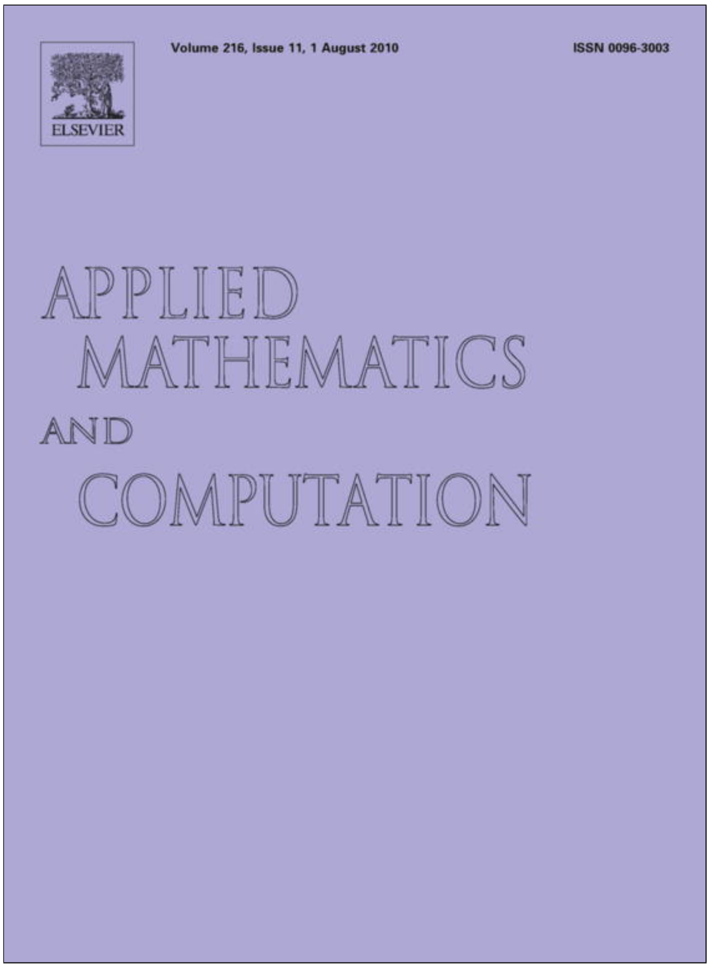
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Some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded

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A R T I C L E I N F O

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A B S T R A C T

We establish some Iyengar-type inequalities on time scales for functions whose second derivatives are bounded by using Steffensen's inequality on time scales.

1. Introduction

In 1938, Iyengar [18] proved the following interesting integral inequality which has received considerable attention from many researchers [2,8–11,19].

Theorem 1. *Let f be a differentiable function on (a,b) and assume that there is a constant $M_1 > 0$ such that $|f'(x)| < M_1$ for $x \in (a,b)$. Then*

$$\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leqslant \frac{M_1(b-a)^2}{4} - \frac{1}{4M_1}(f(b)-f(a))^2. \tag{1}$$

Especially, the authors in [2,11] proved the following inequality involving bounded second-order derivatives.

Theorem 2. *Let $f \in C^2[a,b]$ and $|f''(x)| \leqslant M_2$. Then*

$$\left|\int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \leqslant \frac{M_2}{24}\left((b-a)^3 - \left(\frac{|\Delta|}{M_2}\right)^3\right), \tag{2}$$

where $\Delta = f(a) - 2f((a+b)/2) + f(b)$.

In [12], Franjić et al. proved the following Iyengar-type inequality and show that it is always better than (2).

Theorem 3. *Let $f \in C^2[a,b]$ and $|f''(x)| \leqslant M_2$. Then*

$$\begin{aligned} -\frac{M_2}{24}(b-a)^3 + \frac{M_2}{3}(\lambda_a^3+\lambda_b^3) &\leqslant \int_a^b f(x)dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)) \\ &\leqslant \frac{M_2}{24}(b-a)^3 - \frac{M_2}{3}\left[\left(\frac{b-a}{2}-\lambda_a\right)^3 + \left(\frac{b-a}{2}-\lambda_b\right)^3\right], \end{aligned} \tag{3}$$

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where

$$\lambda_a = \frac{1}{2M_2}\left(f'\left(\frac{a+b}{2}\right) - f'(a)\right) + \frac{b-a}{4}$$

and

$$\lambda_b = \frac{1}{2M_2}\left(f'(b) - f'\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{4}.$$

The development of the theory of time scales was initiated by Hilger [13] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied the theory of certain integral inequalities on time scales. For example, we refer the reader to [1,6,7,15–17,20–22].

In the present paper we shall establish some new Iyengar-type inequalities on time scales for functions whose second derivatives are bounded by using Steffensen's inequality on time scales. Our results (see Theorems 6 and 7) extend the results in [2,11,12] to arbitrary time scales.

