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Ostrowski Type Inequalities on Time Scales for Double Integrals

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Abstract In this paper we first derive an Ostrowski type inequality on time scales for double integrals via $\Delta\Delta$-integral which unify corresponding continuous and discrete versions. We then replace the $\Delta\Delta$-integral by the $\nabla\nabla$-, $\Delta\nabla$-, and $\nabla\Delta$-integrals and get completely analogous results.

Keywords Ostrowski inequality · Double integrals · Time scales

Mathematics Subject Classification (2000) 26D15 · 39A10 · 39A12 · 39A13

1 Introduction

In 1938, Ostrowski [27] proved the following interesting integral inequality which has received considerable attention from many researchers [11, 12, 21, 22, 25, 26, 32].

Theorem 1 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *whose derivative function* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b), *i.e.*, $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$.

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Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty \tag{1}$$

for all $x \in [a, b]$.

In [13], Dragomir et al. proved the following Ostrowski type inequality for double integrals.

Theorem 2 *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ *exist and are continuous on* $[a, b] \times [c, d]$. *Then*

$$\begin{aligned} &\left| f(x, y) - \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(t, s)dtds \right| \\ &\le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\left\|\frac{\partial f}{\partial t}\right\|_\infty + \left(\frac{1}{4} + \frac{\left(y - \frac{c+d}{2}\right)^2}{(d-c)^2} \right)(d-c)\left\|\frac{\partial f}{\partial s}\right\|_\infty \\ &\quad + \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)\left(\frac{1}{4} + \frac{\left(y - \frac{c+d}{2}\right)^2}{(d-c)^2} \right)(b-a)(d-c)\left\|\frac{\partial^2 f}{\partial t \partial s}\right\|_\infty \end{aligned} \tag{2}$$

for all $(x, y) \in [a, b] \times [c, d]$.

The development of the theory of time scales was initiated by Hilger [15] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied certain integral inequalities or dynamic equations on time scales [1, 7, 8, 16, 28–31, 36]. In [8], Bohner and Matthews established the following so-called Ostrowski inequality on time scales which was later generalized by the present authors [18–20, 23].

Theorem 3 (See [8], Theorem 3.5) *Let* $a, b, x, t \in \mathbb{T}$, $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be differentiable. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f^\sigma(t)\Delta t \right| \le \frac{M}{b-a}\Big(h_2(x, a) + h_2(x, b)\Big), \tag{3}$$

where $M = \sup_{a<x<b} |f^\Delta(x)|$ (*see Definition* 3 *below for* $h_2(\cdot, \cdot)$). *This inequality is sharp in the sense that the right-hand side of* (3) *can't be replaced by a smaller one*.

In the present paper, we shall first generalize the above Ostrowski inequality on time scales for double integrals via $\Delta\Delta$-integral which unify corresponding continuous and discrete versions. We then replace the $\Delta\Delta$-integral by the $\nabla\nabla$-, $\Delta\nabla$-, and $\nabla\Delta$-integrals and get completely analogous results.

This paper is organized as follows. In Sect. 2, we give a brief introduction into the (one-variable) time scales theory. In Sect. 3, we recall in brief the so-called Riemann $\Delta\Delta$-integrals and develop Riemann $\nabla\nabla$-integrals, Riemann $\Delta\nabla$-integrals and Riemann $\nabla\Delta$-integrals. In Sect. 4, we first present an Ostrowski inequality on time scales for double integrals via $\Delta\Delta$-integral, then get completely analogous results via the $\nabla\nabla$-, $\Delta\nabla$-, and $\nabla\Delta$-integrals. In the last section, we indicate our further works on deducing our results to the double $\diamond$-integral.

2 The One-Variable Time Scales Theory

Now we briefly introduce the (one-variable) time scales theory and refer the reader to Hilger [15] and the books [9, 10, 17] for further details.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$, and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$, and $\sigma(\inf \mathbb{T}) = \inf \mathbb{T}$. If $\sigma(t) > t$, then we say that t is *right-scattered*, while if $\rho(t) < t$ then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, t is called *right-dense*, and if $\rho(t) = t$ then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called dense. Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ satisfying $\mu(t) := \sigma(t) - t$, $(t) := t - \rho(t)$ are called forward and backward *graininess functions*, respectively.

We now introduce the set $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$, which are derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa := \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_\kappa := \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa := \mathbb{T}$. Finally, we define $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. Given a function $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$ and define the function $f^\rho : \mathbb{T} \to \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$, for some $\delta > 0$) such that for all $s \in U$

$$\left| f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right|.$$

We say that f is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Similarly, for $t \in \mathbb{T}_\kappa$, we define $f^\nabla(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood V of t (i.e., $V = (t - \delta, t + \delta) \cap \mathbb{T}$, for some $\delta > 0$) such that for all $s \in V$

$$\left| f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s) \right| \leq \varepsilon \left| \rho(t) - s \right|.$$

We say that f is ∇-differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist at all right-dense points in $\mathbb{T}$.

Definition 1 A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the Δ-integral of f is defined by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

Definition 2 A function $G : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $g : \mathbb{T} \to \mathbb{R}$ provided $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_\kappa$. Then the ∇-integral of g is defined by

$$\int_a^b g(t)\,\nabla t = G(b) - G(a).$$

Definition 3 Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t,s) = 1 \quad \text{for all } s,t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau \quad \text{for all } s,t \in \mathbb{T}.$$

Definition 4 Let $j_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$j_0(t,s) = 1 \quad \text{for all } s,t \in \mathbb{T}$$

and then recursively by

$$j_{k+1}(t,s) = \int_s^t j_k(\tau,s)\,\nabla\tau \quad \text{for all } s,t \in \mathbb{T}.$$

3 The Two-Variable Time Scales Theory

The two-variable time scales calculus and multiple integration on time scales were introduced in [4, 5] (see also [6]). Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$, which is a complete metric space with the metric d defined by

$$d\left((x,y),\left(x',y'\right)\right) = \sqrt{(x-x')^2 + (y-y')^2}, \quad \forall (x,y),\left(x',y'\right) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

For a given $\delta > 0$, the δ-neighborhood $U_\delta(x_0,y_0)$ of a given point $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((x,y),(x_0,y_0)) < \delta$. Let σ_1, ρ_1 and σ_2, ρ_2 be the forward jump and backward jump operators in $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively.

In the first part of this section, followed from [5], we recall in brief the so-called Riemann $\Delta\Delta$-integrals. Then followed by a note in [5] (see also in [6]), we develop Riemann $\nabla\nabla$-integrals, Riemann $\Delta\nabla$-integrals and Riemann $\nabla\Delta$-integrals.

3.1 Riemann $\Delta\Delta$-Integrals

In this subsection, we will recall the so-called double delta integrals from [5, Sects. 2 and 3].

Definition 5 The first order partial delta derivatives of $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ at a point $(x_0,y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ are defined to be

$$\frac{\partial f(x_0,y_0)}{\Delta_1 x} = \lim_{x \to x_0, x \neq \sigma_1(x_0)} \frac{f(\sigma_1(x_0),y_0) - f(x,y_0)}{\sigma_1(x_0) - x},$$
$$\frac{\partial f(x_0,y_0)}{\Delta_2 y} = \lim_{y \to y_0, y \neq \sigma_2(y_0)} \frac{f(x_0,\sigma_2(y_0)) - f(x_0,y)}{\sigma_2(y_0) - y}.$$

Then, the authors defined the second order partial derivatives

$$\frac{\partial^2 f(x,y)}{\Delta_1 x^2} = \frac{\partial}{\Delta_1 x}\left(\frac{\partial f(x,y)}{\Delta_1 x}\right) \quad \text{and} \quad \frac{\partial^2 f(x,y)}{\Delta_2 y \Delta_1 x} = \frac{\partial}{\Delta_2 y}\left(\frac{\partial f(x,y)}{\Delta_1 x}\right).$$

Next, we recall the so-called double Riemann Δ-integrals (which will be denoted by $\Delta\Delta$-integrals) over regions $\mathbb{T}_1 \times \mathbb{T}_2$ and present some properties of it over rectangles. Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half-closed bounded interval in $\mathbb{T}_2$.

