$.

Moreover, our assumption includes the following situations which could not be handled in [5,4].

(i) $A(x, \xi) = \frac{h(x)}{p}|\xi|^p$ with $p \geqq 2$ and $h \in L^1_{\mathrm{loc}}(\Omega)$.

(ii) $A(x, \xi) = \frac{h(x)}{p}((1 + |\xi|^2)^{\frac{p}{2}} - 1)$ with $p \geqq 2$ and $h \in L^{\frac{p}{p-1}}(\Omega)$.

2. Auxiliary results

In this section we recall certain properties of functionals Λ and J. But firstly, we list here some properties of A.

Lemma 5 (*See [3]*).

(i) *A verifies the growth condition*

$$|A(x, \xi)| \leqq c_0(h_0(x)|\xi| + h_1(x)|\xi|^p)$$

for all $\xi \in \mathbb{R}^N$, *a.e.* $x \in \Omega$.

(ii) $A(x, z\xi) \leqq A(x, \xi)z^p$ *for all* $z \geqq 1$, $x, \xi \in \mathbb{R}^N$.

Due to the presence of h_1, the functional Λ does not belong to $C^1(E, \mathbb{R})$. This means that we cannot apply directly the Mountain-Pass Lemma of Ambrosetti and Rabinowitz. In this situation, we recall the following concept of weakly continuous differentiability. Our approach is based on a weak version of the Mountain-Pass Lemma introduced by Duc [2].

Definition 6. Let $\mathcal{F}$ be a map from a Banach space Y to $\mathbb{R}$. We say that $\mathcal{F}$ is weakly continuous differentiable on Y if and only if the following two conditions are satisfied:

(i) For any $u \in Y$ there exists a linear map $D\mathcal{F}(u)$ from Y to $\mathbb{R}$ such that

$$\lim_{t \to 0} \frac{\mathcal{F}(u + tv) - J(u)}{t} = D\mathcal{F}(u)(v)$$

for every $v \in Y$.

(ii) For any $v \in Y$, the map $u \mapsto D\mathcal{F}(u)(v)$ is continuous on Y.

Denote by $C^1_w(Y)$ the set of weakly continuously differentiable functionals on Y. It is clear that $C^1(Y) \subset C^1_w(Y)$ where we denote by $C^1(Y)$ the set of all continuously Frechet differentiable functionals on Y. For simplicity of notation, we shall denote $D\mathcal{F}(u)$ by $\mathcal{F}'(u)$.

The following lemma concerns the smoothness of the functional Λ.

Lemma 7 (*See [3]*).

(i) *If* $\{u_n\}$ *is a sequence weakly converging to* u *in* X, *denoted by* $u_n \rightharpoonup u$, *then* $\Lambda(u) \leqq \liminf_{n\to\infty} \Lambda(u_n)$.

(ii) *For all* $u, z \in E$,

$$\Lambda\left(\frac{u + z}{2}\right) \leqq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(z) - k_1 \|u - z\|^p_E .$$

(iii) Λ *is continuous on* E.

(iv) Λ *is weakly continuously differentiable on* E *and*

$$\langle \Lambda'(u), v\rangle = \int_\Omega a(x, \nabla u)\nabla v \mathrm{d}x$$

for all $u, v \in E$.

(v) $\Lambda(u) - \Lambda(v) \geqq \langle \Lambda'(v), u - v\rangle$ *for all* $u, v \in E$.

The following lemma concerns the smoothness of the functional J. The proof is standard and simple, so we omit it.

Lemma 8. **(i)** *If* $u_n \rightharpoonup u$ *in* X, *then* $\lim_{n\to\infty} J(u_n) = J(u)$.
(ii) J *is continuous on* E.
(iii) J *is weakly continuously differentiable on* E *and*

$$\langle J'(u), v\rangle = \int_\Omega h(x)|u|^{r-1}uv\,\mathrm{d}x + \int_\Omega g(x)|u|^{s-1}uv\,\mathrm{d}x$$

for all $u, v \in E$.

Our main tool is a variation of the Mountain-Pass Theorem introduced in [2] and the Z_2 version of it introduced in [6].

