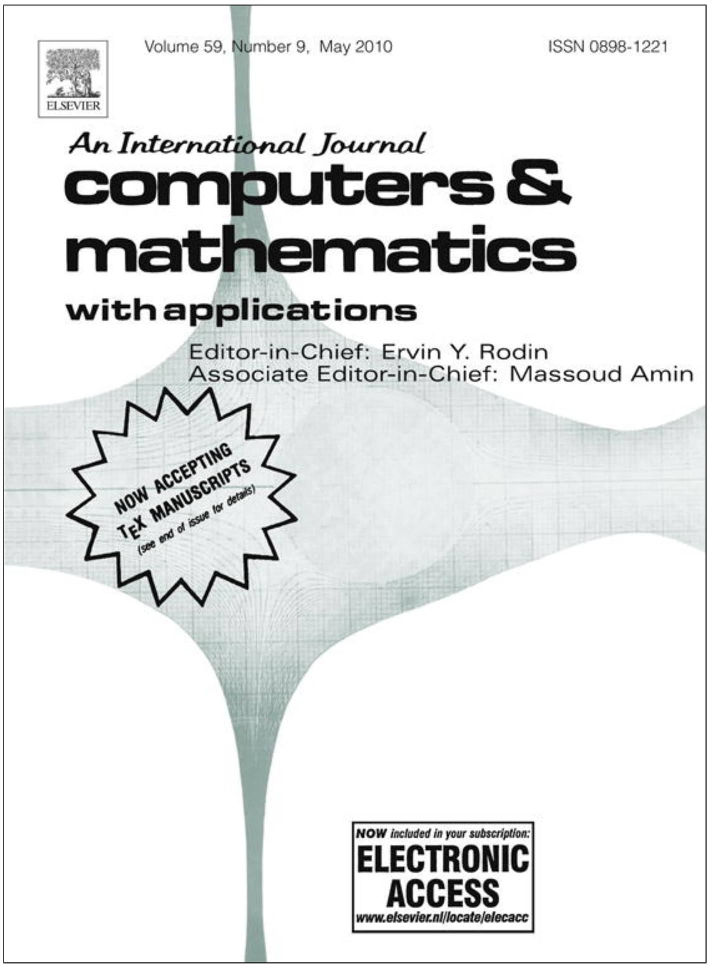
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A new way to think about Ostrowski-like type inequalities

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ABSTRACT

In this present paper, by considering some known inequalities of Ostrowski-like type, we propose a new way to treat a class of Ostrowski-like type inequalities involving n points and m-th derivative. To be precise, the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{n}\sum_{i=1}^{n} f(a + x_i(b-a))\right| \leqq \frac{2m+5}{4}\frac{(b-a)^{m+1}}{(m+1)!}(S-s) \tag{$\star$}$$

holds, where $S := \sup_{a\leq x\leq b} f^{(m)}(x)$, $s := \inf_{a\leq x\leq b} f^{(m)}(x)$ and for suitable $x_1, x_2, \ldots, x_n$. It is worth noticing that n, m are arbitrary numbers. This means that the estimate in ($\star$) is more accurate when m is large enough. Our approach is also elementary.

1. Introduction

In recent years, a number of authors have considered error inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of these formulas, see [1–5] and their references therein where the midpoint and trapezoid quadrature rules are considered.

In [6, Corollary 3] the following Simpson–Grüss type inequalities have been proved. If $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is an absolutely continuous function and

$$\gamma_n \leqq f^{(n)}(t) \leqq \Gamma_n, \quad \text{(a.e.) on } [a, b]$$

for some real constants γ_n and Γ_n, then for $n = 1, 2, 3$, we have

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leqq C_n(\Gamma_n - \gamma_n)(b-a)^{n+1}, \tag{1}$$

where

$$C_1 = \frac{5}{72}, \qquad C_2 = \frac{1}{162}, \qquad C_3 = \frac{1}{1152}.$$

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In [2, Theorem 3], the following results obtained: Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have

$$\left| \int_a^b f(x)\, \mathrm{d}x - \frac{b-a}{2} \left(f\left(\frac{a+b}{2} - \left(2-\sqrt{3}\right)(b-a) \right) + f\left(\frac{a+b}{2} + \left(2-\sqrt{3}\right)(b-a) \right) \right) \right| \leqq \frac{7-4\sqrt{3}}{8} \left\| f'' \right\|_\infty (b-a)^3. \tag{2}$$