Let us introduce a $\Delta\Delta$-rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ by $R_{\Delta\Delta} = [a, b) \times [c, d) = \{(t, s) : t \in [a, b), s \in [c, d)\}$. Let

$$\{x_0, x_1, \ldots, x_n\} \subset [a, b], \quad \text{where } a = x_0 < x_1 < \cdots < x_n = b$$

and

$$\{y_0, y_1, \ldots, y_k\} \subset [c, d], \quad \text{where } c = y_0 < y_1 < \cdots < y_k = d.$$

We call the collection of intervals $P_1 = \{[x_{i-1}, x_i) : 1 \le i \le n\}$ a Δ-partition of $[a, b)$ and denote the set of all Δ-partitions of $[a, b)$ by $\mathcal{P}_\Delta([a, b))$. Similarly, the collection of intervals $P_2 = \{[y_{i-1}, y_i) : 1 \le i \le k\}$ is called a Δ-partition of $[c, d)$ and the set of all Δ-partitions of $[c, d)$ is denoted by $\mathcal{P}_\Delta([c, d))$. Set

$$R_{ij} = [x_{i-1}, x_i) \times [y_{j-1}, y_j), \quad \text{where } 1 \le i \le n,\ j \le 1 \le k.$$

We call the collection $P_{\Delta\Delta} = \{R_{ij} : 1 \le i \le n, 1 \le j \le k\}$ a $\Delta\Delta$-partition of $R_{\Delta\Delta}$, generated by the Δ-partition $P_1 = \{[x_{i-1}, x_i) : 1 \le i \le n\}$ and Δ-partition $P_2 = \{[y_{i-1}, y_i) : 1 \le j \le k\}$ of $[a, b)$ and $[c, d)$, respectively, and write $P_{\Delta\Delta} = P_1 \times P_2$. The rectangles R_{ij}, $1 \le i \le n$, $1 \le j \le k$, are called the subrectangles of the partition $P_{\Delta\Delta}$. The set of all $\Delta\Delta$-partitions of $R_{\Delta\Delta}$ is denoted by $\mathcal{P}_{\Delta\Delta}(R)$.

We need the following auxiliary result. See [10, Lemma 5.7] for the proof.

Lemma 1 *For any $\delta > 0$ there exists at least one $P_1 \in \mathcal{P}_\Delta([a, b))$ generated by a set $\{x_0, x_1, \ldots, x_n\} \subset [a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$ so that for each $i \in \{1, 2, \ldots, n\}$ either $x_i - x_{i-1} \leqq \delta$ or $x_i - x_{i-1} > \delta$ and $\sigma_1(x_{i-1}) = x_i$.*

We denote by $(\mathcal{P}_\Delta)_\delta([a, b))$ the set of all $P_1 \in \mathcal{P}_\Delta([a, b))$ that possess the property indicated in Lemma 1. Similarly, we define $(\mathcal{P}_\Delta)_\delta([c, d))$. Further, by $(\mathcal{P}_{\Delta\Delta})_\delta(R)$ we denote the set of all $P_{\Delta\Delta} \in \mathcal{P}_{\Delta\Delta}(R)$ such that $P_{\Delta\Delta} = P_1 \times P_2$ where $P_1 \in (\mathcal{P}_\Delta)_\delta([a, b))$ and $P_2 \in (\mathcal{P}_\Delta)_\delta([c, d))$.

Definition 6 Let f be a bounded function on R and $P \in \mathcal{P}_{\Delta\Delta}(R)$ be given as above. In each rectangle R_{ij} with $1 \le i \le n, 1 \le j \le k$, choose an arbitrary point (ξ_{ij}, η_{ij}) and form the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{k} f\left(\xi_{ij}, \eta_{ij}\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right).$$

We call S a Riemann $\Delta\Delta$-sum of f corresponding to $P \in \mathcal{P}_{\Delta\Delta}(R)$.

Definition 7 We say that f is Riemann $\Delta\Delta$-integrable over R if there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann $\Delta\Delta$-sum S of f corresponding to any $P \in (\mathcal{P}_{\Delta\Delta})_\delta(R)$ independent of the way in which we choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ for $1 \le i \le n, 1 \le j \le k$. The number I is the Riemann $\Delta\Delta$-integral of f over R, denoted by

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y.$$

We write $I = \lim_{\delta\to 0} S$.

It is worth recalling from [5, Theorems 3.4 and 3.10] the following propositions.

Proposition 1 (Linearity) *Let f, g be $\Delta\Delta$-integrable functions on $R = [a, b) \times [c, d)$ and let $\alpha, \beta \in \mathbb{R}$. Then*

$$\iint_R [\alpha f(x, y) + \beta g(x, y)]\Delta_1 x\Delta_2 y = \alpha \iint_R f(x, y)\Delta_1 x\Delta_2 y + \beta \iint_R g(x, y)\Delta_1 x\Delta_2 y.$$

An effective way for evaluating multiple integrals is to reduce them to iterated (successive) integrations with respect to each of the variables.

Proposition 2 *Let f be $\Delta\Delta$-integrable on $R = [a, b) \times [c, d)$ and suppose that the single integral $I(x) = \int_c^d f(x, y)\Delta_2 y$ exists for each $x \in [a, b)$. Then the iterated integral $\int_a^b I(x)\Delta_1 x$ exists, and*

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y = \int_a^b \Delta_1 x \int_c^d f(x, y)\Delta_2 y.$$

Remark 1 The notation $\Delta\Delta$ means that we take the first Δ as the differentiation of the first variable of function under the integral sign and then we take the second Δ as the differentiation of the second variable of function under the integral sign. In the following parts of this section, for example, by $\Delta\nabla$-integral, we mean that we take the first Δ as the differentiation of the first variable of function under the integral sign and then we take the second ∇ as the differentiation of the second variable of function under the integral sign.

3.2 Riemann $\nabla\nabla$-Integrals

Riemann $\nabla\nabla$-integrals can be defined similarly to Riemann $\Delta\Delta$-integrals as following.

Definition 8 The first order partial nabla derivatives of $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ at a point $(x_0, y_0) \in (\mathbb{T}_1)_\kappa \times (\mathbb{T}_2)_\kappa$ are defined to be

$$\frac{\partial f(x_0, y_0)}{\nabla_1 x} = \lim_{x\to x_0, x\neq\rho_1(x_0)} \frac{f(\rho_1(x_0), y_0) - f(x, y_0)}{\rho_1(x_0) - x},$$
$$\frac{\partial f(x_0, y_0)}{\nabla_2 y} = \lim_{y\to y_0, y\neq\rho_2(y_0)} \frac{f(x_0, \rho_2(y_0)) - f(x_0, y)}{\rho_2(y_0) - y}.$$

Then, we define the second order partial nabla derivatives as follows

$$\frac{\partial^2 f(x, y)}{\nabla_1 x^2} = \frac{\partial}{\nabla_2 x}\left(\frac{\partial f(x, y)}{\nabla_1 x}\right) \quad\text{and}\quad \frac{\partial^2 f(x, y)}{\nabla_2 y\nabla_1 x} = \frac{\partial}{\nabla_2 y}\left(\frac{\partial f(x, y)}{\nabla_1 x}\right).$$

Next, we define the double Riemann $\nabla\nabla$-integrals (which will be called by $\nabla\nabla$-integrals) over regions $\mathbb{T}_1 \times \mathbb{T}_2$ and present some properties of it over rectangles. Suppose

$a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $(a, b]$ is the half-closed bounded interval in $\mathbb{T}_1$, and $(c, d]$ is the half-closed bounded interval in $\mathbb{T}_2$.

