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Existence of positive solution for system of quasilinear elliptic systems on a bounded domain

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Abstract. In this paper we study the existence of positive solution of the following Dirichlet problem for a system of nonlinear equations on a bounded domain

$$\begin{cases} -\Delta_p u = \lambda f(x,u,v) \text{ in } \Omega \\ -\Delta_q v = \mu g(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega. \end{cases}$$

The proof is based on the comparison principle and the Schauder Fixed-Point Theorem.

Keywords: positive solution, bounded domain, maximum principle, comparison principle, Schauder fixed-point theorem

1 Introduction

In this present work we are interested in the study of the following Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u,v) \text{ in } \Omega \\ -\Delta_q v = \mu g(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with a smooth boundary $\partial\Omega$; $p, q > N$; λ, μ are suitable parameters; $f, g : \overline{\Omega} \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions and

$$-\Delta_s u = \operatorname{div}\left(|\nabla u|^{s-2} \nabla u\right)$$

denotes the s-Laplacian as usual.

The study of nonlinear systems, such as (1), naturally arises in the study of various kinds of nonlinear phenomena such as non-Newtonian fluid: pseudo-plastic fluids correspond to $s \in (1,2)$ while dilatant fluids correspond to $s > 2$. The case $s = 2$ expresses Newtonian fluids phenomena such as chemical reactions, pattern formation, polulation evolution. As a consequence, positive solutions of (1) are of interest. Problem (1) covers several important cases. When $p = q = 2$, (1) becomes the semilinear elliptic system:

$$\begin{cases} -\Delta u = \lambda f(x,u,v) \text{ in } \Omega \\ -\Delta v = \mu g(x,u,v) \text{ in } \Omega \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{2}$$

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A special case of (2) is the Lane-Emden system which was treated by Dalmasso [6].

An other special case of (2) is the usual Laplacian problem as follows

$$\begin{cases} -\Delta u = \lambda f(x,u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

which has received extensive investigations in the past several decades, see, e.g., [7] and references therein.

Several methods have been used to treat quasilinear equations and systems. In the scalar case, weak solutions can be obtained through variational methods which provide critical points of the corresponding energy functional, an approach which is also fruitfull in the case of potential systems. However, due to the lack of variational structure, the treatment of nonvariational system, such as (1), is more complicated and is based mostly on topological methods, see, e.g [1].

Recently, in [15] the authors discussed (1) in the more general cases without the presence of parameters to obtain the existence of positive solution under the suitable conditions. The main approach in such papers is based on the comparison principle and the Schauder Fixed-Point Theorem. In this paper, we adopt these methods to extend the existence results obtained in [15].

The aim of the present work is to study the existence of positive solutions of the problem (1) under the above hypothesis and suitable conditions for parameters. We shall show that there exist numbers λ_0 and μ_0 such that (1) has a positive solution for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$. Our paper is organized as follows. Section 2 provides some preliminaries and notations before stating our main results in Section 3.

2 Preliminaries

We recall some basic results for the p-Laplacian. Let ϕ (respectively φ) be the torsion functions relative to Ω and to the operator $-\Delta_p$ (respectively $-\Delta_q$), that is,

$$\begin{cases} -\Delta_p \phi = 1 \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_q \varphi = 1 \text{ in } \Omega \\ \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

We make the following assumptions:

(H1) $f, g : \overline{\Omega} \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions such that

(i) $u \to f(x,u,v)\,, u \to g(x,u,v)$ are nondecreasing for every $x \in \overline{\Omega}$ and $v \geq 0$.

(ii) $v \to f(x,u,v)\,, v \to g(x,u,v)$ are nondecreasing for every $x \in \overline{\Omega}$ and $u \geq 0$.

(H2)

$$\liminf_{z\to+\infty} \frac{F(z,z)}{z} = 0, \quad \liminf_{z\to+\infty} \frac{G(z,z)}{z} = 0,$$

where $F(u,v) := \max_{x\in\overline{\Omega}} f(x,u,v)$, $G(u,v) := \max_{x\in\overline{\Omega}} g(x,u,v)$.

(H3)

$$\liminf_{z\to 0+} \frac{h(z,z)}{z} = +\infty, \quad \liminf_{z\to 0+} \frac{k(z,z)}{z} = +\infty,$$

where $h(u,v) := \min_{x\in\overline{\Omega}} f(x,u,v)$, $k(u,v) := \min_{x\in\overline{\Omega}} g(x,u,v)$.