2. Time scales essentials

Now we briefly introduce the time scales theory and refer the reader to Hilger [13] and the books [4,5,14] for further details.

Definition 1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers.

Definition 2. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*, while if $\rho(t) < t$ then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then t is called *right-dense*, and if $\rho(t) = t$ and $t > \inf \mathbb{T}$, then t is called *left-dense*. Points that are both right-dense and left-dense are called dense.

Definition 3. Let $t \in \mathbb{T}$, then two mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ satisfying

$$\mu(t) := \sigma(t) - t, \quad \nu(t) := t - \rho(t)$$

are called the graininess functions.

We now introduce the set $\mathbb{T}^\kappa$ which is derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t, then $\mathbb{T}^\kappa := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$. Furthermore for a function $f := \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 4. Let $f : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$ there is a neighborhood U of t such that for all $s \in U$

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s|.$$

We say that f is Δ-differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. We talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$.

Definition 5. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) provided if it satisfies

1. f is continuous at each right-dense point or maximal element of $\mathbb{T}$.
2. The left-sided limit $\lim_{s \to t-} f(s) = f(t-)$ exists at each left-dense point t of $\mathbb{T}$.

Remark 1. It follows from Theorem 1.74 of Bohner and Peterson [4] that every rd-continuous function has an anti-derivative.

Definition 6. A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-anti-derivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the Δ-integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Proposition 4. *Let f, g be rd-continuous, $a,\ b,\ c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$,
(ii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,
(iii) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,
(iv) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$,
(v) $\int_a^a f(t)\Delta t = 0$.
(vi) If $f(t) \geqslant 0$ for all $a \leqslant t < b$ then $\int_a^b f(t)\Delta t \geqslant 0$.

Definition 7. Let $h_k : \mathbb{T}^2 \to \mathbb{R},\ k \in \mathbb{N}_0$ be defined by

$$h_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}.$$

Remark 2. It follows from Proposition 4(vi) that if $s \leqslant t$ then $h_{k+1}(t,s) \geqslant 0$ for all $t,\ s \in \mathbb{T}$ and all $k \in \mathbb{N}$.

Remark 3. If we let $h_k^\Delta(t,s)$ denote the derivative of $h_k(t,s)$ with respect to t for each fixed s, then

$$h_k^\Delta(t,s) = h_{k-1}(t,s), \quad \text{for} \quad k \in \mathbb{N},\ t \in \mathbb{T}^\kappa.$$

The following Steffensen's inequality on time scales was established in [3].

Theorem 5 (*Steffensen's Inequality*). *Let $a,\ b \in \mathbb{T}_\kappa^\kappa$ and $F, G : [a,b] \to \mathbb{R}$ be integrable functions, with F decreasing and $0 \leqslant G \leqslant 1$ on $[a,b]$. Assume $\lambda := \int_a^b G(t)\Delta t$ such that $b - \lambda,\ a + \lambda \in \mathbb{T}$. Then*

$$\int_{b-\lambda}^b F(t)\Delta t \leqslant \int_a^b F(t)G(t)\Delta t \leqslant \int_a^{a+\lambda} F(t)\Delta t. \tag{4}$$

Throughout this paper, we suppose that $\mathbb{T}$ is a time scale, $a,\ b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of real interval with the given time scale.

3. Iyengar-type inequalities on time scales

Our first result is embodied in the following.

Theorem 6. *Let $a,\ \frac{a+b}{2},\ b \in \mathbb{T}^\kappa$ and $f : [a,b] \to \mathbb{R}$ be a twice differentiable function and $f^{\Delta\Delta} : [a,b] \cap \mathbb{T}^{\kappa^2} \to \mathbb{R}$ is bounded, i.e.*

$$-\infty < m \leqslant f^{\Delta\Delta}(t) \leqslant M < +\infty, \quad \forall\ t \in [a,b] \cap \mathbb{T}^{\kappa^2}.$$

Then

$$\begin{aligned}
&(M-m)\left[\operatorname{sgn}\left(\lambda - \frac{b-a}{2}\right)h_3\left(a+\lambda, \frac{a+b}{2}\right) + h_3\left(a, \frac{a+b}{2}\right)\right] - \left[Mh_3\left(a, \frac{a+b}{2}\right) - mh_3\left(b, \frac{a+b}{2}\right)\right] \\
&\quad \leqslant \int_a^b f(\sigma^2(x))\Delta x - \frac{b-a}{2}[f^\sigma(a) + f^\sigma(b)] + h_2\left(b, \frac{a+b}{2}\right)f^\Delta(b) - h_2\left(a, \frac{a+b}{2}\right)f^\Delta(a) \\
&\quad \leqslant (M-m)\left[\operatorname{sgn}\left(\lambda - \frac{b-a}{2}\right)h_3\left(b-\lambda, \frac{a+b}{2}\right) + h_3\left(b, \frac{a+b}{2}\right)\right] - \left[Mh_3\left(a, \frac{a+b}{2}\right) - mh_3\left(b, \frac{a+b}{2}\right)\right],
\end{aligned} \tag{5}$$

where

$$\lambda = \frac{b-a}{2} + \frac{1}{M-m}\left[f^\Delta(a) - 2f^\Delta\left(\frac{a+b}{2}\right) + f^\Delta(b)\right]$$

such that $a + \lambda \in \mathbb{T},\ b - \lambda \in \mathbb{T}$.