Let us introduce a $\nabla\nabla$-rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ by $R_{\nabla\nabla} = (a, b] \times (c, d] = \{(t, s) : t \in (a, b], s \in (c, d]\}$. Let

$$\{x_0, x_1, \ldots, x_n\} \subset [a, b], \quad \text{where } a = x_0 < x_1 < \cdots < x_n = b$$

and

$$\{y_0, y_1, \ldots, y_k\} \subset [c, d], \quad \text{where } c = y_0 < y_1 < \cdots < y_k = d.$$

We call the collection of intervals $P_1 = \{(x_{i-1}, x_i] : 1 \leq i \leq n\}$ a ∇-partition of $(a, b]$ and denote the set of all ∇-partitions of $(a, b]$ by $\mathcal{P}_\nabla((a, b])$. Similarly, the collection of intervals $P_2 = \{(y_{i-1}, y_i] : 1 \leq i \leq k\}$ is called a ∇-partition of $(c, d]$ and the set of all ∇-partitions of $(c, d]$ is denoted by $\mathcal{P}_\nabla((c, d])$. Set

$$R_{ij} = \left(x_{i-1}, x_i\right] \times \left(y_{j-1}, y_j\right], \quad \text{where } 1 \leq i \leq n,\, j \leq 1 \leq k.$$

We call the collection $P_{\nabla\nabla} = \{R_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ a $\nabla\nabla$-partition of $R_{\nabla\nabla}$, generated by the ∇-partition $P_1 = \{(x_{i-1}, x_i] : 1 \leq i \leq n\}$ and ∇-partition $P_2 = \{(y_{i-1}, y_i] : 1 \leq j \leq k\}$ of $(a, b]$ and $(c, d]$, respectively, and write $P_{\nabla\nabla} = P_1 \times P_2$. The rectangles R_{ij}, $1 \leq i \leq n$, $1 \leq j \leq k$, are called the subrectangles of the partition P. The set of all $\nabla\nabla$-partitions of $R_{\nabla\nabla}$ is denoted by $\mathcal{P}_{\nabla\nabla}(R)$.

Similar to Lemma 1, we obtain the following lemma.

Lemma 2 *For any $\delta > 0$ there exists at least one $P_1 \in \mathcal{P}_\nabla((a, b])$ generated by a set $\{x_0, x_1, \ldots, x_n\} \subset [a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$ so that for each $i \in \{1, 2, \ldots, n\}$ either $x_i - x_{i-1} \leqq \delta$ or $x_i - x_{i-1} > \delta$ and $\rho_1(x_i) = x_{i-1}$.*

We denote by $(\mathcal{P}_\nabla)_\delta((a, b])$ the set of all $P_1 \in \mathcal{P}_\nabla((a, b])$ that possess the property indicated in Lemma 2. Similarly, we define $(\mathcal{P}_\nabla)_\delta((c, d])$. Further, by $(\mathcal{P}_{\nabla\nabla})_\delta(R)$ we denote the set of all $P_{\nabla\nabla} \in \mathcal{P}_{\nabla\nabla}(R)$ such that $P_{\nabla\nabla} = P_1 \times P_2$ where $P_1 \in (\mathcal{P}_\nabla)_\delta((a, b])$ and $P_2 \in (\mathcal{P}_\nabla)_\delta((c, d])$.

Definition 9 Let f be a bounded function on R and $P \in \mathcal{P}_{\nabla\nabla}(R)$ be given as above. In each rectangle R_{ij} with $1 \leq i \leq n, 1 \leq j \leq k$, choose an arbitrary point (ξ_{ij}, η_{ij}) and form the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{k} f\left(\xi_{ij}, \eta_{ij}\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right).$$

We call S a Riemann $\nabla\nabla$-sum of f corresponding to $P \in \mathcal{P}_{\nabla\nabla}(R)$.

Definition 10 We say that f is Riemann $\nabla\nabla$-integrable over R if there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann $\nabla\nabla$-sum S of f corresponding to any $P \in (\mathcal{P}_{\nabla\nabla})_\delta(R)$ independent of the way in which we choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq k$. The number I is the double Riemann ∇-integral of f over R, denoted by

$$\iint_R f(x, y)\nabla_1 x \nabla_2 y.$$

We write $I = \lim_{\delta \to 0} S$.

Similarly to Proposition 1, we obtain

Proposition 3 (Linearity) *Let f, g be $\nabla\nabla$-integrable functions on $R = (a, b] \times (c, d]$ and let $\alpha, \beta \in \mathbb{R}$. Then*

$$\iint_R [\alpha f(x, y) + \beta g(x, y)]\nabla_1 x\nabla_2 y = \alpha \iint_R f(x, y)\nabla_1 x\nabla_2 y + \beta \iint_R g(x, y)\nabla_1 x\nabla_2 y.$$

An effective way for evaluating $\nabla\nabla$-integrals is to reduce them to iterated (successive) integrations with respect to each of the variables which can be proved similarly to Proposition 2.

Proposition 4 *Let f be $\nabla\nabla$-integrable on $R = (a, b] \times (c, d]$ and suppose that the single integral $I(x) = \int_c^d f(x, y)\nabla_2 y$ exists for each $x \in [a, b)$. Then the iterated integral $\int_a^b I(x)\nabla x$ exists, and*

$$\iint_R f(x, y)\nabla_1 x\nabla_2 y = \int_a^b \nabla_1 x \int_c^d f(x, y)\nabla_2 y.$$

In the next subsection, we can define $\Delta\nabla$-integral over $[a, b) \times (c, d]$ by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times (\gamma, \delta]$.

3.3 Riemann $\Delta\nabla$-Integrals

Riemann $\Delta\nabla$-integrals can be defined similarly to Riemann $\Delta\Delta$-integrals as following.

Definition 11 The first order partial nabla derivatives of $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ at a point $(x_0, y_0) \in (\mathbb{T}_1)_\kappa \times (\mathbb{T}_2)_\kappa$ are defined to be

$$\frac{\partial f(x_0, y_0)}{\Delta_1 x} = \lim_{x\to x_0, x\neq\sigma_1(x_0)} \frac{f(\sigma_1(x_0), y_0) - f(x, y_0)}{\sigma_1(x_0) - x},$$

$$\frac{\partial f(x_0, y_0)}{\nabla_2 y} = \lim_{y\to y_0, y\neq\rho_2(y_0)} \frac{f(x_0, \rho_2(y_0)) - f(x_0, y)}{\rho_2(y_0) - y}.$$

Then, we define the following mixed derivatives obtained by combining both delta and nabla differentiations

$$\frac{\partial^2 f(x, y)}{\Delta_1 x\nabla_2 y} = \frac{\partial}{\Delta_1 x}\left(\frac{\partial f(x, y)}{\nabla_2 y}\right).$$

Next, we define the double Riemann $\Delta\nabla$-integrals (which will be called by $\Delta\nabla$-integrals) over regions $\mathbb{T}_1 \times \mathbb{T}_2$ and present some properties of it over rectangles. Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $(c, d]$ is the half-closed bounded interval in $\mathbb{T}_2$.