3 Main results

Theorem 1. *Assume that (H1)-(H2)-(H3) hold. Then there exist positive numbers λ_0 and $n\mu_0$ such that (1) has a positive solution for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$.*

Theorem 2. *Let (H1) holds and $\lambda, \mu > 0$ and nassume that there exist positive constants r_1, r_2, s_1, s_2 such that for $x \in \overline{\Omega}$*

$$s \mapsto \frac{f(x,s,t)}{s^{r_1}} \quad , \quad t \mapsto \frac{f(x,s,t)}{t^{r_2}}$$

are nondecreasing, and

$$s \mapsto \frac{g(x,s,t)}{s^{s_1}} \quad , \quad t \mapsto \frac{g(x,s,t)}{t^{s_2}}$$

are nondecreasing. If one of the following conditions is satisfied

(i) $\frac{r_1+r_2}{p-1} < 1$ *and* $\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)} < 1$,

(ii) $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{(s_1+s_2)r_2+r_1(q-1)}{(p-1)(q-1)} < 1$,

(iii) $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{r_1+r_2}{p-1} < 1$,

(iv) $\frac{s_1+s_2}{q-1} < 1$ *and* $\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(p-1)(q-1)^2} < 1$,

(v) $\frac{r_1+r_2}{p-1} < 1$ *and* $\frac{(r_1+r_2)(q-1)s_1+(s_1+s_2)(p-1)s_2}{(p-1)^2(q-1)} < 1$,

(vi) $\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)} < 1$, *and* $\frac{(r_1+r_2)s_1+r_2s_2(p-1)+r_1(p-1)(q-1)}{(p-1)^2(q-1)} < 1$,

then (1) admits at most one positive solution.

In view of (H2), (H3), there exist numbers $R > \max\{\|\phi\|_\infty, \|\varphi\|_\infty\}$ and

$$\min\left\{\lambda^{\frac{1}{p-2}}\phi^{\frac{p-1}{p-2}}, \mu^{\frac{1}{q-2}}\phi^{\frac{q-1}{q-2}}\right\} > \varepsilon > 0$$

such that $F(R,R) \leq R, G(R,R) \leq R$ and $h(\varepsilon,\varepsilon) \geq \varepsilon$.

Proof. Let

$$\lambda_0 = \frac{R^{p-2}}{\|\phi\|_\infty^{p-1}} \quad , \quad \mu_0 = \frac{R^{q-2}}{\|\varphi\|_\infty^{q-1}}.$$

We now only consider $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$.

For each $(v_1, v_2) \in C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right)$ we define $(u_1, u_2) = A_{\lambda,\mu}(v_1, v_2)$ by

$$\begin{cases} -\Delta_p u_1 = \lambda f(x, v_1, v_2) \text{ in } \Omega \\ -\Delta_q u_2 = \mu g(x, v_1, v_2) \text{ in } \Omega \\ u_1 = u_2 = 0 \text{ on } \partial\Omega. \end{cases} \tag{4}$$

Then $A_{\lambda,\mu} : C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right)$ is well-defined, completely continuously and fixed points of $A_{\lambda,\mu}$ are solutions of (1). We claim that

$$A_{\lambda,\mu}\left(\overline{B}(\varepsilon,R) \times \overline{B}(\varepsilon,R)\right) \subseteq \overline{B}(\varepsilon,R) \times \overline{B}(\varepsilon,R)$$

where

$$\overline{B}(\varepsilon,R) = \overline{\left\{z \in C\left(\overline{\Omega}\right) \mid \varepsilon < \|z\|_\infty < R\right\}}.$$

Indeed, let v_1, v_2 in $C\left(\overline{\Omega}\right)$ such that $\|v_1\|_\infty \leq R$, $\|v_2\|_\infty \leq R$. By (H1), we have

$$-\Delta_p u = \lambda f(x, v_1, v_2) \leq \lambda f(x, R, R) \leq \lambda F(R,R) \leq \lambda R$$

which implies, by the comparison principle, that

$$u \leq (\lambda R)^{\frac{1}{p-1}}\phi \leq (\lambda R)^{\frac{1}{p-1}}\|\phi\|_\infty \leq R.$$

Consequently, $v \leq R$.

Next, we will show that $\|u\|_\infty \geq \varepsilon$, $\|v\|_\infty \geq \varepsilon$. Indeed, let v_1, v_2 in $C\left(\overline{\Omega}\right)$ such that $\|v_1\|_\infty \geq \varepsilon$, $\|v_2\|_\infty \geq \varepsilon$

$$-\Delta_p u = \lambda f(x, v_1, v_2) \geq \lambda f(x, \varepsilon, \varepsilon) \geq \lambda h(\varepsilon,\varepsilon) \geq \lambda\varepsilon.$$

By the comparison principle, we get:

$$u \geq (\lambda\varepsilon)^{\frac{1}{p-1}}\phi \geq \varepsilon,$$

which gives $\|u\|_\infty \geq \varepsilon$. Consequently, $\|v\|_\infty \geq \varepsilon$.

By the Schauder Fixed-Point Theorem, $A_{\lambda,\mu}$ has a fixed point (u,v) with $\varepsilon \leq \|u\|_\infty \leq R$, $\varepsilon \leq \|v\|_\infty \leq R$. The proof is complete.