Proof. To prove our result, we shall use the Steffensen's inequality on time scales. For this, let

$$F(x) = \begin{cases} h_2\left(x, \frac{a+b}{2}\right) & x \in \left[a, \frac{a+b}{2}\right), \\ -h_2\left(x, \frac{a+b}{2}\right) & x \in \left[\frac{a+b}{2}, b\right], \end{cases} \tag{6}$$

and

$$G(x) = \begin{cases} \frac{M - f^{\Delta\Delta}(x)}{M-m} & x \in \left[a, \frac{a+b}{2}\right), \\ \frac{f^{\Delta\Delta}(x) - m}{M-m} & x \in \left[\frac{a+b}{2}, b\right]. \end{cases} \tag{7}$$

Thus it holds

$$0 \leqslant G(x) \leqslant 1$$

and

$$\lambda = \frac{1}{M-m}\left[\int_a^{\frac{a+b}{2}} (M - f^{\Delta\Delta}(x))\Delta x + \int_{\frac{a+b}{2}}^b (f^{\Delta\Delta}(x) - m)\Delta x\right] = \frac{b-a}{2} + \frac{1}{M-m}\left[f^{\Delta}(a) - 2f^{\Delta}\left(\frac{a+b}{2}\right) + f^{\Delta}(b)\right].$$

Theorem 5 tells us that

$$\underbrace{\int_{b-\lambda}^b F(x)\Delta x}_{\alpha} \leqslant \int_a^b F(x)G(x)\Delta x \leqslant \underbrace{\int_a^{a+\lambda} F(x)\Delta x}_{\beta}. \tag{8}$$

Two cases are possible:

(a) $f^{\Delta}(a) - 2f^{\Delta}\left(\frac{a+b}{2}\right) + f^{\Delta}(b) \leqslant 0$, and
(b) $f^{\Delta}(a) - 2f^{\Delta}\left(\frac{a+b}{2}\right) + f^{\Delta}(b) > 0$.

Case (a). $f^{\Delta}(a) - 2f^{\Delta}\left(\frac{a+b}{2}\right) + f^{\Delta}(b) \leqslant 0$ implies $\lambda \leqslant \frac{b-a}{2}$ and thus $a + \lambda \leqslant \frac{a+b}{2} \leqslant b - \lambda$. The left and right term of inequality (4) are:

$$\begin{aligned}\alpha &= \int_{b-\lambda}^b F(x)\Delta x = -\int_{b-\lambda}^b h_2\left(x, \frac{a+b}{2}\right)\Delta x = -\int_{\frac{a+b}{2}}^b h_2\left(x, \frac{a+b}{2}\right)\Delta x + \int_{\frac{a+b}{2}}^{b-\lambda} h_2\left(x, \frac{a+b}{2}\right)\Delta x \\ &= -h_3\left(b, \frac{a+b}{2}\right) + h_3\left(b-\lambda, \frac{a+b}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\beta &= \int_a^{a+\lambda} F(x)\Delta x = \int_a^{a+\lambda} h_2\left(x, \frac{a+b}{2}\right)\Delta x = \int_a^{\frac{a+b}{2}} h_2\left(x, \frac{a+b}{2}\right)\Delta x - \int_{a+\lambda}^{\frac{a+b}{2}} h_2\left(x, \frac{a+b}{2}\right)\Delta x \\ &= -h_3\left(a, \frac{a+b}{2}\right) + h_3\left(a+\lambda, \frac{a+b}{2}\right).\end{aligned}$$

Case (b). Here it holds that $\lambda > \frac{b-a}{2}$ and thus $b - \lambda < \frac{a+b}{2} < a + \lambda$. A similar calculation gives

$$\alpha = \int_{b-\lambda}^b F(x)\Delta x = -\int_{\frac{a+b}{2}}^b h_2\left(x, \frac{a+b}{2}\right)\Delta x + \int_{b-\lambda}^{\frac{a+b}{2}} h_2\left(x, \frac{a+b}{2}\right)\Delta x = -h_3\left(b, \frac{a+b}{2}\right) - h_3\left(b-\lambda, \frac{a+b}{2}\right),$$

and

$$\beta = \int_a^{a+\lambda} F(x)\Delta x = \int_a^{\frac{a+b}{2}} h_2\left(x, \frac{a+b}{2}\right)\Delta x - \int_{\frac{a+b}{2}}^{a+\lambda} h_2\left(x, \frac{a+b}{2}\right)\Delta x = -h_3\left(a, \frac{a+b}{2}\right) - h_3\left(a+\lambda, \frac{a+b}{2}\right).$$