Lemma 9 (*Mountain-Pass Lemma*). *Let* $\mathcal{F}$ *be a continuous function from a Banach space* E *into* $\mathbb{R}$. *Let* $\mathcal{F}$ *be weakly continuously differentiable on* E *and satisfy the Palais–Smale condition. Assume that* $\mathcal{F}(0) = 0$ *and there exist a positive real number* ρ *and* $z_0 \in E$ *such that*

(i) $\|z_0\|_E > \rho$, $\mathcal{F}(z_0) \leqq \mathcal{F}(0)$.
(ii) $\alpha = \inf\{\mathcal{F}(u) : u \in E, \|u\|_E = \rho\} > 0$.

Put $G = \{\phi \in C([0,1], E) : \phi(0) = 0, \phi(1) = z_0\}$. *Assume that* $G \neq \emptyset$. *Set*

$$\beta = \inf\{\max \mathcal{F}(\phi([0,1])) : \phi \in G\}.$$

Then $\beta \geqq \alpha$ *and* β *is a critical value of* $\mathcal{F}$.

Lemma 10 (*Symmetric Mountain-Pass Lemma*). *Let* E *be an infinite dimensional Banach space. Let* $\mathcal{F}$ *be weakly continuously differentiable on* E *and satisfy the Palais–Smale condition. Assume that* $\mathcal{F}(0) = 0$ *and:*

(i) *There exist a positive real number* α *and* ρ *such that*

$$\inf_{u\in\partial B_\rho} \mathcal{F}(u) \geqq \alpha > 0$$

where B_ρ *is an open ball in* E *of radius* ρ *centered at the origin and* ∂B_ρ *is its boundary.*
(ii) *For each finite dimensional linear subspace* Y *in* E, *the set*

$$\left\{u \in Y : \mathcal{F}(u) \geqq 0\right\}$$

is bounded.

Then $\mathcal{F}$ *possesses an unbounded sequence of critical values.*

3. Proofs

We remark that the critical points of the functional I correspond to the weak solutions of (1). In order to apply Lemma 9 we need to verify the following facts.

Lemma 11. **(i)** I *is a continuous function from* E *to* $\mathbb{R}$.
(ii) I *be weakly continuously differentiable on* E *and*

$$\langle I'(u), v\rangle = \int_\Omega a(x, \nabla u)\nabla v\,\mathrm{d}x - \int_\Omega h(x)|u|^{r-1}uv\,\mathrm{d}x - \int_\Omega g(x)|u|^{s-1}uv\,\mathrm{d}x$$

for all $u, v \in E$.
(iii) $I(0) = 0$.
(iv) *There exist two positive real numbers* ρ *and* α *such that*

$$\inf\{I(u) : u \in E, \|u\|_E = \rho\} > \alpha.$$

(v) *There exists* $\psi \in E$ *such that* $\lim_{t\to\infty} I(t\psi) = -\infty$.
(vi) I *satisfies the Palais–Smale condition on* E.
(vii) *There exists* $z_0 \in E$ *such that* $\|z_0\|_E > \rho$, $I(z_0) \leqq 0$.

(viii) *The set*

$$G = \{\varphi \in C([0,1], E) : \varphi(0) = 0, \varphi(1) = z_0\}$$

is not empty.

Proof. **(i)** This comes from (iii) in Lemma 7 and (ii) in Lemma 8.

(ii) This comes from (iv) in Lemma 7 and (iii) in Lemma 8.

(iii) This comes from the definition of I.

(iv) First, let $\mathcal{S}$ be the best Sobolev constant of the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$, that is,

$$\mathcal{S} = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p \mathrm{d}x}{\left(\int_\Omega |u|^{p^\star} \mathrm{d}x\right)^{\frac{p}{p^\star}}}.$$

Thus, we obtain

$$\mathcal{S}^{\frac{1}{p}} \|v\|_{L^{p^\star}(\Omega)} \leqq \|v\|$$

for all $v \in E$. Since

$$r_0 = \frac{Np}{Np - (r+1)(N-p)}$$

then

$$\frac{1}{r_0} + \frac{1}{\frac{p^\star}{r+1}} = \frac{Np - (r+1)(N-p)}{Np} + (r+1)\frac{N-p}{Np} = 1.$$

