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ON NEW OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS ON TIME SCALES

WENJUN LIU, QUỐC ANH NGÔ, AND WENBING CHEN

College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China
Department of Mathematics, College of Science, Việt Nam National University, Hà Nội, Việt Nam;
Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543
College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China

ABSTRACT. A new Ostrowski type inequality for double integrals on time scales via $\Delta\Delta$-integral is derived which unify corresponding continuous and discrete versions. Analogous results for $\nabla\nabla$-, $\Delta\nabla$- and $\nabla\Delta$-integrals are also discussed.

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1. INTRODUCTION

The following classical integral inequality established by Ostrowski [28] in 1938 has received considerable attention from many researchers [12, 13, 18, 19, 26, 27].

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative function $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty, \quad \forall\, x \in [a,b].$$

In the present paper, we will present a generalization of Ostrowski's inequality using the time scale theory. And thus, for completeness in the paper, we would like to give some basic concepts of the time scale below.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers. The development of the theory of time scales was initiated by Hilger [14] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then, many authors have studied certain integral inequalities or dynamic

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equations on time scales ([1, 10, 11, 15, 16, 29, 31, 32, 33]). We refer the reader to Hilger's Ph.D. thesis [14] and the books [5, 6, 17] for details of the (one-variable) time scales theory. We also refer the reader to [7, 8, 9] for the two dimensional time scales calculus and the so-call Riemann $\Delta\Delta$-integrals. The Riemann $\nabla\nabla$-integrals, Riemann $\Delta\nabla$-integrals and Riemann $\nabla\Delta$-integrals were developed by the present authors in [24].

In [11], Bohner and Matthews established the following Ostrowski inequality on time scales which was later generalized by the present authors ([20, 21, 22, 23]).

Theorem 1.2 (See [11, Theorem 3.5])**.** *Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f^\sigma(t)\Delta t \right| \leq \frac{M}{b-a}\Big(h_2(x,a) + h_2(x,b)\Big), \tag{1.1}$$

where $M = \sup_{a<x<b} |f^\Delta(x)|$ (see Definition 1.4 below for $h_2(\cdot,\cdot)$). This inequality is sharp in the sense that the right-hand side of (1.1) can't be replaced by a smaller one.

Recently, the present authors [24] generalized the above Ostrowski inequality on time scales for double integrals and obtained the following result.

Theorem 1.3. *Let $x, t \in \mathbb{T}_1$, $y, s \in \mathbb{T}_2$ and $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$, $\frac{\partial f(t,s)}{\Delta_2 s}$, $\frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}$ exist and are continuous on $[a, b] \times [c, d]$. Then*

$$\begin{aligned}\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\sigma_1(t), \sigma_2(s))\Delta_1 t \Delta_2 s \right| \\ \leq \frac{M_1}{b-a}\Big(h_2(x,a) + h_2(x,b)\Big) + \frac{M_2}{d-c}\Big(h_2(y,c) + h_2(y,d)\Big) \\ + \frac{M_3}{(b-a)(d-c)}\Big(h_2(x,a) + h_2(x,b)\Big)\Big(h_2(y,c) + h_2(y,d)\Big)\end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$M_1 = \sup_{a<t<b} \left| \frac{\partial f(t,s)}{\Delta_1 t} \right|, \quad M_2 = \sup_{c<s<d} \left| \frac{\partial f(t,s)}{\Delta_2 s} \right|, \quad M_3 = \sup_{a<t<b, c<s<d} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right|.$$

In this paper, we shall first derive a new Ostrowski inequality for double integrals on time scales via $\Delta\Delta$-integral, then get completely analogous results via $\nabla\nabla$-, $\Delta\nabla$- and $\nabla\Delta$-integrals. For a general time scale $\mathbb{T}$, we need the following definitions.

Definition 1.4. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$h_0(t,s) = 1 \quad \text{for all} \quad s, t \in \mathbb{T}$$

and then recursively by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau \quad \text{for all} \quad s, t \in \mathbb{T}.$$

Definition 1.5. Let $j_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$j_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}$$

and then recursively by

$$j_{k+1}(t,s) = \int_s^t j_k(\tau,s)\,\nabla\tau \quad \text{for all} \quad s,t \in \mathbb{T}.$$

2. MAIN RESULTS

In this section, we suppose that $\mathbb{T}_i$ is a time scale, $a_i, b_i \in \mathbb{T}_i$ with $b_i > a_i$, h_k^i and j_k^i are the generalized polynomial defined on $\mathbb{T}_i$, $i = 1, 2$.