In the above mentioned results, constants C_n in (1) and $\frac{7-4\sqrt{3}}{8}$ in (2) are sharp in the sense that these cannot be replaced by smaller ones. We may think the estimate in (1) involves the following six points x_i, $i = \overline{1,6}$ which will be called knots in the sequel

$$a + \underbrace{0}_{x_1} \times (b-a) < a + \underbrace{\frac{1}{2}}_{x_2} (b-a) = \cdots = a + \underbrace{\frac{1}{2}}_{x_5} (b-a) < a + \underbrace{1}_{x_6} \times (b-a).$$

While in (2), we have two knots $x_1 < x_2$ as following

$$a + \underbrace{\left(\frac{1}{2} - \left(2-\sqrt{3}\right) \right)}_{x_1} \times (b-a) < a + \underbrace{\left(\frac{1}{2} + \left(2-\sqrt{3}\right) \right)}_{x_2} (b-a).$$

On the other hand, as can be seen in both (1) and (2) the number of knots in the left hand side reflects the exponent of $b-a$ in the right hand side. This leads us to strengthen (1)–(2) by enlarging the number of knots (six knots in (1) and two knots in (2)).

Before stating our main result, let us introduce the following notation

$$I(f) = \int_a^b f(x)\, \mathrm{d}x.$$

Let $1 \leqq m, n < \infty$. For each $i = \overline{1,n}$, we assume $0 < x_i < 1$ such that

$$\begin{cases} x_1 + x_2 + \cdots + x_n = \dfrac{n}{2}, \\ \cdots \\ x_1^j + x_2^j + \cdots + x_n^j = \dfrac{n}{j+1}, \\ \cdots \\ x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1} = \dfrac{n}{m}, \\ x_1^m + x_2^m + \cdots + x_n^m = \dfrac{n}{m+1}. \end{cases}$$

Put

$$Q(f, n, m, x_1, \ldots, x_n) = \frac{b-a}{n} \sum_{i=1}^n f(a + x_i(b-a)).$$

Remark 1. With the above notations, (1) reads as follows

$$\left| I(f) - Q\left(f, 6, m, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right) \right| \leqq C_m (\Gamma_m - \gamma_m)(b-a)^{m+1}, \quad m = \overline{1,3}, \tag{3}$$

while (2) reads as follows

$$\left| I(f) - Q\left(f, 2, 2, \frac{1}{2} - \left(2-\sqrt{3}\right), \frac{1}{2} + \left(2-\sqrt{3}\right)\right) \right| \leqq \frac{7-4\sqrt{3}}{8} \left\| f'' \right\|_\infty (b-a)^3. \tag{4}$$

We are now in a position to state our main result.

Theorem 2. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a m-th differentiable function. Then we have*

$$|I(f) - Q(f, n, m, x_1, \ldots, x_n)| \leqq \frac{2m+5}{4} \frac{(b-a)^{m+1}}{(m+1)!} (S - s) \tag{5}$$

where $S := \sup_{a \le x \le b} f^{(m)}(x)$ and $s := \inf_{a \le x \le b} f^{(m)}(x)$.

Remark 3. It is worth noticing that the right hand side of (5) does not involve $x_i, i = \overline{1, n}$ and that m can be chosen arbitrarily. This means that our inequality (5) is better in some sense, especially when $b - a \ll 1$.

This work can be considered as a continued and complementary part to a recent paper [7]. More specifically, [7, Theorem 4] provides a similar estimate as (5). However, in contrast to the result presented here our estimate in (5) depends only on the L^p-norm of $f^{(m)}(x)$. There is one thing we should mention here; both Theorem 2 presented here and Theorem 4 in [7] are not optimal. This is because of the restriction of the technique that we use. It is better if we leave these to be solved by the interested reader.