By Hölder's inequality and the above relation we deduce

$$\int_\Omega h(x)|u|^{r+1}\mathrm{d}x \leqq \|h\|_{L^{r_0}(\Omega)} \|u\|^{r+1}_{L^{p^\star}(\Omega)} \tag{4}$$

$$\leqq \|h\|_{L^{r_0}(\Omega)} \frac{1}{\mathcal{S}^{\frac{r+1}{p}}} \left(S^{\frac{1}{p}} \|h\|_{L^{p^\star}(\Omega)}\right)^{r+1} \tag{5}$$

$$\leqq \|h\|_{L^{r_0}(\Omega)} \frac{1}{\mathcal{S}^{\frac{r+1}{p}}} \|u\|^{r+1}_E =: (r+1)\,\mu \|u\|^{r+1}_E. \tag{6}$$

With similar arguments, we have

$$\int_\Omega g(x)|u|^{s+1}\mathrm{d}x \leqq (s+1)\,\nu\|u\|^{s+1}_E.$$

Thus, we obtain

$$I(u) \geqq \min\left\{k_0, \frac{1}{p}\right\} \|u\|^p_E - \mu\|u\|^{r+1}_E - \nu\|u\|^{s+1}_E \tag{7}$$

$$=: \left(\lambda - \mu\|u\|^{r+1-p}_E - \nu\|u\|^{s+1-p}_E\right)\|u\|^p_E. \tag{8}$$

We show that there exists $t_0 > 0$ such that

$$\lambda - \mu t_0^{r+1-p} - \nu t_0^{s+1-p} > 0. \tag{9}$$

To do that, we define the function

$$Q(t) = \mu t^{r+1-p} + \nu t^{s+1-p}, \quad t > 0.$$

Since $\lim_{t\to\infty} Q(t) = \lim_{t\to 0} Q(t) = \infty$, it follows that Q possesses a positive minimum, say $t_0 > 0$. In order to find t_0, we have to solve equation $Q'(t_0) = 0$, where

$$Q'(t) = (r+1-p)\,\mu t^{r-p} + (s+1-p)\,\nu t^{s-p}.$$

A simple computation yields

$$t_0 = \left(\frac{p-r-1}{s+1-p}\frac{\mu}{\nu}\right)^{\frac{1}{s-r}}.$$

Thus, relation (9) holds provided that

$$\mu\left(\frac{p-r-1}{s+1-p}\frac{\mu}{\nu}\right)^{\frac{r+1-p}{s-r}}+\nu\left(\frac{p-r-1}{s+1-p}\frac{\mu}{\nu}\right)^{\frac{s+1-p}{s-r}}<\lambda$$

or equivalently

$$\frac{\mu^{\frac{s+1-p}{s-r}}}{\nu^{\frac{r+1-p}{s-r}}}\left(\frac{p-r-1}{s+1-p}\right)^{\frac{r+1-p}{s-r}}+\frac{\mu^{\frac{s+1-p}{s-r}}}{\nu^{\frac{r+1-p}{s-r}}}\left(\frac{p-r-1}{s+1-p}\right)^{\frac{s+1-p}{s-r}}<\lambda$$

or

$$\|h\|_{L^{r_0}(\Omega)}^{\frac{s+1-p}{s-r}}\|g\|_{L^{s_0}(\Omega)}^{\frac{p-r-1}{s-r}}$$

is small enough.

(v) Let $\psi\in C_0^\infty(\Omega)$, $\psi\geqq 0$, $\psi\not\equiv 0$. Then using Lemma 5, we have

$$I(t\psi)=\int_\Omega A(x,t\nabla\psi)\,\mathrm{d}x-\frac{t^{r+1}}{r+1}\int_\Omega h(x)|\psi|^{r+1}\mathrm{d}x \tag{10}$$

$$-\frac{t^{s+1}}{s+1}\int_\Omega g(x)|\psi|^{s+1}\mathrm{d}x \tag{11}$$

$$\leqq t^p\int_\Omega A(x,\nabla\psi)\,\mathrm{d}x-\frac{t^{s+1}}{s+1}\int_\Omega g(x)|\psi|^{s+1}\mathrm{d}x. \tag{12}$$

Thus, $\lim_{t\to\infty} I(t\psi)=-\infty$.