Thus, (8) implies that

$$\begin{aligned}\operatorname{sgn}\left(\lambda - \frac{b-a}{2}\right)h_3\left(b-\lambda, \frac{a+b}{2}\right) + h_3\left(b, \frac{a+b}{2}\right) &\geqslant -\int_a^b F(t)G(t)\Delta t \\ &\geqslant \operatorname{sgn}\left(\lambda - \frac{b-a}{2}\right)h_3\left(a+\lambda, \frac{a+b}{2}\right) + h_3\left(a, \frac{a+b}{2}\right).\end{aligned} \tag{9}$$

Now, we only need to calculate the middle term in Stefenssen's inequality. Obviously,

$$I = \int_a^b F(t)G(t)\Delta t = \frac{1}{M-m}\left[\int_a^{\frac{a+b}{2}} h_2\left(x, \frac{a+b}{2}\right)(M - f^{\Delta\Delta}(x))\Delta x - \int_{\frac{a+b}{2}}^b h_2\left(x, \frac{a+b}{2}\right)(f^{\Delta\Delta}(x) - m)\Delta x\right].$$

Using Proposition 4(iv) and Remark 3, we have

$$\begin{aligned}\int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)(M-f^{\Delta\Delta}(x))\Delta x &= M\int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)\Delta x - \int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)f^{\Delta\Delta}(x)\Delta x\\ &= -M\int_{\frac{a+b}{2}}^{a} h_2\left(x,\frac{a+b}{2}\right)\Delta x - \left[h_2\left(x,\frac{a+b}{2}\right)f^{\Delta}(x)\right]\Big|_a^{\frac{a+b}{2}}\\ &\quad + \int_a^{\frac{a+b}{2}}\left[h_2\left(x,\frac{a+b}{2}\right)\right]^{\Delta} f^{\Delta}(\sigma(x))\Delta x\\ &= -Mh_3\left(a,\frac{a+b}{2}\right) + h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a) + \int_a^{\frac{a+b}{2}} h_1\left(x,\frac{a+b}{2}\right)f^{\Delta}(\sigma(x))\Delta x.\end{aligned}$$

By using Proposition 4(iv) and Remark 3 again, we obtain

$$\begin{aligned}\int_a^{\frac{a+b}{2}} h_1\left(x,\frac{a+b}{2}\right)f^{\Delta}(\sigma(x))\Delta x &= \left[h_1\left(x,\frac{a+b}{2}\right)f(\sigma(x))\right]\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}}\left[h_1\left(x,\frac{a+b}{2}\right)\right]^{\Delta} f(\sigma(\sigma(x)))\Delta x\\ &= -h_1\left(a,\frac{a+b}{2}\right)f(\sigma(a)) - \int_a^{\frac{a+b}{2}} h_0\left(x,\frac{a+b}{2}\right)f(\sigma^2(x))\Delta x\\ &= -h_1\left(a,\frac{a+b}{2}\right)f(\sigma(a)) - \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x = \frac{b-a}{2}f(\sigma(a)) - \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x\end{aligned}$$

which implies that

$$\int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)(M-f^{\Delta\Delta}(x))\Delta x = -Mh_3\left(a,\frac{a+b}{2}\right) + h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a) + \frac{b-a}{2}f(\sigma(a)) - \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x.$$

Similarly,

$$-\int_{\frac{a+b}{2}}^{b} h_2\left(x,\frac{a+b}{2}\right)(f^{\Delta\Delta}(x)-m)\Delta = mh_3\left(b,\frac{a+b}{2}\right) - h_2\left(b,\frac{a+b}{2}\right)f^{\Delta}(b) + \frac{b-a}{2}f(\sigma(b)) - \int_{\frac{a+b}{2}}^{b} f(\sigma^2(x))\Delta x.$$

