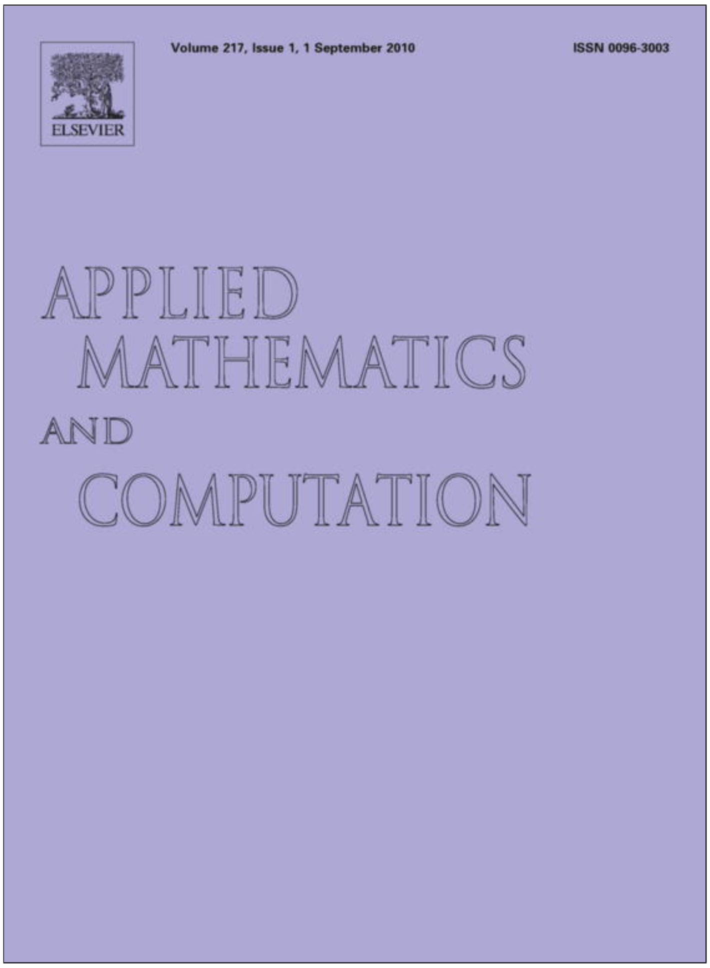
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On an Iyengar-type inequality involving quadratures in n knots

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ABSTRACT

In this short note, we give an Iyengar-type inequality involving quadratures in n knots, where n is an arbitrary natural number.

1. Introduction

In 1938, Iyengar [6] proved the following interesting integral inequality which has received considerable attention from many researchers.

Theorem 1 (See [6]). *Let f be differentiable on* $[a,b]$ *and* $|f'(x)| \leqq M$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{f(a)+f(b)}{2}\right| \leqq \frac{M(b-a)}{4} - \frac{(f(b)-f(a))^2}{4M(b-a)}. \tag{1}$$

Through the years, Iyengar's inequality (1) has been generalized in various ways. Set

$$I = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{b-a}{2}(f(a)+f(b)) + \frac{(b-a)^2}{8}(f'(b)-f'(a)),$$

in [1,4], the following Iyengar-type inequality was obtained.

Theorem 2 (See [1,4]). *Let* $f \in C^2[a,b]$ *and* $|f''(x)| \leqq M$. *Then*

$$|I| \leqq \frac{M}{24}(b-a)^3 - \frac{|\Delta|^3}{24M^2}, \tag{2}$$

where

$$\Delta = f'(a) - 2f'\left(\frac{a+b}{2}\right) + f'(b).$$

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Since then, Theorem 2 was generalized and improved by lots of mathematicians, let us mention the works of Cheng in [3] and Franjić et al. [5] in the literature. In those papers, the authors tried to estimate the left hand side of (2) by various ways. In contrast to [3,5], we will generalize the left hand side of (2) into a general form and then obtain some new estimates. Before stating our main result, let us introduce the following notation. For each $i = \overline{1,n}$, we assume $0 \leqq x_i \leqq 1$, put

$$Q(f; x_1, \ldots, x_n) = \frac{1}{n}\sum_{k=1}^{n} f(a + (b-a)x_k).$$

We are in a position to state our main result.