Let us introduce a $\Delta\nabla$-rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ by $R_{\Delta\nabla} = [a, b) \times (c, d] = \{(t, s) : t \in [a, b), s \in (c, d]\}$. Let

$$\{x_0, x_1, \dots, x_n\} \subset [a, b], \quad \text{where } a = x_0 < x_1 < \cdots < x_n = b$$

and

$$\{y_0, y_1, \dots, y_k\} \subset [c, d], \quad \text{where } c = y_0 < y_1 < \cdots < y_k = d.$$

We call the collection of intervals $P_1 = \{[x_{i-1}, x_i) : 1 \le i \le n\}$ a Δ-partition of $[a, b)$ and denote the set of all Δ-partitions of $[a, b)$ by $\mathcal{P}_\Delta([a, b))$. Similarly, the collection of intervals $P_2 = \{(y_{i-1}, y_i] : 1 \le i \le k\}$ is called a ∇-partition of $(c, d]$ and the set of all ∇-partitions of $(c, d]$ is denoted by $\mathcal{P}_\nabla((c, d])$. Set

$$R_{ij} = \left[x_{i-1}, x_i\right) \times \left(y_{j-1}, y_j\right], \quad \text{where } 1 \le i \le n,\ j \le 1 \le k.$$

We call the collection $P_{\Delta\nabla} = \{R_{ij} : 1 \le i \le n, 1 \le j \le k\}$ a $\Delta\nabla$-partition of $R_{\Delta\nabla}$, generated by the Δ-partition $P_1 = \{[x_{i-1}, x_i) : 1 \le i \le n\}$ and ∇-partition $P_2 = \{(y_{i-1}, y_i] : 1 \le j \le k\}$ of $[a, b)$ and $(c, d]$, respectively, and write $P_{\Delta\nabla} = P_1 \times P_2$. The rectangles R_{ij}, $1 \le i \le n$, $1 \le j \le k$, are called the subrectangles of the partition P. The set of all $\Delta\nabla$-partitions of $R_{\Delta\nabla}$ is denoted by $\mathcal{P}_{\Delta\nabla}(R)$.

We denote by $(\mathcal{P}_\Delta)_\delta([a, b))$ the set of all $P_1 \in \mathcal{P}_\Delta([a, b))$ that possess the property indicated in Lemma 1. Similarly, we define $(\mathcal{P}_\nabla)_\delta((c, d])$ to be the set of all $P_2 \in \mathcal{P}_\nabla((c, d])$ that possess the property indicated in Lemma 2. Further, by $(\mathcal{P}_{\Delta\nabla})_\delta(R)$ we denote the set of all $P_{\Delta\nabla} \in \mathcal{P}_{\Delta\nabla}(R)$ such that $P_{\Delta\nabla} = P_1 \times P_2$ where $P_1 \in (\mathcal{P}_\Delta)_\delta([a, b))$ and $P_2 \in (\mathcal{P}_\nabla)_\delta([c, d))$.

Definition 12 Let f be a bounded function on R and $P \in \mathcal{P}_{\Delta\nabla}(R)$ be given as above. In each rectangle R_{ij} with $1 \le i \le n, 1 \le j \le k$, choose an arbitrary point (ξ_{ij}, η_{ij}) and form the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{k} f\left(\xi_{ij}, \eta_{ij}\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right).$$

We call S a Riemann $\Delta\nabla$-sum of f corresponding to $P \in \mathcal{P}_{\Delta\nabla}(R)$.

Definition 13 We say that f is Riemann $\Delta\nabla$-integrable over R if there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann $\Delta\nabla$-sum S of f corresponding to any $P \in (\mathcal{P}_{\Delta\nabla})_\delta(R)$ independent of the way in which we choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ for $1 \le i \le n, 1 \le j \le k$. The number I is the double Riemann $\Delta\nabla$-integral of f over R, denoted by

$$\iint_R f(x, y)\Delta_1 x \nabla_2 y.$$

We write $I = \lim_{\delta \to 0} S$.

Proposition 5 (Linearity) *Let f, g be $\Delta\nabla$-integrable functions on $R = [a, b) \times (c, d]$ and let $\alpha, \beta \in \mathbb{R}$. Then*

$$\iint_R [\alpha f(x, y) + \beta g(x, y)]\Delta_1 x \nabla_2 y = \alpha \iint_R f(x, y)\Delta_1 x \nabla_2 y + \beta \iint_R g(x, y)\Delta_1 x \nabla_2 y.$$

An effective way for evaluating $\Delta\nabla$-integrals is to reduce them to iterated (successive) integrations with respect to each of the variables which can be proved similarly to Proposition 2.

Proposition 6 *Let f be $\Delta\nabla$-integrable on $R = [a, b) \times (c, d]$ and suppose that the single integral $I(x) = \int_c^d f(x, y)\nabla_2 y$ exists for each $x \in [a, b)$. Then the iterated integral $\int_a^b I(x)\nabla x$*

exists, and

$$\iint_R f(x,y)\Delta_1 x\nabla_2 y = \int_a^b \Delta_1 x \int_c^d f(x,y)\nabla_2 y.$$

3.4 Riemann $\nabla\Delta$-Integrals

Riemann $\nabla\Delta$-integrals which is denoted by

$$\iint_R f(x,y)\nabla_1 x\Delta_2 y$$

can be defined similarly to Riemann $\Delta\nabla$-integrals where R is a $\nabla\Delta$-rectangle of the form $(a,b]\times[c,d)$. We omit it in details.

4 Ostrowski's Inequality on time Scales for Double Integrals

In this section, we suppose that

(a) $\mathbb{T}_1$ is a time scale, $a<b$ are points in $\mathbb{T}_1$;
(b) $\mathbb{T}_2$ is a time scale, $c<d$ are points in $\mathbb{T}_2$.

4.1 Ostrowski's Inequality for Double Integrals via $\Delta\Delta$-Integral

We first derive the following Ostrowski type inequality on time scales for double integrals via $\Delta\Delta$-integral.

Theorem 4 *Let $x,t\in\mathbb{T}_1$, $y,s\in\mathbb{T}_2$ and $f:[a,b]\times[c,d]\to\mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$, $\frac{\partial f(t,s)}{\Delta_2 s}$, $\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}$ exist and are continuous on $[a,b]\times[c,d]$. Then*

$$\begin{aligned}\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s\right|\\ \le \frac{M_1}{b-a}\Big(h_2(x,a)+h_2(x,b)\Big)+\frac{M_2}{d-c}\Big(h_2(y,c)+h_2(y,d)\Big)\\ +\frac{M_3}{(b-a)(d-c)}\Big(h_2(x,a)+h_2(x,b)\Big)\Big(h_2(y,c)+h_2(y,d)\Big) \qquad (4)\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$M_1=\sup_{a<t<b}\left|\frac{\partial f(t,s)}{\Delta_1 t}\right|,\qquad M_2=\sup_{c<s<d}\left|\frac{\partial f(t,s)}{\Delta_2 s}\right|,\qquad M_3=\sup_{a<t<b,c<s<d}\left|\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\right|.$$

To prove Theorem 4, we need the following two Lemmas. This is partly motivated by the ideas of Dragomir et al. in [13].