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 4 (*See [8]*). *Let $f : [a, b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a, b]$, $x_0 \in (a, b)$ then for all $x \in (a, b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x)$$

where $T_{r-1}(f, x_0, \cdot)$ is Taylor's polynomial of degree $r - 1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x - t)^{r-1} f^{(r)}(t)}{(r - 1)!} \, dt. \tag{6}$$

By a simple calculation, the remainder in (6) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x - x_0 - t)^{r-1} f^{(r)}(x_0 + t)}{(r - 1)!} \, dt$$

which helps us to deduce a similar representation of f as follows

$$f(x + u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u - t)^{r-1}}{(r - 1)!} f^{(r)}(x + t) \, dt. \tag{7}$$

Before proving Theorem 2, we see that

$$\frac{1}{b - a}\left(\int_a^b \frac{(b - x)^m}{m!} \, dx\right)\left(\int_a^b f^{(m)}(x) \, dx\right) = \frac{(b - a)^m}{(m + 1)!}\left(f^{(m-1)}(b) - f^{(m-1)}(a)\right).$$

Since

$$(b - a) s \leqq f^{(m-1)}(b) - f^{(m-1)}(a) \leqq (b - a) S$$

then

$$\frac{(b - a)^{m+1}}{(m + 1)!} s \leqq \frac{1}{b - a}\left(\int_a^b \frac{(b - x)^m}{m!} \, dx\right)\left(\int_a^b f^{(m)}(x) \, dx\right) \leqq \frac{(b - a)^{m+1}}{(m + 1)!} S.$$

Besides,

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\left[\left(\int_a^b \frac{x_i^m (b - x)^{m-1}}{(m - 1)!} \, dx\right)\left(\int_a^b f^{(m)}((1 - x_i)a + x_i x) \, dx\right)\right] \\ &\quad = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{x_i^m (b - a)^m}{m!}\left(\int_a^b f^{(m)}((1 - x_i)a + x_i x) \, dx\right)\right]. \end{aligned}$$

Clearly,

$$(b - a) s \leqq \int_a^b f^{(m)}((1 - x_i)a + x_i x) \, dx \leqq (b - a) S$$

which implies that

$$\frac{(b - a)^{m+1}}{(m + 1)!} s \leqq \frac{1}{n}\sum_{i=1}^{n}\left(\int_a^b \frac{x_i^m (b - x)^{m-1}}{(m - 1)!} \, dx\right)\left(\int_a^b f^{(m)}((1 - x_i)a + x_i x) \, dx\right) \leqq \frac{(b - a)^{m+1}}{(m + 1)!} S.$$

Lemma 5 (*Grüss Inequality, See [9]*). *Let f and g be two functions defined and integrable over $[a, b]$. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right)\right| \leqq \frac{1}{4}(S_1 - s_1)(S_2 - s_2)$$

where $s_1 \leqq f(x) \leqq S_1$ and $s_2 \leqq g(x) \leqq S_2$ for all $x \in [a, b]$.

Proof of Theorem 2. Denote

$$F(x) = \int_a^x f(x)\,\mathrm{d}x.$$

By the Fundamental Theorem of Calculus

$$I(f) = F(b) - F(a).$$

Applying Lemma 4 to $F(x)$ with $x = a$ and $u = b - a$, we get

$$F(b) = F(a) + \sum_{k=1}^{m}\frac{(b-a)^k}{k!}F^{(k)}(a) + \int_a^b \frac{(b-t)^m}{m!}F^{(m+1)}(t)\,\mathrm{d}t$$

which yields

$$I(f) = \sum_{k=1}^{m}\frac{(b-a)^k}{k!}F^{(k)}(a) + \int_a^b \frac{(b-t)^m}{m!}F^{(m+1)}(t)\,\mathrm{d}t.$$

Equivalently,

$$I(f) = \sum_{k=0}^{m-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a) + \int_a^b \frac{(b-t)^m}{m!}f^{(m)}(t)\,\mathrm{d}t. \tag{8}$$