Theorem 3. *Let $I \subset \mathbb{R}$ be an open interval such that $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that, for all $x \in [a,b]$, $\gamma \leqq f'(x) \leqq \Gamma$ for some positive γ and Γ. Assume $\{x_k\}_{k=1}^n \subset [0,1)$ is such that*

$$x_1 + x_2 + \cdots + x_n = \frac{n}{2}, \tag{3}$$

and

$$x_1^2 + x_2^2 + \cdots + x_n^2 = nq, \tag{4}$$

where $q \in [0, \frac{1}{2}]$ is a given number. Then the following estimate holds

$$A_{p,q}(b-a)^2 \leqq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f; x_1, x_2, \ldots, x_n) + (b-a)p(f'(b) - f'(a)) \leqq B_{p,q}(b-a)^2, \tag{5}$$

where p is an arbitrary number and

$$A_{p,q} = \begin{cases} -\frac{q}{2}\Gamma + (p + \frac{1}{6})\gamma, & \frac{q}{2} - \frac{1}{6} \geqq 0, \quad p - \frac{q}{2} + \frac{1}{6} \geqq 0, \\ -\frac{1}{6}\Gamma + (p - \frac{q}{2} + \frac{1}{3})\gamma, & \frac{q}{2} - \frac{1}{6} < 0, \quad p - \frac{q}{2} + \frac{1}{6} \geqq 0, \\ (p - q + \frac{1}{6})\Gamma + \frac{q}{2}\gamma, & \frac{q}{2} - \frac{1}{6} \geqq 0, \quad p - \frac{q}{2} + \frac{1}{6} < 0, \\ (p - \frac{q}{2})\Gamma + \frac{1}{6}\gamma, & \frac{q}{2} - \frac{1}{6} < 0, \quad p - \frac{q}{2} + \frac{1}{6} < 0, \end{cases}$$

and

$$B_{p,q} = \begin{cases} -\frac{q}{2}\gamma + (p + \frac{1}{6})\Gamma, & \frac{q}{2} - \frac{1}{6} \geqq 0, \quad p - \frac{q}{2} + \frac{1}{6} \geqq 0, \\ -\frac{1}{6}\gamma + \Gamma(p - \frac{q}{2} + \frac{1}{3}), & \frac{q}{2} - \frac{1}{6} < 0, \quad p - \frac{q}{2} + \frac{1}{6} \geqq 0 \\ (p - q + \frac{1}{6})\gamma + \frac{q}{2}\Gamma, & \frac{q}{2} - \frac{1}{6} \geqq 0, \quad p - \frac{q}{2} + \frac{1}{6} < 0, \\ (p - \frac{q}{2})\gamma + \frac{1}{6}\Gamma, & \frac{q}{2} - \frac{1}{6} < 0, \quad p - \frac{q}{2} + \frac{1}{6} < 0. \end{cases}$$

Remark 1. If we take $n = 2$, $p = \frac{1}{8}$ and $x_1 = 0$, $x_2 = 1$ then by (4) one has $q = \frac{1}{2}$. Thus (5) tells us that

$$\left(-\frac{1}{4}\Gamma + \frac{7}{24}\gamma\right)(b-a)^2 \leqq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{f(a)+f(b)}{2} + \frac{b-a}{8}(f'(b) - f'(a)) \leqq \left(\frac{7}{24}\Gamma - \frac{1}{4}\gamma\right)(b-a)^2,$$

which is nothing but an Iyengar-type inequality of kind (2).

Theorem 4. *Let $I \subset \mathbb{R}$ be an open interval such that $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be a thrice differentiable function such that $f''' \in L^r[a,b]$ for some $1 < r < \infty$. The given set $\{x_k\}_{k=1}^n \subset [0,1)$ is as in Theorem 3. Then the following estimate holds*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f; x_1, x_2, \ldots, x_n) + (b-a)\left(\frac{q}{2} - \frac{1}{6}\right)(f'(b) - f'(a))\right| \leqq K_{r,q}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r, \tag{6}$$

where

$$K_{r,q} = \frac{1}{6}\left(\frac{r-1}{4r-1}\right)^{\frac{r-1}{r}} + \frac{q}{2}\left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}} + \left(\frac{q}{2} - \frac{1}{6}\right)\left(\frac{r-1}{2r-1}\right)^{\frac{r-1}{r}}.$$

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 5. [See [2]]*Let $f : [a,b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a,b]$, $x_0 \in (a,b)$ then for all $x \in (a,b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x),$$

where $T_{r-1}(f, x_0, \cdot)$ *is Taylor's polynomial of degree* $r-1$, *that is,*

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!},$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!} \mathrm{d}t. \tag{7}$$

By a simple calculation, the remainder in (7) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_{0}^{x-x_0} \frac{(x-x_0-t)^{r-1} f^{(r)}(x_0+t)}{(r-1)!} \mathrm{d}t,$$

which helps us to deduce a similar representation of f as follows

$$f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t) \mathrm{d}t. \tag{8}$$