Therefore,

$$\begin{aligned}I &= \int_a^b F(t)G(t)\Delta t\\ &= \frac{1}{M-m}\left[-Mh_3\left(a,\frac{a+b}{2}\right) + h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a) + \frac{b-a}{2}f^{\sigma}(a)\right.\\ &\quad\left. - \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x + mh_3\left(b,\frac{a+b}{2}\right) - h_2\left(b,\frac{a+b}{2}\right)f^{\Delta}(b) + \frac{b-a}{2}f^{\sigma}(b) - \int_{\frac{a+b}{2}}^{b} f(\sigma^2(x))\Delta x\right]\\ &= -\frac{1}{M-m}\left\{\int_a^b f(\sigma^2(x))\Delta x - \frac{b-a}{2}[f^{\sigma}(a)+f^{\sigma}(b)] + h_2\left(b,\frac{a+b}{2}\right)f^{\Delta}(b) - h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a)\right.\\ &\quad\left. + \left[Mh_3\left(a,\frac{a+b}{2}\right) - mh_3\left(b,\frac{a+b}{2}\right)\right]\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_a^b f(\sigma^2(x))\Delta x - \frac{b-a}{2}[f^{\sigma}(a)+f^{\sigma}(b)] + h_2\left(b,\frac{a+b}{2}\right)f^{\Delta}(b) - h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a)\\ &\quad = -(M-m)\int_a^b F(t)G(t)\Delta t - \left[Mh_3\left(a,\frac{a+b}{2}\right) - mh_3\left(b,\frac{a+b}{2}\right)\right]\end{aligned}$$

which implies that

$$\begin{aligned}&(M-m)\left[\operatorname{sgn}\left(\lambda-\frac{b-a}{2}\right)h_3\left(a+\lambda,\frac{a+b}{2}\right) + h_3\left(a,\frac{a+b}{2}\right)\right] - \left[Mh_3\left(a,\frac{a+b}{2}\right) - mh_3\left(b,\frac{a+b}{2}\right)\right]\\ &\quad\leqslant \int_a^b f(\sigma^2(x))\Delta x - \frac{b-a}{2}[f^{\sigma}(a)+f^{\sigma}(b)] + h_2\left(b,\frac{a+b}{2}\right)f^{\Delta}(b) - h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a)\\ &\quad\leqslant (M-m)\left[\operatorname{sgn}\left(\lambda-\frac{b-a}{2}\right)h_3\left(b-\lambda,\frac{a+b}{2}\right) + h_3\left(b,\frac{a+b}{2}\right)\right] - \left[Mh_3\left(a,\frac{a+b}{2}\right) - mh_3\left(b,\frac{a+b}{2}\right)\right].\end{aligned} \tag{10}$$

This proves the theorem. □

Remark 4. If we apply the inequality (5) to different time scales, we can get some well-known and some new results. We only give an example of the case $\mathbb{T}=\mathbb{R}$ here. The interested reader can investigate the case $\mathbb{T}=\mathbb{Z},\ \mathbb{T}=q^{\mathbb{N}_0}$ (see also [15–17]).

Corollary 1 (*Continuous case*). *Let $\mathbb{T}=\mathbb{R}$. Then it is known that*

$$h_k(t,s)=\frac{(t-s)^k}{k!}$$

for all $t,\ s\in\mathbb{R}$ and all $1\leqslant k\in\mathbb{N}$. Moreover,

$$\lambda=\frac{b-a}{2}+\frac{1}{M-m}\left[\underbrace{f'(a)-2f'\left(\frac{a+b}{2}\right)+f'(b)}_{\Delta}\right].$$

Thus, (5) reads

$$\begin{aligned}&(M-m)\left(\operatorname{sgn}\left(\underbrace{\lambda-\frac{b-a}{2}}_{\frac{1}{M-m}\Delta}\right)\underbrace{h_3\left(a+\lambda,\frac{a+b}{2}\right)}_{\frac{\left(\frac{1}{M-m}\Delta\right)^3}{6}}+\underbrace{h_3\left(a,\frac{a+b}{2}\right)}_{\frac{(a-b)^3}{48}}\right)-\left[M\underbrace{h_3\left(a,\frac{a+b}{2}\right)}_{\frac{(a-b)^3}{48}}-m\underbrace{h_3\left(b,\frac{a+b}{2}\right)}_{\frac{(b-a)^3}{48}}\right]\\
&\leqslant\int_a^b\underbrace{f(\sigma^2(x))}_{f(x)}\Delta x-\frac{b-a}{2}\underbrace{[f^\sigma(a)+f^\sigma(b)]}_{f(a)+f(b)}+\underbrace{h_2\left(b,\frac{a+b}{2}\right)f^\Delta(b)}_{\frac{(b-a)^2}{8}f'(b)}-\underbrace{h_2\left(a,\frac{a+b}{2}\right)f^\Delta(a)}_{\frac{(b-a)^2}{8}f'(a)}\\
&\leqslant(M-m)\left(\operatorname{sgn}\left(\underbrace{\lambda-\frac{b-a}{2}}_{\frac{1}{M-m}\Delta}\right)\underbrace{h_3\left(b-\lambda,\frac{a+b}{2}\right)}_{-\frac{\left(\frac{1}{M-m}\Delta\right)^3}{6}}+\underbrace{h_3\left(b,\frac{a+b}{2}\right)}_{\frac{(b-a)^3}{48}}\right)-\left[M\underbrace{h_3\left(a,\frac{a+b}{2}\right)}_{\frac{(a-b)^3}{48}}-m\underbrace{h_3\left(b,\frac{a+b}{2}\right)}_{\frac{(b-a)^3}{48}}\right]\qquad(11)\end{aligned}$$