2.1. **A new Ostrowski's inequality for double integrals via $\Delta\Delta$-integral.** We first derive the following new Ostrowski type inequality for double integrals on time scales via $\Delta\Delta$-integral.

Theorem 2.1. *Let $x, t \in \mathbb{T}_1$, $y, s \in \mathbb{T}_2$ and $f : [a_1, b_1]\times[a_2, b_2] \to \mathbb{R}$ be such that $f(\cdot,\cdot)$ is integrable on $[a_1, b_1]\times[a_2, b_2]$, $f(x,\cdot)$ is integrable on $[a_2, b_2]$ for any $x \in [a_1, b_1]$ and $f(\cdot, y)$ is integrable on $[a_1, b_1]$ for any $y \in [a_2, b_2]$, the partial derivative $\frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}$ exists and is continuous on $[a_1, b_1] \times [a_2, b_2]$. Then*

$$\begin{aligned}
&\bigg|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s \\
&\quad -\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,\sigma_2(s))\Delta_2 s - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(\sigma_1(t),y)\Delta_1 t + f(x,y)\bigg| \\
&\quad \le \frac{M}{(b_1-a_1)(b_2-a_2)}\Big(h_2^1(x,a_1)+h_2^1(x,b_1)\Big)\Big(h_2^2(y,a_2)+h_2^2(y,b_2)\Big)
\end{aligned} \tag{2.1}$$

for all $(x, y) \in [a_1, b_1] \times [a_2, b_2]$, where

$$M = \sup_{a_1<t<b_1, a_2<s<b_2}\left|\frac{\partial^2 f(t,s)}{\Delta_1 t \Delta_2 s}\right|.$$

For easiness in the proof of our main result Theorem 2.1, we need to prove the following lemma of which proof is partially motivated by the key idea employed by Barnett and Dragomir in [4].

Lemma 2.2 (See [30, Lemma 2.3])**.** *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned}
&\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} p_1(x,t)p_2(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_1 t\Delta_2 s \\
&= \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s \\
&\quad -\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,\sigma_2(s))\Delta_2 s - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(\sigma_1(t),y)\Delta_1 t + f(x,y),
\end{aligned}$$

where

$$p_1(x,t) = \begin{cases} t-a_1, & a_1 \le t < x, \\ t-b_1, & x \le t \le b_1, \end{cases} \qquad p_2(y,s) = \begin{cases} s-a_2, & a_2 \le s < y, \\ s-b_2, & y \le s \le b_2. \end{cases}$$

If we apply Lemma 2.2 to the discrete and continuous cases, we have the following results.

Corollary 2.3 (Continuous case)**.** *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Then*

$$\begin{aligned} &\frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} p_1(x,t) p_2(y,s) \frac{\partial^2 f(t,s)}{\partial s \partial t} ds dt \\ &= \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t,s) ds dt \\ &\quad - \left(\frac{1}{b_2-a_2} \int_{a_2}^{b_2} f(x,s) ds + \frac{1}{b_1-a_1} \int_{a_1}^{b_1} f(t,y) dt \right) + f(x,y), \end{aligned}$$

which is exactly the integral identity shown in [4]*.*

Corollary 2.4 (Discrete case)**.** *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a_1 = 0$, $b_1 = n$, $a_2 = 0$, $b_2 = m$, $x = i$, $y = j$, $t = k$, $s = l$, and $f(p,q) = x_p y_q$. Then*

$$\frac{1}{mn} \sum_{k=1}^{n} \sum_{l=1}^{m} p_1(i,k) p_2(j,l) \Delta x_k \Delta y_l = \frac{1}{mn} \sum_{k=1}^{n} \sum_{l=1}^{m} x_k y_l - \frac{1}{m} \sum_{l=1}^{m} x_i y_l - \frac{1}{n} \sum_{k=1}^{n} x_k y_j + x_i y_j,$$

where

$$\begin{aligned} p_1(i,0) &= 0, \\ p_1(1,k) &= k-n, \quad \text{for } 1 \le k \le n-1; \\ p_1(n,k) &= k, \quad \text{for } 0 \le k \le n-1; \\ p_1(i,k) &= \begin{cases} k, & 0 \le k < i, \\ k-n, & i \le k \le n-1, \end{cases} \end{aligned}$$

and

$$\begin{aligned} p_2(j,0) &= 0; \\ p_2(1,l) &= l-n, \quad \text{for } 1 \le l \le m-1; \\ p_2(m,l) &= l, \quad \text{for } 0 \le l \le m-1; \\ p_2(j,l) &= \begin{cases} l, & 0 \le l < j, \\ l-m, & j \le l \le m-1. \end{cases} \end{aligned}$$