Proof of Theorem 3. Put

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Applying Lemma 5 to $F(x)$ with $x = a$ and $u = b - a$ and the Fundamental Theorem of Calculus, we get

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) = (b-a)f(a) + \frac{(b-a)^2}{2} f'(a) + \int_a^b \frac{(b-x)^2}{2} f''(x)\mathrm{d}x.$$

Similarly, one has

$$\begin{aligned} f(a+(b-a)x_k) &= f(a) + \frac{f'(a)}{1!}(b-a)x_k + \int_0^{(b-a)x_k} ((b-a)x_k - u) f''(a+u)\mathrm{d}u \overset{u=x_k(x-a)}{=} f(a) + f'(a)(b-a)x_k \\ &\quad + \int_a^b x_k^2 (b-x) f''((1-x_k)a + x_k x)\mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} Q(f; x_1, x_2, \ldots, x_n) &= f(a) + (b-a)\underbrace{\frac{\sum_{k=1}^n x_k}{n}}_{=\frac{1}{2}} f'(a) + \frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2 (b-x) f''((1-x_k)a + x_k x)\mathrm{d}x \\ &= f(a) + \frac{b-a}{2} f'(a) + \frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2 (b-x) f''((1-x_k)a + x_k x)\mathrm{d}x, \end{aligned}$$

and also

$$f'(b) - f'(a) = \int_a^b f''(x)\mathrm{d}x.$$

Then

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f; x_1, x_2, \ldots, x_n) + (b-a)p(f'(b) - f'(a)) \\ &= \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f; x_1, x_2, \ldots, x_n) + (b-a)\left(\frac{q}{2} - \frac{1}{6}\right)(f'(b) - f'(a)) + (b-a)\left(p - \frac{q}{2} + \frac{1}{6}\right)(f'(b) - f'(a)) \\ &= \frac{1}{b-a}\int_a^b \frac{(b-x)^2}{2} f''(x)\mathrm{d}x - \frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2 (b-x) f''((1-x_k)a + x_k x)\mathrm{d}x + (b-a)\left(\frac{q}{2} - \frac{1}{6}\right)\int_a^b f''(x)\mathrm{d}x \\ &\quad + (b-a)\left(p - \frac{q}{2} + \frac{1}{6}\right)(f'(b) - f'(a)) \\ &= \frac{1}{b-a}\int_a^b \frac{(b-x)^2}{2}[f''(x) - \gamma]\mathrm{d}x - \frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2 (b-x)[f''((1-x_k)a + x_k x) - \gamma]\mathrm{d}x + (b-a)\left(\frac{q}{2} - \frac{1}{6}\right)\int_a^b [f''(x) - \gamma]\mathrm{d}x \\ &\quad + (b-a)\left(p - \frac{q}{2} + \frac{1}{6}\right)(f'(b) - f'(a)). \end{aligned}$$

We can estimate further as follows.

$$0 \leqq \frac{1}{b-a}\int_a^b \frac{(b-x)^2}{2}[f''(x)-\gamma]\mathrm{d}x \leqq \frac{1}{b-a}\int_a^b \frac{(b-x)^2}{2}[\Gamma-\gamma]\mathrm{d}x = \frac{1}{b-a}(\Gamma-\gamma)\int_a^b \frac{(b-x)^2}{2}\mathrm{d}x = \frac{1}{6}(b-a)^2(\Gamma-\gamma),$$

while

$$\begin{aligned} 0 &\geqq -\frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2(b-x)[f''((1-x_k)a+x_kx)-\gamma]\mathrm{d}x \geqq -\frac{1}{n}\sum_{k=1}^n \int_a^b x_k^2(b-x)[\Gamma-\gamma]\mathrm{d}x \\ &= -\frac{1}{n}\sum_{k=1}^n x_k^2\left[\int_a^b (b-x)(\Gamma-\gamma)\mathrm{d}x\right] = -\frac{1}{n}\sum_{k=1}^n \left[\frac{1}{2}(b-a)^2(\Gamma-\gamma)x_k^2\right] = -\frac{q}{2}(b-a)^2(\Gamma-\gamma). \end{aligned}$$