which implies

$$\begin{aligned}-\frac{M-m}{6}\left[\left(\frac{b-a}{2}\right)^3-\left(\frac{|\Delta|}{M-m}\right)^3\right]-\frac{(b-a)^3}{48}(M+m)&\leqslant\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]+\frac{f'(b)-f'(a)}{8}(b-a)^2\\
&\leqslant\frac{M-m}{6}\left[\left(\frac{b-a}{2}\right)^3-\left(\frac{|\Delta|}{M-m}\right)^3\right]-\frac{(b-a)^3}{48}(M+m).\qquad(12)\end{aligned}$$

Thus we obtain

$$\left|\int_a^b f(x)dx-\frac{b-a}{2}[f(a)+f(b)]+\frac{f'(b)-f'(a)}{8}(b-a)^2+\frac{(b-a)^3}{48}(M+m)\right|\leqslant\frac{M-m}{6}\left[\left(\frac{b-a}{2}\right)^3-\left(\frac{|\Delta|}{M-m}\right)^3\right]$$

which is exactly the inequality shown in Theorem 1 *of* [11]. *We get the inequality* (2) *in* Theorem 2 *if we set $M=-m=M_2$.*

In our next result, we shall generalize Theorem 3 to arbitrary time scales.

Theorem 7. *Let $a,\ \frac{a+b}{2},\ b\in\mathbb{T}^\kappa$ and $f:[a,b]\to\mathbb{R}$ be a twice differentiable function and $f^{\Delta\Delta}:[a,b]\cap\mathbb{T}^{\kappa^2}\to\mathbb{R}$ is bounded, i.e.*

$$M:=\sup_{t\in[a,b]\cap\mathbb{T}^{\kappa^2}}|f^{\Delta\Delta}(t)|<\infty.$$

Then

$$\begin{aligned}&2M\int_{\frac{a+b}{2}-\lambda_a}^{\frac{a+b}{2}+\lambda_b}h_2\left(x,\frac{a+b}{2}\right)\Delta x+M\left[h_3\left(a,\frac{a+b}{2}\right)-h_3\left(b,\frac{a+b}{2}\right)\right]\\
&\leqslant\int_a^b f(\sigma^2(x))\Delta x-\frac{b-a}{2}[f^\sigma(a)+f^\sigma(b)]+h_2\left(b,\frac{a+b}{2}\right)f^\Delta(b)-h_2\left(a,\frac{a+b}{2}\right)f^\Delta(a)\\
&\leqslant2M\left[\int_a^{a+\lambda_a}h_2\left(x,\frac{a+b}{2}\right)\Delta x+\int_{b-\lambda_b}^b h_2\left(x,\frac{a+b}{2}\right)\Delta x\right]+M\left[h_3\left(a,\frac{a+b}{2}\right)-h_3\left(b,\frac{a+b}{2}\right)\right],\qquad(13)\end{aligned}$$

where

$$\lambda_a = \frac{b-a}{4} + \frac{1}{2M}\left[f^{\Delta}\left(\frac{a+b}{2}\right) - f^{\Delta}(a)\right]$$

and

$$\lambda_b = \frac{b-a}{4} + \frac{1}{2M}\left[f^{\Delta}(b) - f^{\Delta}\left(\frac{a+b}{2}\right)\right]$$

are such that $a+\lambda_a \in \mathbb{T}$ *and* $b-\lambda_a \in \mathbb{T}$.

Proof. We shall apply Theorem 5 for $F(x) = h_2\left(x, \frac{a+b}{2}\right)$, $G(x) = \frac{f^{\Delta\Delta}(x)+M}{2M}$. We have $0 \leqslant G(x) \leqslant 1$ for each $x \in [a,b] \cap \mathbb{T}^{\kappa^2}$. So, on $[a, \frac{a+b}{2}] \cap \mathbb{T}^{\kappa^2}$, Theorem 5 tells us that

$$\int_{\frac{a+b}{2}-\lambda_a}^{\frac{a+b}{2}} F(x)\Delta x \leqslant \int_a^{\frac{a+b}{2}} F(x)G(x)\Delta x \leqslant \int_a^{a+\lambda_a} F(x)\Delta x,$$

which is equivalent to

$$2M\int_{\frac{a+b}{2}-\lambda_a}^{\frac{a+b}{2}} F(x)\Delta x \leqslant 2M\int_a^{\frac{a+b}{2}} F(x)G(x)\Delta x \leqslant 2M\int_a^{a+\lambda_a} F(x)\Delta x, \tag{14}$$

where

$$\lambda_a = \int_a^{\frac{a+b}{2}} G(x)\Delta x = \frac{1}{2M}\int_a^{\frac{a+b}{2}} (f^{\Delta\Delta}(x)+M)\Delta x = \frac{b-a}{4} + \frac{1}{2M}\left[f^{\Delta}\left(\frac{a+b}{2}\right) - f^{\Delta}(a)\right].$$