Corollary 2.5 (Quantum calculus case). *Let $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ and $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ with $q_1 > 1$ and $q_2 > 1$. Suppose $a_1 = q_1^i, b_1 = q_1^j, a_2 = q_2^k, b_2 = q_2^l$ for some $i < j$ and $k < l$. Then*

$$\frac{\sum_{r=i}^{j-1}\sum_{s=k}^{l-1}\left(\frac{f(q_1^{r+1},q_2^{s+1})-f(q_1^r,q_2^{s+1})-f(q_1^{r+1},q_2^s)+f(q_1^r,q_2^s)}{(q_1-1)(q_2-1)q_1^r q_2^s}\right)p_1(q_1^m,q_1^r)p_2(q_2^n,q_2^s)}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s}$$

$$= \frac{\sum_{r=i}^{j-1}\sum_{s=k}^{l-1} q_1^r q_2^s f(q_1^{r+1},q_2^{s+1})}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} - \frac{\sum_{r=i}^{j-1} q_1^r f(q_1^{r+1},q_2^n)}{\sum_{r=i}^{j-1} q_1^r} - \frac{\sum_{s=k}^{l-1} q_2^s f(q_1^m,q_2^{s+1})}{\sum_{s=k}^{l-1} q_2^s} + f(q_1^m,q_2^n).$$

Proof of Theorem 2.1. By applying Lemma 2.2, we can state that

$$\begin{aligned}
&\Bigg|\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(\sigma_1(t),\sigma_2(s))\Delta_1 t\Delta_2 s \\
&\quad - \left(\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,\sigma_2(s))\Delta_2 s + \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(\sigma_1(t),y)\Delta_1 t\right) + f(x,y)\Bigg| \\
&\quad \le \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} |p_1(x,t)||p_2(y,s)|\left|\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\right|\Delta_1 t\Delta_2 s \\
&\quad \le \frac{M}{(b_1-a_1)(b_2-a_2)}\left(\int_{a_1}^{b_1} |p_1(x,t)|\Delta_1 t\right)\left(\int_{a_2}^{b_2} |p_2(y,s)|\Delta_2 s\right)
\end{aligned}$$

for all $(x,y) \in [a_1,b_1]\times[a_2,b_2]$. By a simple calculation we get

$$\begin{aligned}
\int_{a_1}^{b_1} |p_1(x,t)|\Delta_1 t &= \int_{a_1}^{x} |t-a_1|\Delta_1 t + \int_{x}^{b_1} |t-b_1|\Delta_1 t \\
&= \int_{a_1}^{x} (t-a_1)\Delta_1 t + \int_{x}^{b_1} (b_1-t)\Delta_1 t = h_2^1(x,a_1) + h_2^1(x,b_1)
\end{aligned}$$

and

$$\int_{a_2}^{b_2} |p_2(y,s)|\Delta_2 s = h_2^2(y,a_2) + h_2^2(y,b_2)$$

Therefore, we obtain (2.1). □

If we apply the new Ostrowski type inequality for double integrals to different time scales, we will get some well-known and some new results.

Corollary 2.6 (Continuous case). *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. Then our delta integral is the usual Riemann integral from calculus. Hence,*

$$h_2(t,s) = \frac{(t-s)^2}{2}, \quad \text{for all} \quad t,s \in \mathbb{R}.$$

This leads us to obtain exactly the Ostrowski type inequality for double integrals shown in Theorem 2.1 of [4].

Corollary 2.7 (Discrete case)**.** *Let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a_1 = 0$, $b_1 = n$, $a_2 = 0$, $b_2 = m$, $x = i$, $y = j$, $t = k$, $s = l$, *and* $f(p,q) = x_p y_q$. *With these, it is known that*

$$h_k(t,s) = \binom{t-s}{k}, \qquad \text{for all} \quad t,s \in \mathbb{Z}.$$

Therefore,

$$h_2^1(x,a_1) = \binom{i}{2} = \frac{i(i-1)}{2},$$
$$h_2^1(x,b_1) = \binom{i-n}{2} = \frac{(i-n)(i-n-1)}{2},$$

and

$$h_2^2(y,a_2) = \binom{j}{2} = \frac{j(j-1)}{2},$$
$$h_2^2(y,b_2) = \binom{j-m}{2} = \frac{(j-m)(j-m-1)}{2}.$$