Lemma 3 (Montgomery Identity, see [8]) *Let $\alpha,\beta,u,z\in\mathbb{T}$, $\alpha<\beta$ and $g:[\alpha,\beta]\to\mathbb{R}$ be differentiable. Then*

$$g(u)=\frac{1}{\beta-\alpha}\int_\alpha^\beta g^\sigma(z)\Delta z+\frac{1}{\beta-\alpha}\int_\alpha^\beta k(u,z)g^\Delta(z)\Delta z, \qquad (5)$$

where

$$k(u,z)=\begin{cases} z-\alpha, & \alpha\le z<u,\\ z-\beta, & u\le z\le \beta.\end{cases}$$

Lemma 4 *Under the assumptions of Theorem 4, we have*

$$\begin{aligned} f(x,y)&=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s\\ &\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_1 t\Delta_2 s\\ &\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\Delta_1 t\Delta_2 s\\ &\quad+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_1 t\Delta_2 s, \end{aligned}\tag{6}$$

where

$$p(x,t)=\begin{cases} t-a, & a\le t<x,\\ t-b, & x\le t\le b,\end{cases}\qquad q(y,s)=\begin{cases} s-c, & c\le s<y,\\ s-d, & y\le s\le d.\end{cases}$$

Proof Applying Lemma 3 for the partial delta map $f(\cdot,y)$, $y\in[c,d]$, we obtain

$$f(x,y)=\frac{1}{b-a}\int_a^b f(\sigma_1(t),y)\Delta_1 t+\frac{1}{b-a}\int_a^b p(x,t)\frac{\partial f(t,y)}{\Delta_1 t}\Delta_1 t \tag{7}$$

for all $(x,y)\in[a,b]\times[c,d]$. Also, if we apply Lemma 3 for the partial delta map $f(\sigma_1(t),\cdot)$, $t\in[a,b]$, we obtain

$$f(\sigma_1(t),y)=\frac{1}{d-c}\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_2 s+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\Delta_2 s \tag{8}$$

for all $(t,y)\in[a,b]\times[c,d]$. Applying the same Lemma 3 for the partial delta map $\frac{\partial f(t,\cdot)}{\Delta_1 t}$, we have

$$\frac{\partial f(t,y)}{\Delta_1 t}=\frac{1}{d-c}\int_c^d \frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_2 s+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_2 s \tag{9}$$

for all $(t,y)\in[a,b]\times[c,d]$. Substituting (8) and (9) in (7), we have

$$\begin{aligned} f(x,y)&=\frac{1}{b-a}\int_a^b\biggl[\frac{1}{d-c}\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_2 s\\ &\quad+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\Delta_2 s\biggr]\Delta_1 t\\ &\quad+\frac{1}{b-a}\int_a^b p(x,t)\biggl[\frac{1}{d-c}\int_c^d \frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_2 s \end{aligned}$$

$$+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_2 s\Bigg]\Delta_1 t$$

$$=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\Delta_1 t\Delta_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_1 t\Delta_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_1 t\Delta_2 s,$$

i.e. (6) holds. □

If we apply Lemma 4 to the discrete and continuous cases, we have the following results.

Corollary 1 (Continuous case) *Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Then*

$$f(x,y)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dtds$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dtds$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dtds$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dtds,$$

which is exactly the integral identity shown in Theorem 2 *of* [13].

Corollary 2 (Discrete case) *Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $a=0$, $b=n$, $c=0$, $d=m$, $x=i$, $y=j$, $t=k$, $s=l$, and $f(p,q)=x_p y_q$. Then*

$$x_i y_j=\frac{1}{mn}\sum_{k=1}^n\sum_{l=1}^m x_k y_l+\frac{1}{mn}\sum_{k=1}^n\sum_{l=1}^m p(i,k)y_l\Delta x_k+\frac{1}{mn}\sum_{k=1}^n\sum_{l=1}^m q(j,l)x_k\Delta y_l$$
$$+\frac{1}{mn}\sum_{k=1}^n\sum_{l=1}^m p(i,k)q(j,l)\Delta x_k\Delta y_l,$$

where

$$p(i,0)=0,$$

$$p(1,k) = k - n, \quad for\ 1 \le k \le n-1;$$

$$p(n,k) = k, \quad for\ 0 \le k \le n-1;$$

$$p(i,k) = \begin{cases} k, & 0 \le k < i, \\ k - n, & i \le k \le n-1, \end{cases}$$

and

$$q(j,0) = 0;$$

$$q(1,l) = l - n, \quad for\ 1 \le l \le m-1;$$

$$q(m,l) = l, \quad for\ 0 \le l \le m-1;$$

$$q(j,l) = \begin{cases} l, & 0 \le l < j, \\ l - m, & j \le l \le m-1. \end{cases}$$

Corollary 3 (Quantum calculus case) *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ *with* $q_1 > 1$ *and* $q_2 > 1$. *Suppose* $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ *for some* $i < j$ *and* $k < l$. *Then*

$$\begin{aligned} f(q_1^m, q_2^n) &= \frac{\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} q_1^r q_2^s f(q_1^{r+1}, q_2^{s+1})}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \\ &+ \frac{\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} (\frac{f(q_1^{r+1}, q_2^{s+1}) - f(q_1^r, q_2^{s+1})}{(q_1-1)q_1^r}) p(q_1^m, q_1^r)}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \\ &+ \frac{\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} (\frac{f(q_1^{r+1}, q_2^{s+1}) - f(q_1^{r+1}, q_2^s)}{(q_2-1)q_2^s}) q(q_2^n, q_2^s)}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \\ &+ \frac{\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} (\frac{f(q_1^{r+1}, q_2^{s+1}) - f(q_1^r, q_2^{s+1}) - f(q_1^{r+1}, q_2^s) + f(q_1^r, q_2^s)}{(q_1-1)(q_2-1)q_1^r q_2^s}) p(q_1^m, q_1^r) q(q_2^n, q_2^s)}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s}. \end{aligned}$$

Proof of Theorem 4 By applying Lemma 4, we can state that

$$\begin{aligned} &\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\sigma_1(t), \sigma_2(s)) \Delta_1 t \Delta_2 s \right| \\ &\le \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) \frac{\partial f(t, \sigma_2(s))}{\Delta_1 t} \Delta_1 t \Delta_2 s \right. \\ &\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d q(y,s) \frac{\partial f(\sigma_1(t), s)}{\Delta_2 s} \Delta_1 t \Delta_2 s \\ &\quad \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \Delta_1 t \Delta_2 s \right| \\ &\le \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d |p(x,t)| \left| \frac{\partial f(t, \sigma_2(s) s)}{\Delta_1 t} \right| \Delta_1 t \Delta_2 s \end{aligned}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |q(y,s)|\left|\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\right|\Delta_1 t\Delta_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |p(x,t)||q(y,s)|\left|\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\right|\Delta_1 t\Delta_2 s$$
$$\le \frac{M_1}{b-a}\int_a^b |p(x,t)|\Delta_1 t+\frac{M_2}{d-c}\int_c^d |q(y,s)|\Delta_2 s$$
$$+\frac{M_3}{(b-a)(d-c)}\int_a^b |p(x,t)|\Delta_1 t\int_c^d |q(y,s)|\Delta_2 s$$

for all $(x,y)\in[a,b]\times[c,d]$. By a simple calculation we get

$$\begin{aligned}\int_a^b |p(x,t)|\Delta_1 t &= \int_a^x |t-a|\Delta_1 t+\int_x^b |t-b|\Delta_1 t\\ &= \int_a^x (t-a)\Delta_1 t+\int_x^b (b-t)\Delta_1 t\\ &= h_2(x,a)+h_2(x,b)\end{aligned}$$

and

$$\int_c^d |q(y,s)|\Delta_2 s = h_2(y,c)+h_2(y,d).$$

Therefore, we obtain (4). □

If we apply the Ostrowski type inequality for double integrals to different time scales, we will get some well-known and some new results.