(vi) Let $\{u_n\}$ be a sequence in X and β be a real number such that $I(u_n)\to\beta$ and $I'(u_n)\to 0$ in $E^\star$. We prove that $\{u_n\}$ is bounded in E. We assume by contradiction that $\|u_n\|_E\to\infty$ as $n\to\infty$. It follows from conditions ($\mathbf{A}_3$) and ($\mathbf{A}_4$) that for n large enough

$$\beta+1+\|u_n\|_E\geqq I(u_n)-\frac{1}{s+1}\left\langle I'(u_n),u_n\right\rangle \tag{13}$$

$$=\int_\Omega\left(A(x,\nabla u_n)-\frac{1}{s+1}a(x,\nabla u_n)\nabla u_n\right)\mathrm{d}x \tag{14}$$

$$-\frac{s-r}{(r+1)(s+1)}\int_\Omega h(x)|u_n|^{r+1}\mathrm{d}x \tag{15}$$

or

$$\beta+1+\|u_n\|_E+\frac{s-r}{(r+1)(s+1)}\int_\Omega h(x)|u_n|^{r+1}\mathrm{d}x \tag{16}$$

$$\geqq\left(1-\frac{p}{s+1}\right)k_0\int_\Omega|\nabla u_n|^p\mathrm{d}x. \tag{17}$$

Hence

$$\beta+1+\|u_n\|_E+\frac{s-r}{(r+1)(s+1)}\|h\|_{L^{r_0}(\Omega)}\frac{1}{\mathcal{S}^{\frac{r+1}{p}}}\|u_n\|_E^{r+1} \tag{18}$$

$$\geqq\left(1-\frac{p}{s+1}\right)k_0\|u_n\|_E^p. \tag{19}$$

Since $1<r<p-1$ and $\|u_n\|_E\to\infty$ as $n\to\infty$, dividing the above inequality by $\|u_n\|_E^p$ and passing to the limit as $n\to\infty$ we obtain a contradiction. Hence $\{u_n\}$ is bounded in E. By Remark 2, we deduce that $\{u_n\}$ is bounded in X. Since X is reflexive, then by passing to a subsequence, still denoted by $\{u_n\}$, we can assume that the sequence $\{u_n\}$ converges weakly to some u in X. We shall prove that the sequence $\{u_n\}$ converges strongly to u in E.

We observe by Remark 2 that $u\in E$. Hence $\{\|u_n-u\|_E\}$ is bounded. Since $\left\{\left\|I'(u_n-u)\right\|_E\right\}$ converges to 0, then $\left\langle I'(u_n-u),u_n-u\right\rangle$ converges to 0.

Since E is continuously embedded in $L^{p^\star}(\Omega)$, then we deduce that u_n converges weakly to u in $L^{p^\star}(\Omega)$. Then it is clear that $|u_n|^{r-1}u_n$ converges weakly to $|u|^{r-1}u$ in $L^{\frac{p^\star}{r}}(\Omega)$.

Now define the operator $U : L^{\frac{p^\star}{r}}(\Omega) \to \mathbb{R}$ by

$$\langle U, w\rangle = \int_\Omega h(x)uw\mathrm{d}x.$$

We remark that U is linear. Moreover, it follows from

$$\frac{1}{p^\star} + \frac{r}{p^\star} + \frac{1}{r_0} = 1$$

that U is continuous since $h \in L^{r_0}(\Omega)$. All the above pieces of information imply

$$\left\langle U, |u_n|^{r-1}u_n\right\rangle \to \left\langle U, |u|^{r-1}u\right\rangle,$$

that is,

$$\lim_{n\to\infty}\int_\Omega h(x)|u_n|^{r-1}u_n u\mathrm{d}x = \int_\Omega h(x)|u|^{r+1}\mathrm{d}x.$$

With the same arguments we can show that

$$\lim_{n\to\infty}\int_\Omega g(x)|u_n|^{s-1}u_n u\mathrm{d}x = \int_\Omega g(x)|u|^{s+1}\mathrm{d}x, \tag{20}$$

$$\lim_{n\to\infty}\int_\Omega h(x)|u_n|^{r+1}\mathrm{d}x = \int_\Omega h(x)|u|^{r+1}\mathrm{d}x, \tag{21}$$

$$\lim_{n\to\infty}\int_\Omega g(x)|u_n|^{s+1}\mathrm{d}x = \int_\Omega g(x)|u|^{s+1}\mathrm{d}x. \tag{22}$$