Thus, we have

$$\left|\sum_{k=1}^{n}\sum_{l=1}^{m} x_k y_l - m\sum_{k=1}^{n} f(k,j) - n\sum_{l=1}^{m} f(i,l) + mnx_ix_j\right|$$
$$\leq \frac{M}{mn}\left(\left|i - \frac{n+1}{2}\right|^2 + \frac{n^2-1}{4}\right)\left(\left|j - \frac{m+1}{2}\right|^2 + \frac{m^2-1}{4}\right)$$

for all $i = \overline{1,n}$ *and* $j = \overline{1,m}$, *where*

$$M = \max_{1\leq k\leq n-1, 1\leq l\leq m-1} |\Delta x_k \Delta y_l|.$$

Corollary 2.8 (Quantum calculus case)**.** *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ *with* $q_1 > 1$ *and* $q_2 > 1$. *Suppose* $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ *for some* $i < j$ *and* $k < l$. *In this situation, one has*

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}, \qquad \text{for all} \quad t,s \in \mathbb{T}.$$

Therefore,

$$h_2^1(x,a_1) = \frac{(q_1^m - q_1^i)(q_1^m - q_1^{i+1})}{1+q_1},$$
$$h_2^1(x,b_1) = \frac{(q_1^m - q_1^j)(q_1^m - q_1^{j+1})}{1+q_1},$$

and

$$h_2^2(y,a_2) = \frac{(q_2^n - q_2^k)(q_2^n - q_2^{k+1})}{1+q_2},$$
$$h_2^2(y,b_2) = \frac{(q_2^n - q_2^l)(q_2^n - q_2^{l+1})}{1+q_2}.$$

Then

$$\left| \frac{\sum_{r=i}^{j-1}\sum_{s=k}^{l-1} q_1^r q_2^s f(q_1^{r+1}, q_2^{s+1})}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} - \frac{\sum_{r=i}^{j-1} q_1^r f(q_1^{r+1}, q_2^n)}{\sum_{r=i}^{j-1} q_1^r} - \frac{\sum_{s=k}^{l-1} q_2^s f(q_1^m, q_2^{s+1})}{\sum_{s=k}^{l-1} q_2^s} + f(q_1^m, q_2^n) \right|$$
$$\le \frac{M}{\sum_{r=i}^{j-1} q_1^r \sum_{s=k}^{l-1} q_2^s} \left(\frac{(q_1^m - q_1^i)(q_1^m - q_1^{i+1}) + (q_1^m - q_1^j)(q_1^m - q_1^{j+1})}{1+q_1} \times \frac{(q_2^n - q_2^k)(q_2^n - q_2^{k+1}) + (q_2^n - q_2^l)(q_2^n - q_2^{l+1})}{1+q_2} \right),$$

where

$$M = \sup_{i<m<j-1,\ k<n<l-1} \left| \frac{f(q_1^{m+1}, q_2^{n+1}) - f(q_1^m, q_2^{n+1}) - f(q_1^{m+1}, q_2^n) + f(q_1^m, q_2^n)}{(q_1-1)(q_2-1)q_1^m q_2^n} \right|.$$

Remark 2.9. We note that in the special case, if $f(x,y)$ in Theorem 2.1 does not depend on y, we get back the Ostrowski inequality (1.1) on (one-variable) time scales.

2.2. **Ostrowski's inequality for double integrals via $\Delta\nabla$-integral.** By a completely analogous method, we can derive the following similar result via $\Delta\nabla$-integral. This would be interesting, since the calculus on time scales via ∇ derivatives seem to have many interesting application [2, 3, 25].

Theorem 2.10. *Let $x, t \in \mathbb{T}_1$, $y, s \in \mathbb{T}_2$ and $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be such that $f(\cdot,\cdot)$ is integrable on $[a_1,b_1]\times[a_2,b_2]$, $f(x,\cdot)$ is integrable on $[a_2,b_2]$ for any $x \in [a_1,b_1]$ and $f(\cdot,y)$ is integrable on $[a_1,b_1]$ for any $y \in [a_2,b_2]$, the partial derivative $\frac{\partial^2 f(t,s)}{\nabla_2 s \Delta_1 t}$ exists and is continuous on $[a_1,b_1]\times[a_2,b_2]$. Then*

$$\begin{aligned} &\Bigg| \frac{1}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(\sigma_1(t), \rho_2(s))\Delta_1 t \nabla_2 s \\ &\quad - \left(\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,\rho_2(s))\nabla_2 s + \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(\sigma_1(t), y)\Delta_1 t \right) + f(x,y) \Bigg| \\ &\le \frac{M}{(b_1-a_1)(b_2-a_2)} \Big(h_2^1(x,a_1) + h_2^1(x,b_1) \Big) \Big(j_2^2(y,a_2) + j_2^2(y,b_2) \Big) \end{aligned} \tag{2.2}$$

for all $(x,y) \in [a_1,b_1]\times[a_2,b_2]$, where

$$M = \sup_{a_1<t<b_1, a_2<s<b_2} \left| \frac{\partial^2 f(t,s)}{\Delta_1 t \nabla_2 s} \right|.$$

In fact, to prove Theorem 2.10, we need the following lemma which can be proved similarly to Lemma 2.2.