The middle term of inequality (14) is

$$\begin{aligned}2M\int_a^{\frac{a+b}{2}} F(x)G(x)\Delta x &= \int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)(f^{\Delta\Delta}(x)+M)\Delta x = M\int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)\Delta x + \int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)f^{\Delta\Delta}(x)\Delta x\\ &= -M\int_{\frac{a+b}{2}}^{a} h_2\left(x,\frac{a+b}{2}\right)\Delta x + \int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)f^{\Delta\Delta}(x)\Delta x\\ &= -Mh_3\left(a,\frac{a+b}{2}\right) + \int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)f^{\Delta\Delta}(x)\Delta x.\end{aligned}$$

With the help of Proposition 4(iv) and Remark 3, one has

$$\begin{aligned}\int_a^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)f^{\Delta\Delta}(x)\Delta x &= \left[h_2\left(x,\frac{a+b}{2}\right)f^{\Delta}(x)\right]\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}}\left[h_2\left(x,\frac{a+b}{2}\right)\right]^{\Delta} f^{\Delta}(\sigma(x))\Delta x\\ &= -h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a) - \int_a^{\frac{a+b}{2}} h_1\left(x,\frac{a+b}{2}\right)f^{\Delta}(\sigma(x))\Delta x.\end{aligned}$$

Again by Proposition 4(iv) and Remark 3, one obtains

$$\begin{aligned}\int_a^{\frac{a+b}{2}} h_1\left(x,\frac{a+b}{2}\right)f^{\Delta}(\sigma(x))\Delta x &= \left[h_1\left(x,\frac{a+b}{2}\right)f(\sigma(x))\right]\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}}\left[h_1\left(x,\frac{a+b}{2}\right)\right]^{\Delta} f(\sigma(\sigma(x)))\Delta x\\ &= -h_1\left(a,\frac{a+b}{2}\right)f(\sigma(a)) - \int_a^{\frac{a+b}{2}} h_0\left(x,\frac{a+b}{2}\right)f(\sigma^2(x))\Delta x = \frac{b-a}{2}f(\sigma(a)) - \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x.\end{aligned}$$

Thus,

$$2M\int_a^{\frac{a+b}{2}} F(x)G(x)\Delta x = -Mh_3\left(a,\frac{a+b}{2}\right) - h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a) - \frac{b-a}{2}f^{\sigma}(a) + \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x.$$

Therefore, we get

$$\begin{aligned}2M\int_{\frac{a+b}{2}-\lambda_a}^{\frac{a+b}{2}} h_2\left(x,\frac{a+b}{2}\right)\Delta x + Mh_3\left(a,\frac{a+b}{2}\right) &\leqslant \int_a^{\frac{a+b}{2}} f(\sigma^2(x))\Delta x - \frac{b-a}{2}f^{\sigma}(a) - h_2\left(a,\frac{a+b}{2}\right)f^{\Delta}(a)\\ &\leqslant 2M\int_a^{a+\lambda_a} h_2\left(x,\frac{a+b}{2}\right)\Delta x + Mh_3\left(a,\frac{a+b}{2}\right).\end{aligned} \tag{15}$$

On $\left[\frac{a+b}{2}, b\right] \cap \mathbb{T}^{\kappa^2}$, similar to (14) one has

$$2M \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\lambda_b} F(x)\Delta x \leqslant 2M \int_{\frac{a+b}{2}}^{b} F(x)G(x)\Delta x \leqslant 2M \int_{b-\lambda_b}^{b} F(x)\Delta x, \tag{16}$$

where

$$\lambda_b = \int_{\frac{a+b}{2}}^{b} G(x)\Delta x = \frac{1}{2M}\int_{\frac{a+b}{2}}^{b} (f^{\Delta\Delta}(x) + M)\Delta x = \frac{b-a}{4} + \frac{1}{2M}\left[f^{\Delta}(b) - f^{\Delta}\left(\frac{a+b}{2}\right)\right].$$