Therefore,

$$\lim_{n\to\infty}\int_\Omega h(x)|u_n|^{r-1}u_n\left(u_n - u\right)\mathrm{d}x = \lim_{n\to\infty}\left(\int_\Omega h(x)|u_n|^{r+1}dx - \int_\Omega h(x)|u|^{r+1}\mathrm{d}x\right)$$
$$- \lim_{n\to\infty}\left(\int_\Omega h(x)|u_n|^{r-1}u_n u\mathrm{d}x - \int_\Omega h(x)|u|^{r+1}\mathrm{d}x\right)$$

which yields

$$\lim_{n\to\infty}\int_\Omega h(x)|u_n|^{r-1}u_n\left(u_n - u\right)\mathrm{d}x = 0.$$

Similarly, we obtain

$$\lim_{n\to\infty}\int_\Omega g(x)|u_n|^{s-1}u_n\left(u_n - u\right)\mathrm{d}x = 0.$$

On the other hand,

$$\left\langle J'\left(u_n\right), u_n - u\right\rangle = \int_\Omega h(x)|u_n|^{r-1}u_n\left(u_n - u\right)\mathrm{d}x \tag{23}$$

$$+ \int_\Omega g(x)|u_n|^{s-1}u_n\left(u_n - u\right)\mathrm{d}x. \tag{24}$$

Thus

$$\lim_{n\to\infty}\left\langle J'\left(u_n\right), u_n - u\right\rangle = 0.$$

This and the fact that

$$\left\langle \Lambda'\left(u_n\right), u_n - u\right\rangle = \left\langle I'\left(u_n\right), u_n - u\right\rangle + \left\langle J'\left(u_n\right), u_n - u\right\rangle$$

gives

$$\lim_{n\to\infty}\left\langle \Lambda'\left(u_n\right), u_n - u\right\rangle = 0.$$

By using (v) in Lemma 7, we get

$$\Lambda(u) - \lim_{n\to\infty} \Lambda(u_n) = \lim_{n\to\infty} (\Lambda(u) - \Lambda(u_n)) \geqq \lim_{n\to\infty} \left\langle \Lambda'(u_n), u - u_n \right\rangle = 0.$$

This and (i) in Lemma 7 give

$$\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u).$$

Now if we assume by contradiction that $\|u_n - u\|_E$ does not converge to 0, then there exists $\varepsilon > 0$ and a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ such that

$$\left\|u_{n_m} - u\right\|_E \geqq \varepsilon.$$

By using relation (ii) in Lemma 7, we get

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda\left(u_{n_m}\right) - \Lambda\left(\frac{u_{n_m}+u}{2}\right) \geqq k_1 \left\|u_{n_m} - u\right\|_E^p \geqq k_1\varepsilon^p.$$

Letting $m \to \infty$ we find that

$$\limsup_{m\to\infty} \Lambda\left(\frac{u_{n_m}+u}{2}\right) \leqq \Lambda(u) - k_1\varepsilon^p.$$

We also have that $\frac{u_{n_m}+u}{2}$ converges weakly to u in E. Using (i) in Lemma 7 again, we get

$$\Lambda(u) \leqq \liminf_{m\to\infty} \Lambda\left(\frac{u_{n_m}+u}{2}\right).$$

That is a contradiction. Therefore $\{u_n\}$ converges strongly to u in E.

(vii) The existence of $z_0 \in E$ such that $\|z_0\|_E > \rho$, $I(z_0) \leqq 0$ is followed from the fact that $\lim_{t\to\infty} I(t\psi) = -\infty$.

(viii) We consider a function $\varphi \in C([0,1], E)$ defined by $\varphi(t) = tz_0$, for every $t \in [0,1]$. It is clear that $\varphi \in G$. □

Proof of Theorem 3. Using Lemmas 9 and 11 we deduce the existence of $u_1 \in E$ as a nontrivial generalized solution of (1). We prove now that there exists a second weak solution $u_2 \in E$ such that $u_2 \neq u_1$.