For each $1 \leqq i \leqq n$, applying Lemma 4 to $f(x)$ with $x = a$ and $u = x_i(b - a)$, we get

$$\begin{aligned} f(a + x_i(b-a)) &= \sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a) + \int_0^{x_i(b-a)}\frac{(x_i(b-a)-t)^{m-1}}{(m-1)!}f^{(m)}(a+t)\,\mathrm{d}t \\ &= \sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a) + \int_0^{b-a}\frac{x_i^m(b-a-u)^{m-1}}{(m-1)!}f^{(m)}(a+x_iu)\,\mathrm{d}u \\ &= \sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a) + \int_a^b\frac{x_i^m(b-u)^{m-1}}{(m-1)!}f^{(m)}(a(1-x_i)+x_iu)\,\mathrm{d}u. \end{aligned} \tag{9}$$

By applying (9) to $i = \overline{1, n}$ and then summing up, we deduce that

$$\begin{aligned} \sum_{i=1}^{n} f(a + x_i(b-a)) &= \sum_{i=1}^{n}\sum_{k=0}^{m-1}\frac{x_i^k(b-a)^k}{k!}f^{(k)}(a) + \sum_{i=1}^{n}\int_a^b\frac{x_i^m(b-u)^{m-1}}{(m-1)!}f^{(m)}(a(1-x_i)+x_iu)\,\mathrm{d}u \\ &= \sum_{k=0}^{m-1}\frac{\sum_{i=1}^{n}x_i^k(b-a)^k}{k!}f^{(k)}(a) + \sum_{i=1}^{n}\int_a^b\frac{x_i^m(b-u)^{m-1}}{(m-1)!}f^{(m)}(a(1-x_i)+x_iu)\,\mathrm{d}u \\ &= \sum_{k=0}^{m-1}\frac{n(b-a)^k}{(k+1)!}f^{(k)}(a) + \sum_{i=1}^{n}\int_a^b\frac{x_i^m(b-u)^{m-1}}{(m-1)!}f^{(m)}(a(1-x_i)+x_iu)\,\mathrm{d}u. \end{aligned} \tag{10}$$

Thus,

$$Q(f, n, m, x_1, \ldots, x_n) = \sum_{k=0}^{m-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a) + \frac{b-a}{n}\sum_{i=1}^{n}\int_a^b\frac{x_i^m(b-u)^{m-1}}{(m-1)!}f^{(m)}(a(1-x_i)+x_iu)\,\mathrm{d}u. \tag{11}$$

Therefore, by combining (8) and (11), we get

$$I(f) - Q(f, n, m, x_1, \ldots, x_n) = \int_a^b\frac{(b-x)^m}{m!}f^{(m)}(x)\,\mathrm{d}x - \frac{b-a}{n}\sum_{i=1}^{n}\int_a^b\frac{x_i^m(b-x)^{m-1}}{(m-1)!}f^{(m)}((1-x_i)a + x_ix)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{(b-a)^m}{(m+1)!}(f^{(m-1)}(b) - f^{(m-1)}(a)) + \left[\int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)\,\mathrm{d}x - \frac{1}{b-a}\left(\int_a^b \frac{(b-x)^m}{m!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}(x)\,\mathrm{d}x\right)\right] \\
&\quad - \frac{b-a}{n}\sum_{i=1}^n \left[\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right. \\
&\quad \left. - \frac{1}{b-a}\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right] \\
&\quad - \frac{1}{n}\sum_{i=1}^n \left[\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right] \\
&= \frac{(b-a)^m}{(m+1)!}(f^{(m-1)}(b) - f^{(m-1)}(a)) - \frac{1}{n}\sum_{i=1}^n \left[\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right] \\
&\quad + \left[\int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)\,\mathrm{d}x - \frac{1}{b-a}\left(\int_a^b \frac{(b-x)^m}{m!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}(x)\,\mathrm{d}x\right)\right] \\
&\quad - \frac{b-a}{n}\sum_{i=1}^n \left[\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right. \\
&\quad \left. - \frac{1}{b-a}\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right].
\end{aligned}
$$