The middle term of inequality (16) is

$$\begin{aligned} 2M \int_{\frac{a+b}{2}}^{b} F(x)G(x)\Delta x &= \int_{\frac{a+b}{2}}^{b} h_2\left(x, \frac{a+b}{2}\right)(f^{\Delta\Delta}(x) + M)\Delta x \\ &= Mh_3\left(b, \frac{a+b}{2}\right) + h_2\left(b, \frac{a+b}{2}\right)f^{\Delta}(b) - \int_{\frac{a+b}{2}}^{b}\left(x - \frac{a+b}{2}\right)f^{\Delta}(\sigma(x))\Delta x \\ &= Mh_3\left(b, \frac{a+b}{2}\right) + h_2\left(b, \frac{a+b}{2}\right)f^{\Delta}(b) - \frac{b-a}{2}f^{\sigma}(b) + \int_{\frac{a+b}{2}}^{b} f(\sigma^2(x))\Delta x. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 2M \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\lambda_b} h_2\left(x, \frac{a+b}{2}\right)\Delta x - Mh_3\left(b, \frac{a+b}{2}\right) &\leqslant \int_{\frac{a+b}{2}}^{b} f(\sigma^2(x))\Delta x - \frac{b-a}{2}f^{\sigma}(b) + h_2\left(b, \frac{a+b}{2}\right)f^{\Delta}(b) \\ &\leqslant 2M \int_{b-\lambda_b}^{b} h_2\left(x, \frac{a+b}{2}\right)\Delta x - Mh_3\left(b, \frac{a+b}{2}\right). \end{aligned} \tag{17}$$

Addition of (15) and (17) implies (13). □

Remark 5. If we apply the inequality (13) to different time scales, we can get some well-known and some new results. For example, in the special case $\mathbb{T} = \mathbb{R}$, we get the inequality (3) in Theorem 3. To be precise, we refer the reader to Corollary 1. The interested reader can investigate the case $\mathbb{T} = \mathbb{Z}, \mathbb{T} = q^{\mathbb{N}_0}$.

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References

[1] R. Agarwal, M. Bohner, A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl. 4 (2001) 535–557.
[2] R.P. Agarwal, V. Culjak, J. Pečarić, Some integral inequalities involving bounded higher order derivatives, Math. Comput. Model. 28 (1998) 51–57.
[3] D.R. Anderson, Time-scale integral inequalities, JIPAM. J. Inequal. Pure Appl. Math. 6 (2005). Art. ID 66 , 15pp..
[4] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkhäuser, Boston, 2001.
[5] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[6] M. Bohner, T. Matthews, The Grüss inequality on time scales, Commun. Math. Anal. 3 (2007) 1–8.
[7] M. Bohner, T. Matthews, Ostrowski inequalities on time scales, JIPAM. J. Inequal. Pure Appl. Math. 9 (2008). Art. 6, 8pp..
[8] P. Cerone, S.S. Dragomir, On a weighted generalization of Iyengar type inequalities involving bounded first derivative, Math. Inequal. Appl. 3 (1) (2000) 35–44.
[9] P. Cerone, Generalized trapezoidal rules with error involving bounds of the nth derivative, Math. Inequal. Appl. 5 (2002) 451–462.
[10] X.L. Cheng, The Iyengar type inequality, Appl. Math. Lett. 14 (2001) 975–978.
[11] N. Elezović, J. Pečarić, Steffensen's inequality and estimates of error in trapezoidal rule, Appl. Math. Lett. 11 (1998) 63–69.
[12] I. Franjić, J. Pečarić, I. Perić, Note on an Iyengar type inequality, Appl. Math. Lett. 19 (2006) 657–660.
[13] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universität. Würzburg, 1988.
[14] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic Systems on Measure Chains, Kluwer Academic Publishers, 1996.
[15] W.J. Liu, Q.A. Ngô, A generalization of Ostrowski inequality on time scales for k points, Appl. Math. Comput. 203 (2008) 754–760.
[16] W.J. Liu, Q.A. Ngô, W.B. Chen, A perturbed Ostrowski type inequality on time scales for k points for functions whose second derivatives are bounded, J. Inequal. Appl. (2008). Art. ID 597241, 12pp..
[17] W.J. Liu, Q.A. Ngô, W.B. Chen, Ostrowski type inequalities on time scales for double integrals, Acta Appl. Math. 110 (2010) 477–497.
[18] K.S.K. Iyengar, Note on an inequality, Math. Student 6 (1938) 75–76.
[19] P.M. Vasić, J. Pečarić, Note on the Steffensen inequality, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 716–734 (1981) 80–82.
[20] H. Roman, A time scales version of a Wirtinger-type inequality and applications, dynamic equations on time scales, J. Comput. Appl. Math. 141 (2002) 219–226.
[21] M.Z. Sarikaya, On weighted Iyengar type inequalities on time scales, Appl. Math. Lett. 22 (2009) 1340–1344.
[22] F.-H. Wong, S.-L. Yu, C.-C. Yeh, Anderson's inequality on time scales, Appl. Math. Lett. 19 (2007) 931–935.