By Lemma 11, it follows that there exists a ball centered at the origin $B \subset E$, such that

$$\inf_{\partial B} I > 0.$$

On the other hand, by the lemma there exists $\phi \in E$ such that $I(t\phi) < 0$, for all $t > 0$ small enough. Recalling that relation (7) holds for all $u \in E$, that is,

$$I(u) \geqq \lambda\|u\|_E^p - \mu\|u\|_E^{r+1} - \nu\|u\|_E^{s+1}$$

we get that

$$-\infty < \underline{c} := \inf_B I < 0.$$

We let now

$$0 < \varepsilon < \inf_{\partial B} I - \inf_B I.$$

Applying Ekeland's Variational Principle for the functional $I : \overline{B} \to \mathbb{R}$, there exists $u_\varepsilon \in \overline{B}$ such that

$$I(u_\varepsilon) < \inf_{\overline{B}} I + \varepsilon \tag{25}$$

$$I(u_\varepsilon) < I(u) + \varepsilon \|u - u_\varepsilon\|_E, \quad u \neq u_\varepsilon. \tag{26}$$

Since

$$I(u_\varepsilon) < \inf_{\overline{B}} I + \varepsilon < \inf_B I + \varepsilon < \inf_{\partial B} I$$

it follows that $u_\varepsilon \in B$. Now we define $M : \overline{B} \to \mathbb{R}$ by $M(u) = I(u) + \varepsilon \|u - u_\varepsilon\|_E$. It is clear that u_ε is a minimum point of M and thus

$$\frac{M(u_\varepsilon + tv) - M(u_\varepsilon)}{t} \geqq 0$$

for a small $t > 0$ and ν in the unit sphere of E. The above relation yields

$$\frac{I(u_\varepsilon + t\nu) - I(u_\varepsilon)}{t} + \varepsilon \|\nu\|_E \geqq 0.$$

Letting $t \to 0$ it follows that

$$\left\langle I'(u_\varepsilon), \nu \right\rangle + \varepsilon \|\nu\|_E > 0$$

and we infer that $\left\| I'(u_\varepsilon) \right\|_E \leqq \varepsilon$. We deduce that there exists $\{u_n\} \subset B$ such that $I(u_n) \to \underline{c}$ and $I'(u_n) \to 0$. Using the fact that J satisfies the Palais–Smale condition on E we deduce that $\{u_n\}$ converges strongly to u_2 in E. Thus, u_2 is a weak solution for (1) and since $0 > \underline{c} = I(u_2)$ it follows that u_2 is nontrivial.

Finally, we point out the fact that $u_1 \neq u_2$ since

$$I(u_1) = \overline{c} > 0 > \underline{c} = I(u_2).$$

The proof is complete. □

Proof of Theorem 4. In view of ($\mathbf{A}_6$), I is even. In order to apply Lemma 10, it is enough to verify condition (ii) in Lemma 10. It is known that

$$\int_\Omega A(x, \nabla u_n)\,\mathrm{d}x \leqq c_0 \left(\left(\int_\Omega |h_0(x)| \mathrm{d}x \right)^{\frac{p-1}{p}} \|u_n\|_E + \|u_n\|_E^p \right) \tag{27}$$

$$=: c_1 \|u_n\|_E + c_0 \|u_n\|_E^p. \tag{28}$$

This gives

$$I(u) \leqq c_1 \|u_n\|_E + c_0 \|u_n\|_E^p - \frac{1}{s+1} \int_\Omega g(x)|u|^{s+1} \mathrm{d}x.$$

Suppose that $\widetilde{E}$ is a finite dimensional subspace of E. Setting

$$\|u\|_{\widetilde{E}} = \left(\int_\Omega g(x)|u|^{s+1} \mathrm{d}x \right)^{\frac{1}{s+1}}$$

for all $u \in \widetilde{E}$, we see that $\|.\|_{\widetilde{E}}$ is a norm in $\widetilde{E}$. We also note that in $\widetilde{E}$ the norms are equivalent. Thus, there exists a positive constant K such that

$$\|u\|_E \leqq K \|u\|_{\widetilde{E}}.$$

This implies that

$$I(u) \leqq c_1 \|u_n\|_E + c_0 \|u_n\|_E^p - \frac{K}{s+1} \|u\|_E^{s+1}.$$

Since $p < s + 1$ then $\left\{ u \in \widetilde{E} : I(u) \geqq 0 \right\}$ is bounded. Hence, I possesses an unbounded sequence of critical values. Therefore, I possesses infinitely many critical points in E. This completes the proof. □

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Further reading

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