Proof. Let $(u,v) \not\equiv (u_1,v_1)$ be positive solutions of (1). We will provide the proof only for the cases (i), (iii) and (iv). We define

$$\delta = \inf\left\{\varepsilon \geq 1 \,\middle|\, \varepsilon u_1 \geq u, \varepsilon v_1 \geq v \text{ in } \overline{\Omega}\right\}.$$

We will show that $\delta = 1$. Assume that $\delta > 1$. Let (i) holds, then

$$\begin{aligned}-\Delta_p u = \lambda f(x,u,v) &\leq \lambda f(x,\delta u_1,v) \leq \delta^{r_1}\lambda f(x,u_1,v)\\ &\leq \delta^{r_1}\lambda f(x,u_1,\delta v_1) \leq \delta^{r_1+r_2}\lambda f(x,u_1,v_1)\end{aligned}$$

which gives

$$-\Delta_p u \leq -\Delta_p\left(\delta^{\frac{r_1+r_2}{p-1}} u_1\right),$$

which yields, by the comparison principle, that

$$u \leq \delta^{\frac{r_1+r_2}{p-1}} u_1. \tag{5}$$

Using (5) in the equation for v we get:

$$\begin{aligned}-\Delta_q v = \mu g(x,u,v) &\leq \mu g\left(x, \delta^{\frac{r_1+r_2}{p-1}} u_1, v\right) \leq \delta^{\frac{r_1+r_2}{p-1}s_1}\mu g(x,u_1,v)\\ &\leq \delta^{\frac{r_1+r_2}{p-1}s_1}\mu g(x,u_1,\delta v_1) \leq \delta^{\frac{r_1+r_2}{p-1}s_1+s_2}\mu g(x,u_1,v_1)\end{aligned}$$

which gives

$$-\Delta_q v \leq -\Delta_q\left(\delta^{\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)}} v_1\right)$$

which yields, by the comparison principle, that

$$v \leq \delta^{\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)}} v_1, \tag{6}$$

contradicting the definition of δ due to $\frac{r_1+r_2}{p-1} < 1$ and $\frac{(r_1+r_2)s_1+s_2(p-1)}{(p-1)(q-1)} < 1$.

Let (iii) holds, then

$$\begin{aligned}-\Delta_q v = \mu g(x,u,v) &\leq \mu g(x,\delta u_1,v) \leq \delta^{s_1}\mu g(x,u_1,v)\\ &\leq \delta^{s_1}\mu g(x,u_1,\delta v_1) \leq \delta^{s_1+s_2}\mu g(x,u_1,v_1),\end{aligned}$$

which gives

$$-\Delta_q v \leq -\Delta_q\left(\delta^{\frac{s_1+s_2}{q-1}} v_1\right)$$

which yields, by the comparison principle, that

$$v \leq \delta^{\frac{s_1+s_2}{q-1}} v_1, \tag{7}$$

In view of inequalities (5) and (7) we have a contradiction with the definition of δ.

Let (iii) holds, then working as (5) and (7) we get

$$\begin{aligned}-\Delta_p u &\leq \lambda f(x, \delta^{\frac{r_1+r_2}{p-1}} u_1, \delta^{\frac{s_1+s_2}{q-1}} v_1)\\ &\leq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(q-1)(p-1)}}\lambda f(x,u_1,v_1).\end{aligned}$$

Thus,

$$u \leq \delta^{\frac{(r_1+r_2)(q-1)r_1+(s_1+s_2)(p-1)r_2}{(p-1)^2(q-1)}} u_1$$

contradicting the definition of δ.

Thus $\delta = 1$, i.e., $v \leq v_1$ and $u \leq u_1$. Similarly, $v \geq v_1$ and $u \geq u_1$. Consequently, $u = u_1$ and $v = v_1$.

4 Some applications

In this section, we will give some examples to demonstate our results.

Example 1. Consider the following problem:

$$-\Delta_p u = \lambda v^{\alpha} \text{ in } \Omega,\ -\Delta_q u = \mu u^{\beta} \text{ in } \Omega,\ u = v = 0 \text{ on } \partial\Omega \tag{8}$$

where $2 > \alpha, \beta \geq 0$. Then, there exist numbers λ_0, μ_0 such that (8) has a positive solution for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$.

Example 2. Consider the following problem:

$$-\Delta_p u = \lambda \left(u^{\alpha} + v^{\beta}\right) \text{ in } \Omega,\ -\Delta_q u = \mu \left(u^{\gamma} + v^{\delta}\right) \text{ in } \Omega,\ u = v = 0 \text{ on } \partial\Omega \tag{9}$$

where $1 > \alpha, \beta, \gamma, \delta \geq 0$. Then, there exist numbers λ_0, μ_0 such that (9) has a positive solution for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$.

Example 3. Consider the following problem:

$$-\Delta_p u = \lambda u^{\alpha} v^{\beta} \text{ in } \Omega,\ -\Delta_q u = \mu u^{\gamma} v^{\delta} \text{ in } \Omega,\ u = v = 0 \text{ on } \partial\Omega \tag{10}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ satisfying $\alpha + \beta < 1$ and $\gamma + \delta < 1$. Then, there exist numbers λ_0, μ_0 such that (10) has a positive solution for $0 < \lambda < \lambda_0$ and $0 < \mu < \mu_0$.

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