Then it follows from using the Grüss inequality that

$$
\left|\int_a^b \frac{(b-x)^m}{m!} f^{(m)}(x)\,\mathrm{d}x - \frac{1}{b-a}\left(\int_a^b \frac{(b-x)^m}{m!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}(x)\,\mathrm{d}x\right)\right| \leqq \frac{1}{4}\frac{(b-a)^{m+1}}{m!}(S-s)
$$

and

$$
\begin{aligned}
&\sum_{i=1}^n \left|\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x - \frac{1}{b-a}\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right| \\
&\quad \leqq \frac{1}{4}\sum_{i=1}^n \frac{(b-a)^m x_i^m}{(m-1)!}(S-s) \\
&\quad = \frac{1}{4}\frac{n(b-a)^m}{(m-1)!(m+1)}(S-s).
\end{aligned}
$$

We know that

$$
\frac{(b-a)^{m+1}}{(m+1)!}s \leqq \frac{(b-a)^m}{(m+1)!}\left(f^{(m-1)}(b) - f^{(m-1)}(a)\right) \leqq \frac{(b-a)^{m+1}}{(m+1)!}S
$$

and

$$
\frac{(b-a)^{m+1}}{(m+1)!}s \leqq \frac{1}{n}\sum_{i=1}^n \left[\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right] \leqq \frac{(b-a)^{m+1}}{(m+1)!}S.
$$

Therefore,

$$
\begin{aligned}
&\left|\frac{(b-a)^m}{(m+1)!}\left(f^{(m-1)}(b) - f^{(m-1)}(a)\right) - \frac{1}{n}\sum_{i=1}^n \left[\left(\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\,\mathrm{d}x\right)\left(\int_a^b f^{(m)}((1-x_i)a + x_i x)\,\mathrm{d}x\right)\right]\right| \\
&\quad \leqq \frac{(b-a)^{m+1}}{(m+1)!}(S-s).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|I(f) - Q(f,n,m,x_1,\ldots,x_n)| &\leqq \frac{1}{4}\frac{(b-a)^{m+1}}{m!}(S-s) + \frac{1}{4}\frac{(b-a)^{m+1}}{(m-1)!(m+1)}(S-s) + \frac{(b-a)^{m+1}}{(m+1)!}(S-s) \\
&= \left(\frac{1}{4m} + \frac{1}{4(m+1)} + \frac{1}{m(m+1)}\right)\frac{(b-a)^{m+1}}{(m-1)!}(S-s) \\
&= \frac{2m+5}{4}\frac{(b-a)^{m+1}}{(m+1)!}(S-s)
\end{aligned}
$$

where

$$S := \sup_{a \leqq x \leqq b} f^{(m)}(x) \quad \text{and} \quad s := \inf_{a \leqq x \leqq b} f^{(m)}(x)$$

which completes our proof. □

3. Examples

In this section, by applying our main theorem, we will obtain some new inequalities which cannot be easy obtained by [2,3]. Actually, our result covers several known results in the numerical integration.

Example 6. Assume $n = 6$, $m = 1, 2$, or 3. Clearly $x_1 = 0$, $x_2 = x_3 = x_4 = x_5 = \frac{1}{2}$, and $x_6 = 1$ satisfy the following linear system

$$\begin{cases} x_1 + x_2 + \cdots + x_6 = \dfrac{6}{2}, \\ \cdots \\ x_1^j + x_2^j + \cdots + x_6^j = \dfrac{6}{j+1}, \\ \cdots \\ x_1^m + x_2^m + \cdots + x_6^m = \dfrac{6}{m+1}. \end{cases}$$

Therefore, we obtain the following inequalities

$$\left| \int_a^b f(t)\, dt - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right) \right| \leqq C_m \left(S_m - s_m \right) (b-a)^{m+1} \tag{12}$$

where $S_m = \sup_{a \leq x \leq b} f^{(m)}(x)$ and $s_m = \inf_{a \leq x \leq b} f^{(m)}(x)$ and

$$C_1 = \frac{7}{8}, \qquad C_2 = \frac{9}{24}, \qquad C_3 = \frac{11}{96}.$$

Clearly, the left hand side of (12) is similar to the Simpson rule.