Lemma 2.11. *Under the assumptions of Theorem* 2.10, *we have*

$$\begin{aligned}
&\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} p_1(x,t)p_2(y,s)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s\\
&= \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s\\
&\quad -\left(\frac{1}{b_2-a_2}\int_{a_2}^{b_2} f(x,\rho_2(s))\nabla_2 s + \frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(\sigma_1(t),y)\Delta_1 t\right) + f(x,y).
\end{aligned}\tag{2.3}$$

Proof. We have the following equality:

$$\begin{aligned}
&\int_{a_1}^{x}\int_{a_2}^{y}(t-a_1)(s-a_2)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s\\
&=\int_{a_1}^{x}(t-a_1)\left((y-a_2)\frac{\partial f(t,y)}{\Delta_1 t}-\int_{a_2}^{y}\frac{\partial f(t,\rho_2(s))}{\Delta_1 t}\nabla_2 s\right)\Delta_1 t\\
&=(y-a_2)\int_{a_1}^{x}(t-a_1)\frac{\partial f(t,y)}{\Delta_1 t}\Delta_1 t-\int_{a_2}^{y}\left(\int_{a_1}^{x}(t-a_1)\frac{\partial f(t,\rho_2(s))}{\Delta_1 t}\Delta_1 t\right)\nabla_2 s\\
&= (y-a_2)\left((x-a_1)f(x,y)-\int_{a_1}^{x} f(\sigma_1(t),y))\Delta_1 t\right)\\
&\quad -\int_{a_2}^{y}\left((x-a_1)f(x,\rho_2(s))-\int_{a_1}^{x} f(\sigma_1(t),\rho_2(s))\Delta_1 t\right)\nabla_2 s\\
&= (y-a_2)(x-a_1)f(x,y)-(y-a_2)\int_{a_1}^{x} f(\sigma_1(t),y))\Delta_1 t\\
&\quad -(x-a_1)\int_{a_2}^{y} f(x,\rho_2(s))\nabla_2 s+\int_{a_1}^{x}\int_{a_2}^{y} f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s.
\end{aligned}\tag{2.4}$$

Also, by similar computations we have

$$\begin{aligned}
&\int_{a_1}^{x}\int_{y}^{b_2}(t-a_1)(s-b_2)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s\\
&\qquad= (b_2-y)(x-a_1)f(x,y)-(b_2-y)\int_{a_1}^{x} f(\sigma_1(t),y))\Delta_1 t\\
&\qquad\quad -(x-a_1)\int_{y}^{b_2} f(x,\rho_2(s))\nabla_2 s+\int_{a_1}^{x}\int_{y}^{b_2} f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s,
\end{aligned}\tag{2.5}$$

$$\begin{aligned}
&\int_{x}^{b_1}\int_{y}^{b_2}(t-b_2)(s-b_2)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s\\
&\qquad= (b_2-y)(b_2-x)f(x,y)-(b_2-y)\int_{x}^{b_1} f(\sigma_1(t),y))\Delta_1 t\\
&\qquad\quad -(b_2-x)\int_{y}^{b_2} f(x,\rho_2(s))\nabla_2 s+\int_{x}^{b_1}\int_{y}^{b_2} f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s
\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}
&\int_x^{b_1}\int_{a_2}^{y}(t-b_2)(s-a_2)\frac{\partial^2 f(t,s)}{\nabla_2 s\Delta_1 t}\Delta_1 t\nabla_2 s \\
&= (y-a_2)(b_2-x)f(x,y)-(y-a_2)\int_x^{b_1} f(\sigma_1(t),y))\Delta_1 t \\
&\quad -(b_2-x)\int_{a_2}^{y} f(x,\rho_2(s))\nabla_2 s+\int_x^{b_1}\int_{a_2}^{y} f(\sigma_1(t),\rho_2(s))\Delta_1 t\nabla_2 s.
\end{aligned} \tag{2.7}$$

If we add the equalities (2.4)-(2.7), we can easily get the integral identity (2.3). □

Remark 2.12. The correspondigs of the results in the previous sections can be easily adopted for $\nabla\nabla$-integral and $\nabla\Delta$-integral.

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