Corollary 4 (Continuous case) *Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Hence,*

$$h_2(t,s)=\frac{(t-s)^2}{2},\quad \textit{for all}\ \ t,s\in\mathbb{R}.$$

This leads us to obtain exactly the Ostrowski type inequality for double integrals shown in Theorem 2.

Corollary 5 (Discrete case) *Let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $a=0$, $b=n$, $c=0$, $d=m$, $x=i$, $y=j$, $t=k$, $s=l$, and $f(p,q)=x_p y_q$. With these, it is known that*

$$h_k(t,s)=\binom{t-s}{k},\quad \textit{for all}\ \ t,s\in\mathbb{Z}.$$

Therefore,

$$h_2(x,a)=\binom{i}{2}=\frac{i(i-1)}{2},$$
$$h_2(x,b)=\binom{i-n}{2}=\frac{(i-n)(i-n-1)}{2},$$

and

$$h_2(y,c) = \binom{j}{2} = \frac{j(j-1)}{2},$$

$$h_2(y,d) = \binom{j-m}{2} = \frac{(j-m)(j-m-1)}{2}.$$

Thus, we have

$$\left| x_i y_j - \frac{1}{mn} \sum_{k=1}^{n} \sum_{l=1}^{m} x_k y_l \right| \le \frac{M_1}{n} \left(\left| i - \frac{n+1}{2} \right|^2 + \frac{n^2-1}{4} \right) + \frac{M_2}{m} \left(\left| j - \frac{m+1}{2} \right|^2 + \frac{m^2-1}{4} \right) + \frac{M_3}{mn} \left(\left| i - \frac{n+1}{2} \right|^2 + \frac{n^2-1}{4} \right) \left(\left| j - \frac{m+1}{2} \right|^2 + \frac{m^2-1}{4} \right) \tag{10}$$

for all $i = \overline{1,n}$ *and* $j = \overline{1,m}$, *where*

$$M_1 = \max_{1 \le k \le n-1} |\Delta x_k|, \qquad M_2 = \max_{1 \le l \le m-1} |\Delta y_l|, \qquad M_3 = \max_{1 \le k \le n-1, 1 \le l \le m-1} |\Delta x_k \Delta y_l|.$$

Corollary 6 (Quantum calculus case) *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ *with* $q_1 > 1$ *and* $q_2 > 1$. *Suppose* $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ *for some* $i < j$ *and* $k < l$. *In this situation, one has*

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}, \quad \text{for all } \ t, s \in \mathbb{T}.$$

Therefore,

$$h_2(x,a) = \frac{(q_1^m - q_1^i)(q_1^m - q_1^{i+1})}{1+q_1},$$

$$h_2(x,b) = \frac{(q_1^m - q_1^j)(q_1^m - q_1^{j+1})}{1+q_1},$$

and

$$h_2(y,c) = \frac{(q_2^n - q_2^k)(q_2^n - q_2^{k+1})}{1+q_2},$$

$$h_2(y,d) = \frac{(q_2^n - q_2^l)(q_2^n - q_2^{l+1})}{1+q_2}.$$

Then

$$\left| f(q_1^m, q_2^n) - \frac{\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} q_1^r q_2^s f(q_1^{r+1}, q_2^{s+1})}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \right|$$

$$
\begin{aligned}
&\leq \frac{M_1}{\sum_{r=i}^{j-1} q_1^r} \frac{(q_1^m - q_1^i)(q_1^m - q_1^{i+1}) + (q_1^m - q_1^j)(q_1^m - q_1^{j+1})}{1+q_1} \\
&\quad + \frac{M_2}{\sum_{s=k}^{l-1} q_2^s} \frac{(q_2^n - q_2^k)(q_2^n - q_2^{k+1}) + (q_2^n - q_2^l)(q_2^n - q_2^{l+1})}{1+q_2} \\
&\quad + \frac{M_3}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \left[\frac{(q_1^m - q_1^i)(q_1^m - q_1^{i+1}) + (q_1^m - q_1^j)(q_1^m - q_1^{j+1})}{1+q_1} \right. \\
&\quad \left. \times \frac{(q_2^n - q_2^k)(q_2^n - q_2^{k+1}) + (q_2^n - q_2^l)(q_2^n - q_2^{l+1})}{1+q_2} \right],
\end{aligned}
$$

where

$$
M_1 = \sup_{i<m<j-1} \left| \frac{f(q_1^{m+1}, s) - f(q_1^m, s)}{(q_1 - 1) q_1^m} \right|, \qquad M_2 = \sup_{k<n<l-1} \left| \frac{f(t, q_2^{n+1}) - f(t, q_2^n)}{(q_2 - 1) q_2^n} \right|,
$$

and

$$
M_3 = \sup_{i<m<j-1,\ k<n<l-1} \left| \frac{f(q_1^{m+1}, q_2^{n+1}) - f(q_1^m, q_2^{n+1}) - f(q_1^{m+1}, q_2^n) + f(q_1^m, q_2^n)}{(q_1 - 1)(q_2 - 1) q_1^m q_2^n} \right|.
$$

Remark 2 We note that in the special case, if $f(x, y)$ in Theorem 4 does not depend on y, we get back the Ostrowski inequality (3) on (one-variable) time scales.

4.2 Ostrowski's Inequality for Double Integrals via $\nabla\nabla$-Integral

By a completely analogous method, we can derive the following similar result via $\nabla\nabla$-integral. This would be interesting, since the calculus on time scales via ∇ derivatives seem to have many interesting application [2, 3, 24].

Theorem 5 *Let $x, t \in \mathbb{T}_1$, $y, s \in \mathbb{T}_2$ and $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\nabla_1 t}$, $\frac{\partial f(t,s)}{\nabla_2 s}$, $\frac{\partial^2 f(t,s)}{\nabla_2 s \nabla_1 t}$ exist and are continuous on $[a, b] \times [c, d]$. Then*

$$
\begin{aligned}
&\left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\rho_1(t), \rho_2(s)) \nabla_1 t \nabla_2 s \right| \\
&\quad \leq \frac{M_1}{b-a} \Big(j_2(x, a) + j_2(x, b) \Big) + \frac{M_2}{d-c} \Big(j_2(y, c) + j_2(y, d) \Big) \\
&\qquad + \frac{M_3}{(b-a)(d-c)} \Big(j_2(x, a) + j_2(x, b) \Big) \Big(j_2(y, c) + j_2(y, d) \Big) \qquad (11)
\end{aligned}
$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$
M_1 = \sup_{a<t<b} \left| \frac{\partial f(t,s)}{\nabla_1 t} \right|, \qquad M_2 = \sup_{c<s<d} \left| \frac{\partial f(t,s)}{\nabla_2 s} \right|, \qquad M_3 = \sup_{a<t<b, c<s<d} \left| \frac{\partial^2 f(t,s)}{\nabla_2 s \nabla_1 t} \right|.
$$

In fact, to prove Theorem 5, we need the following two Lemmas which can be proved similarly to Lemmas 3 and 4.