Example 7. Assume $n = 3$, $m = 3$. By solving the following linear system

$$\begin{cases} x_1 + x_2 + x_3 = \dfrac{3}{2}, \\ x_1^2 + x_2^2 + x_3^2 = \dfrac{3}{3}, \\ x_1^3 + x_2^3 + x_3^3 = \dfrac{3}{4}, \end{cases}$$

we obtain $\{x_1, x_2, x_3\}$ is a permutation of

$$\left\{ \frac{1}{2}, 1 - \frac{1}{2}\left(1 \pm \frac{\sqrt{2}}{2} \right), \frac{1}{2}\left(1 \pm \frac{\sqrt{2}}{2} \right) \right\}.$$

Therefore, we obtain the following inequalities

$$\begin{aligned} &\left| \int_a^b f(x)\, \mathrm{d}x - \frac{b-a}{3} \left(f\left(a + \left(1 - \frac{1}{2}\left(1 \pm \frac{\sqrt{2}}{2} \right) \right)(b-a) \right) \right. \right. \\ &\left. \left. + \; f\left(a + \frac{1}{2}(b-a) \right) + f\left(a + \frac{1}{2}\left(1 \pm \frac{\sqrt{2}}{2} \right)(b-a) \right) \right) \right| \leqq \frac{11(b-a)^4}{96} (S - s) \end{aligned} \tag{13}$$

where $S = \sup_{a \leqq x \leqq b} f'''(x)$ and $s = \inf_{a \leqq x \leqq b} f'''(x)$.

Example 8. If $n = 2$, $m = 2$, then by solving the following system

$$\begin{cases} x_1 + x_2 = \dfrac{2}{2}, \\ x_1^2 + x_2^2 = \dfrac{2}{3}, \end{cases}$$

we obtain

$$(x_1, x_2) = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}, \frac{1}{2} \mp \frac{\sqrt{3}}{6}\right).$$

We then obtain a similar 2-point Gaussian quadrature rule

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{2}\left(f\left(a + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)(b-a)\right) + f\left(a + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)(b-a)\right)\right)\right| \leqq \frac{9(b-a)^3}{24}(S-s) \tag{14}$$

where $S = \sup_{a \leqq x \leqq b} f''(x)$ and $s = \inf_{a \leqq x \leqq b} f''(x)$.

Remark 9. Note that using (13) provides a better result than using (14) (or the 2-point Gaussian quadrature rule). For example, let us consider the following function $f(x) = xe^{\sin x}$. Then

$$\int_0^1 f(x)\,\mathrm{d}x \approx 0.9291567730.$$

If we use (13), we then have

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{3}\left(f\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) + f\left(\frac{1}{2}\right) + f\left(f\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)\right)\right) \approx 0.9301849429.$$

If we use (14), we then have

$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{1}{2}\left(f\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) + f\left(f\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\right)\right) \approx 0.9319357678.$$

Example 10. If $m = 2$ and $n = 3$, then by solving the following system

$$\begin{cases} x_1 + x_2 + x_3 = \dfrac{3}{2}, \\ x_1^2 + x_2^2 + x_3^2 = \dfrac{3}{3}, \end{cases}$$

we obtain $\{x_1, x_2, x_3\}$ is a permutation of $\left\{t, \frac{3}{2} - t - k, k\right\}$, where k is a solution of the following algebraic equation

$$8x^2 + (8t - 12)\,x + \left(8t^2 - 12t + 5\right) = 0$$

with

$$t \in \left[\frac{1}{2} - \frac{\sqrt{6}}{6}, \frac{1}{2} + \frac{\sqrt{6}}{6}\right].$$

We then obtain

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{3}\left(f(a + t(b-a)) + f\left(a + \left(\frac{3}{2} - t - k\right)(b-a)\right) + f(a + k(b-a))\right)\right| \leqq \frac{9(b-a)^3}{24}(S-s)$$

where $S = \sup_{a \leqq x \leqq b} f''(x)$ and $s = \inf_{a \leqq x \leqq b} f''(x)$.

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