Moreover, if $\frac{q}{2}-\frac{1}{6} \geqq 0$ then

$$0 \leqq (b-a)\left(\frac{q}{2}-\frac{1}{6}\right)\int_a^b [f''(x)-\gamma]\mathrm{d}x \leqq (b-a)\left(\frac{q}{2}-\frac{1}{6}\right)\int_a^b [\Gamma-\gamma]\mathrm{d}x = \left(\frac{q}{2}-\frac{1}{6}\right)(b-a)^2(\Gamma-\gamma),$$

otherwise,

$$\left(\frac{q}{2}-\frac{1}{6}\right)(b-a)^2(\Gamma-\gamma) \leqq (b-a)\left(\frac{q}{2}-\frac{1}{6}\right)\int_a^b [f''(x)-\gamma]\mathrm{d}x \leqq 0.$$

Finally, if $p-\frac{q}{2}+\frac{1}{6} \geqq 0$ then

$$\gamma(b-a)^2\left(p-\frac{q}{2}+\frac{1}{6}\right) \leqq (b-a)\left(p-\frac{q}{2}+\frac{1}{6}\right)(f'(b)-f'(a)) \leqq \Gamma(b-a)^2\left(p-\frac{q}{2}+\frac{1}{6}\right),$$

and

$$\Gamma(b-a)^2\left(p-\frac{q}{2}+\frac{1}{6}\right) \leqq (b-a)\left(p-\frac{q}{2}+\frac{1}{6}\right)(f'(b)-f'(a)) \leqq \gamma(b-a)^2\left(p-\frac{q}{2}+\frac{1}{6}\right),$$

provided $p-\frac{q}{2}+\frac{1}{6} \leqq 0$. Thus

$$A_{p,q}(b-a)^2 \leqq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f;x_1,x_2,\ldots,x_n) + (b-a)p(f'(b)-f'(a)) \leqq B_{p,q}(b-a)^2,$$

where

$$A_{p,q} = \begin{cases} -\frac{q}{2}(\Gamma-\gamma)+\gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} \geqq 0, \quad p-\frac{q}{2}+\frac{1}{6} \geqq 0, \\ -\frac{q}{2}(\Gamma-\gamma)+\left(\frac{q}{2}-\frac{1}{6}\right)(\Gamma-\gamma)+\gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} < 0, \quad p-\frac{q}{2}+\frac{1}{6} \geqq 0, \\ -\frac{q}{2}(\Gamma-\gamma)+\Gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} \geqq 0, \quad p-\frac{q}{2}+\frac{1}{6} < 0, \\ -\frac{q}{2}(\Gamma-\gamma)+\left(\frac{q}{2}-\frac{1}{6}\right)(\Gamma-\gamma)+\Gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} < 0, \quad p-\frac{q}{2}+\frac{1}{6} < 0, \end{cases}$$

and

$$B_{p,q} = \begin{cases} \frac{1}{6}(\Gamma-\gamma)+\left(\frac{q}{2}-\frac{1}{6}\right)(\Gamma-\gamma)+\Gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} \geqq 0, \quad p-\frac{q}{2}+\frac{1}{6} \geqq 0, \\ \frac{1}{6}(\Gamma-\gamma)+\Gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} < 0, \quad p-\frac{q}{2}+\frac{1}{6} \geqq 0, \\ \frac{1}{6}(\Gamma-\gamma)+\left(\frac{q}{2}-\frac{1}{6}\right)(\Gamma-\gamma)+\gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} \geqslant 0, \quad p-\frac{q}{2}+\frac{1}{6} < 0, \\ \frac{1}{6}(\Gamma-\gamma)+\gamma\left(p-\frac{q}{2}+\frac{1}{6}\right), & \frac{q}{2}-\frac{1}{6} < 0, \quad p-\frac{q}{2}+\frac{1}{6} < 0. \end{cases}$$

The proof is now complete. □

Proof of Theorem 4. We now apply Lemma 5 to $F(x)$ with $x = a$ and $u = b - a$, we obtain

$$\int_a^b f(x)\mathrm{d}x = F(b)-F(a) = (b-a)f(a)+\frac{(b-a)^2}{2}f'(a)+\frac{(b-a)^3}{6}f''(a)+\int_a^b \frac{(b-x)^3}{6}f'''(x)\mathrm{d}x.$$

Similarly, one has

$$\begin{aligned} f(a+(b-a)x_k) &= f(a)+\frac{f'(a)}{1!}(b-a)x_k+\frac{f''(a)}{2!}[(b-a)x_k]^2+\int_0^{(b-a)x_k}\frac{((b-a)x_k-u)^2}{2!}f'''(a+u)\mathrm{d}u \overset{u=x_k(x-a)}{=} f(a) \\ &\quad + f'(a)(b-a)x_k+\frac{f''(a)}{2}(b-a)^2x_k^2+\int_a^b \frac{x_k^3(b-x)^2}{2}f'''((1-x_k)a+x_kx)\mathrm{d}x. \end{aligned}$$