Lemma 5 (Montgomery Identity) *Let $\alpha, \beta, u, z \in \mathbb{T}$, $\alpha < \beta$ and $g : [\alpha, \beta] \to \mathbb{R}$ be differentiable. Then*

$$g(u) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g^{\rho}(z) \nabla z + \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} k(u, z) g^{\nabla}(z) \nabla z, \tag{12}$$

where

$$k(u, z) = \begin{cases} z - \alpha, & \alpha \le z < u, \\ z - \beta, & u \le z \le \beta. \end{cases}$$

Lemma 6 *Under the assumptions of Theorem 5, we have*

$$\begin{aligned} f(x, y) = {} & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\rho_1(t), \rho_2(s)) \nabla_1 t \nabla_2 s \\ & + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x, t) \frac{\partial f(t, \rho_2(s))}{\nabla_1 t} \nabla_1 t \nabla_2 s \\ & + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d q(y, s) \frac{\partial f(\rho_1(t), s)}{\nabla_2 s} \nabla_1 t \nabla_2 s \\ & + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x, t) q(y, s) \frac{\partial^2 f(t, s)}{\nabla_2 s \nabla_1 t} \nabla_1 t \nabla_2 s, \end{aligned} \tag{13}$$

where

$$p(x, t) = \begin{cases} t - a, & a \le t < x, \\ t - b, & x \le t \le b, \end{cases} \qquad q(y, s) = \begin{cases} s - c, & c \le s < y, \\ s - d, & y \le s \le d. \end{cases}$$

4.3 Ostrowski's Inequality for Double Integrals via $\nabla\Delta$-Integral

By a completely analogous method, we can derive the following similar result via $\nabla\Delta$-integral.

Theorem 6 *Let $x, t \in \mathbb{T}_1$, $y, s \in \mathbb{T}_2$ and $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\nabla_1 t}$, $\frac{\partial f(t,s)}{\Delta_2 s}$, $\frac{\partial^2 f(t,s)}{\Delta_2 s \nabla_1 t}$ exist and are continuous on $[a, b] \times [c, d]$. Then*

$$\begin{aligned} & \left| f(x, y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\rho_1(t), \sigma_2(s)) \nabla_1 t \Delta_2 s \right| \\ & \le \frac{M_1}{b-a} \Big(j_2(x, a) + j_2(x, b) \Big) + \frac{M_2}{d-c} \Big(h_2(y, c) + h_2(y, d) \Big) \\ & \quad + \frac{M_3}{(b-a)(d-c)} \Big(j_2(x, a) + j_2(x, b) \Big) \Big(h_2(y, c) + h_2(y, d) \Big) \end{aligned} \tag{14}$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$M_1 = \sup_{a<t<b} \left| \frac{\partial f(t, s)}{\nabla_1 t} \right|, \qquad M_2 = \sup_{c<s<d} \left| \frac{\partial f(t, s)}{\Delta_2 s} \right|, \qquad M_3 = \sup_{a<t<b, c<s<d} \left| \frac{\partial^2 f(t, s)}{\Delta_2 s \nabla_1 t} \right|.$$

In fact, to prove Theorem 6, we need the following two Lemma.

Lemma 7 *Under the assumptions of Theorem 6, we have*

$$\begin{aligned} f(x,y) &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\rho_1(t),\sigma_2(s))\nabla_1 t\Delta_2 s \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(\rho_1(t),s)}{\nabla_1 t}\nabla_1 t\Delta_2 s \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(t,\sigma_2(s))}{\Delta_2 s}\nabla_1 t\Delta_2 s \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\nabla_1 t}\nabla_1 t\Delta_2 s, \end{aligned} \tag{15}$$

where

$$p(x,t)=\begin{cases} t-a, & a\le t<x,\\ t-b, & x\le t\le b,\end{cases} \qquad q(y,s)=\begin{cases} s-c, & c\le s<y,\\ s-d, & y\le s\le d.\end{cases}$$

Proof Applying Lemma 5 for the partial delta map $f(\cdot,y)$, $y\in[c,d]$, we obtain

$$f(x,y)=\frac{1}{b-a}\int_a^b f(\rho_1(t),y)\nabla_1 t+\frac{1}{b-a}\int_a^b p(x,t)\frac{\partial f(t,y)}{\nabla_1 t}\nabla_1 t \tag{16}$$

for all $(x,y)\in[a,b]\times[c,d]$. Also, if we apply Lemma 3 for the partial delta map $f(\rho_1(t),\cdot)$, $t\in[a,b]$, we obtain

$$f(\rho_1(t),y)=\frac{1}{d-c}\int_c^d f(\rho_1(t),\sigma_2(s))\Delta_2 s+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(\rho_1(t),s)}{\Delta_2 s}\Delta_2 s \tag{17}$$

for all $(t,y)\in[a,b]\times[c,d]$. Applying the same Lemma 3 for the partial delta map $\frac{\partial f(t,\cdot)}{\nabla_1 t}$, we have

$$\frac{\partial f(t,y)}{\nabla_1 t}=\frac{1}{d-c}\int_c^d \frac{\partial f(t,\sigma_2(s))}{\nabla_1 t}\Delta_2 s+\frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\nabla_1 t}\Delta_2 s \tag{18}$$

for all $(t,y)\in[a,b]\times[c,d]$. Substituting (18) and (17) in (16), we have

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b\Bigg[\frac{1}{d-c}\int_c^d f(\rho_1(t),\sigma_2(s))\Delta_2 s \\ &\quad + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial f(\rho_1(t),s)}{\Delta_2 s}\Delta_2 s\Bigg]\nabla_1 t \\ &\quad + \frac{1}{b-a}\int_a^b p(x,t)\Bigg[\frac{1}{d-c}\int_c^d \frac{\partial f(t,\sigma_2(s))}{\nabla_1 t}\Delta_2 s \\ &\quad + \frac{1}{d-c}\int_c^d q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\nabla_1 t}\Delta_2 s\Bigg]\nabla_1 t \\ &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\rho_1(t),\sigma_2(s))\nabla_1 t\Delta_2 s \\ &\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(\rho_1(t),s)}{\Delta_2 s}\nabla_1 t\Delta_2 s \end{aligned}$$

$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(t,\sigma_2(s))}{\nabla_1 t}\nabla_1 t\Delta_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\nabla_1 t}\nabla_1 t\Delta_2 s,$$

i.e. (15) holds. □

4.4 Ostrowski's Inequality for Double Integrals via $\Delta\nabla$-Integral

By a completely analogous method, we can derive the following similar result via $\Delta\nabla$-integral.

Theorem 7 *Let $x,t\in\mathbb{T}_1$, $y,s\in\mathbb{T}_2$ and $f:[a,b]\times[c,d]\to\mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$, $\frac{\partial f(t,s)}{\nabla_2 s}$, $\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}$ exist and are continuous on $[a,b]\times[c,d]$. Then*

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s\right|$$
$$\le\frac{M_1}{b-a}\Big(h_2(x,a)+h_2(x,b)\Big)+\frac{M_2}{d-c}\Big(j_2(y,c)+j_2(y,d)\Big)$$
$$+\frac{M_3}{(b-a)(d-c)}\Big(h_2(x,a)+h_2(x,b)\Big)\Big(j_2(y,c)+j_2(y,d)\Big) \tag{19}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$M_1=\sup_{a<t<b}\left|\frac{\partial f(t,s)}{\Delta_1 t}\right|,\qquad M_2=\sup_{c<s<d}\left|\frac{\partial f(t,s)}{\nabla_2 s}\right|,\qquad M_3=\sup_{a<t<b,c<s<d}\left|\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\right|.$$

In fact, to prove Theorem 7, we need the following lemma which can be proved similarly to Lemma 7.