Thus,

$$Q(f;x_1,x_2,\ldots,x_n) = f(a) + (b-a)\underbrace{\frac{\sum_{k=1}^n x_k}{n}}_{=\frac{1}{2}} f'(a) + \frac{f''(a)}{2}(b-a)^2\underbrace{\frac{\sum_{k=1}^n x_k^2}{n}}_{q} + \frac{1}{n}\sum_{k=1}^n \int_a^b \frac{x_k^3(b-x)^2}{2} f'''((1-x_k)a + x_k x)\mathrm{d}x$$

$$= f(a) + \frac{b-a}{2}f'(a) + \frac{q}{2}f''(a)(b-a)^2 + \frac{1}{n}\sum_{k=1}^n \int_a^b \frac{x_k^3(b-x)^2}{2} f'''((1-x_k)a + x_k x)\mathrm{d}x,$$

and

$$f'(b) - f'(a) = (b-a)f''(a) + \int_a^b (b-x)f'''(x)\mathrm{d}x.$$

Then

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f;x_1,x_2,\ldots,x_n) + (b-a)\left(\frac{q}{2}-\frac{1}{6}\right)(f'(b)-f'(a))\right|$$

$$= \left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f;x_1,x_2,\ldots,x_n) + \left(\frac{q}{2}-\frac{1}{6}\right)(b-a)^2 f''(a) + \left(\frac{q}{2}-\frac{1}{6}\right)(b-a)\int_a^b (b-x)f'''(x)\mathrm{d}x\right|$$

$$= \left|\frac{1}{b-a}\int_a^b \frac{(b-x)^3}{6}f'''(x)\mathrm{d}x - \frac{1}{n}\sum_{k=1}^n \int_a^b \frac{x_k^3(b-x)^2}{2}f'''((1-x_k)a + x_k x)\mathrm{d}x + \left(\frac{q}{2}-\frac{1}{6}\right)(b-a)\int_a^b (b-x)f'''(x)\mathrm{d}x\right|.$$

We can estimate further as follows.

$$\left|\frac{1}{b-a}\int_a^b \frac{(b-x)^3}{6}f'''(x)\mathrm{d}x\right| \leqq \frac{1}{6(b-a)}\left(\int_a^b |(b-x)^3|^{\frac{r}{r-1}}\mathrm{d}x\right)^{\frac{r-1}{r}}\|f'''\|_r = \frac{1}{6}\left(\frac{r-1}{4r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r,$$

while

$$\left|\frac{1}{n}\sum_{k=1}^n \int_a^b \frac{x_k^3(b-x)^2}{2}f'''((1-x_k)a + x_k x)\mathrm{d}x\right| \leqq \frac{1}{n}\sum_{k=1}^n\left[x_k^3\left|\int_a^b \frac{(b-x)^2}{2}f'''((1-x_k)a + x_k x)\mathrm{d}x\right|\right]$$

$$\leqq \frac{1}{2n}\sum_{k=1}^n\left[x_k^3\left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r\right] = \left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r\frac{\sum_{k=1}^n x_k^3}{2n}$$

$$\leqq \left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r\frac{\sum_{k=1}^n x_k^2}{2n} = \frac{q}{2}\left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r,$$

and

$$\left|\left(\frac{q}{2}-\frac{1}{6}\right)(b-a)\int_a^b (b-x)f'''(x)\mathrm{d}x\right| \leqq \left(\frac{q}{2}-\frac{1}{6}\right)(b-a)\left|\int_a^b (b-x)f'''(x)\mathrm{d}x\right| \leqq \left(\frac{q}{2}-\frac{1}{6}\right)\left(\frac{r-1}{2r-1}\right)^{\frac{r-1}{r}}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r.$$

Thus

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - Q(f;x_1,x_2,\ldots,x_n) + (b-a)\left(\frac{q}{2}-\frac{1}{6}\right)(f'(b)-f'(a))\right| \leqq K_{r,q}(b-a)^{\frac{3r-1}{r}}\|f'''\|_r,$$

where

$$K_{r,q} = \frac{1}{6}\left(\frac{r-1}{4r-1}\right)^{\frac{r-1}{r}} + \frac{q}{2}\left(\frac{r-1}{3r-1}\right)^{\frac{r-1}{r}} + \left(\frac{q}{2}-\frac{1}{6}\right)\left(\frac{r-1}{2r-1}\right)^{\frac{r-1}{r}}.$$

The proof is now complete. □

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