Lemma 8 *Under the assumptions of Theorem 7, we have*

$$f(x,y)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)\frac{\partial f(\sigma_1(t),s)}{\Delta_1 t}\Delta_1 t\nabla_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d q(y,s)\frac{\partial f(t,\rho_2(s))}{\nabla_2 s}\Delta_1 t\nabla_2 s$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(x,t)q(y,s)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s, \tag{20}$$

where

$$p(x,t)=\begin{cases}t-a, & a\le t<x,\\ t-b, & x\le t\le b,\end{cases}\qquad q(y,s)=\begin{cases}s-c, & c\le s<y,\\ s-d, & y\le s\le d.\end{cases}$$

5 Further Works

The theory and applications of dynamic derivatives which are a linear combination of delta and nabla dynamic derivatives on the (one-variable) time scales have recently received considerable attention [14, 33–35]. We hope that we can develop a similar double Riemann $\diamond$-integrals on time scales, which will be our further work.

One can obtain some similar results by combining all $\Delta\Delta$-, $\Delta\nabla$-, $\nabla\Delta$- and $\nabla\nabla$-integrals, although such results are very complicated and are less interesting than Theorems 4–7. However, on the left hand side of (4), the function f contains forward jump operator while in other situations (see (11), (14) and (19)), f contains backward jump operator. Therefore, our approach cannot deduce to the double $\diamond$-integral directly. Maybe we can achieve it by deriving a new Montgomery Identity instead of Lemma 3. This will also be our further work.

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References

1. Agarwal, R., Bohner, M., Peterson, A.: Inequalities on time scales: A survey. Math. Inequal. Appl. **4**(4), 535–557 (2001)
2. Almeida, R., Torres, D.F.M.: Isoperimetric problems on time scales with nabla derivatives, J. Vibr. Control (in press). http://arxiv.org/abs/0811.3650
3. Atici, F.M., Biles, D.C., Lebedinsky, A.: An application of time scales to economics. Math. Comput. Model. **43**(7–8), 718–726 (2006)
4. Bohner, M., Guseinov, G.S.: Partial differentiation on time scales. Dyn. Syst. Appl. **13**(3–4), 351–379 (2004)
5. Bohner, M., Guseinov, G.S.: Multiple integration on time scales. Dyn. Syst. Appl. **14**(3–4), 579–606 (2005)
6. Bohner, M., Guseinov, G.S.: Double integral calculus of variations on time scales. Comput. Math. Appl. **54**, 45–57 (2007)
7. Bohner, M., Matthews, T.: The Grüss inequality on time scales. Commun. Math. Anal. **3**(1), 1–8 (2007)
8. Bohner, M., Matthews, T.: Ostrowski inequalities on time scales. J. Inequal. Pure Appl. Math. **9**(1), 8 (2008)
9. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkhäuser, Basel (2001)
10. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Basel (2003)
11. Dragomir, S.S., Wang, S.: A new inequality of Ostrowski's type in L1 norm and applications to some special means and to some numerical quadrature rules. Tamkang J. Math. **28**, 239–244 (1997)
12. Dragomir, S.S., Wang, S.: Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules. Appl. Math. Lett. **11**, 105–109 (1998)
13. Dragomir, S.S., Cerone, P., Barnett, N.S., Roumeliotis, J.: An inequality of the Ostrowski type for double integrals and applications for cubature formulae. RGMIA Res. Rep. Collect. **2**, 781–796 (1999)
14. Ferreira, R.A.C., Sidi Ammi, M.R., Torres, D.F.M.: Diamond-alpha integral inequalities on time scales. Int. J. Math. Stat. **5**(A09), 52–59 (2009)
15. Hilger, S.: Ein Maßkettenkalkül mit Anwendung auf Zentrmsmannigfaltingkeiten. Ph.D. thesis, Univarsi. Würzburg (1988)
16. Hilscher, R.: A time scales version of a Wirtinger-type inequality and applications. J. Comput. Appl. Math. **141**, 219–226 (2002)
17. Lakshmikantham, V., Sivasundaram, S., Kaymakcalan, B.: Dynamic Systems on Measure Chains. Kluwer Academic, Dordrecht (1996)
18. Liu, W.J., Ngô, Q.A.: An Ostrowski-Grüss type inequality on time scales. arXiv:0804.3231
19. Liu, W.J., Ngô, Q.A.: A generalization of Ostrowski inequality on time scales for k points. Appl. Math. Comput. **203**, 754–760 (2008)
20. Liu, W.J., Ngô, Q.A.: An Ostrowski type inequality on time scales for functions whose second derivatives are bounded. In: Inequality Theory and Applications, vol 6. Nova Science Publishers, New York (2009, to appear)

21. Liu, W.J., Xue, Q.L., Wang, S.F.: Several new perturbed Ostrowski-like type inequalities. J. Inequal. Pure Appl. Math. **8**(4), 6 (2007)
22. Liu, W.J., Li, C.C., Hao, Y.M.: Further generalization of some double integral inequalities and applications. Acta. Math. Univ. Comen. **77**(1), 147–154 (2008)
23. Liu, W.J., Ngô, Q.A., Chen, W.B.: A perturbed Ostrowski type inequality on time scales for k points for functions whose second derivatives are bounded. J. Inequal. Appl. **2008**. Article ID 597241, 12 p.
24. Martins, N., Torres, D.F.M.: Calculus of variations on time scales with nabla derivatives. Nonlinear Anal. (2008). doi:10.1016/j.na.2008.11.035
25. Mitrinović, D.S., Pecarić, J.E., Fink, A.M.: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Dordrecht (1991)
26. Mitrinović, D.S., Pecarić, J., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic, Dordrecht (1993)
27. Ostrowski, A.M.: Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert. Comment. Math. Helv. **10**, 226–227 (1938)
28. Ozkan, U.M., Yildirim, H.: Ostrowski type inequality for double integrals on time scales. Acta Appl. Math. (2009). doi:10.1007/s10440-008-9407-z
29. Ozkan, U.M., Yildirim, H.: Steffensen's integral inequality on time scales. J. Inequal. Appl. **2007**. Article ID 46524, 10 p.
30. Ozkan, U.M., Yildirim, H.: Hardy-knopp-type inequalities on time scales. Dyn. Syst. Appl. **17**, 477–486 (2008)
31. Ozkan, U.M., Sarikaya, M.Z., Yildirim, H.: Extensions of certain integral inequalities on time scales. Appl. Math. Lett. **21**, 993–1000 (2008)
32. Pachpatte, B.G.: On an inequality of Ostrowski type in three independent variables. J. Math. Anal. Appl. **249**, 583–591 (2000)
33. Rogers Jr., J.W., Sheng, Q.: Notes on the diamond-α dynamic derivative on time scales. J. Math. Anal. Appl. **326**(1), 228–241 (2007)
34. Sheng, Q., Fadag, M., Henderson, J., Davis, J.M.: An exploration of combined dynamic derivatives on time scales and their applications. Nonlinear Anal.: Real World Appl. **7**(3), 395–413 (2008)
35. Sidi Ammi, M.R., Ferreira, R.A.C., Torres, D.F.M.: Diamond-α Jensen's inequality on time scales. J. Inequal. Appl. **2008** (2008). Article ID 576876, 13 p.
36. Wong, F.-H., Yu, S.-L., Yeh, C.-C.: Anderson's inequality on time scales. Appl. Math. Lett. **19**, 931–935 (2007)