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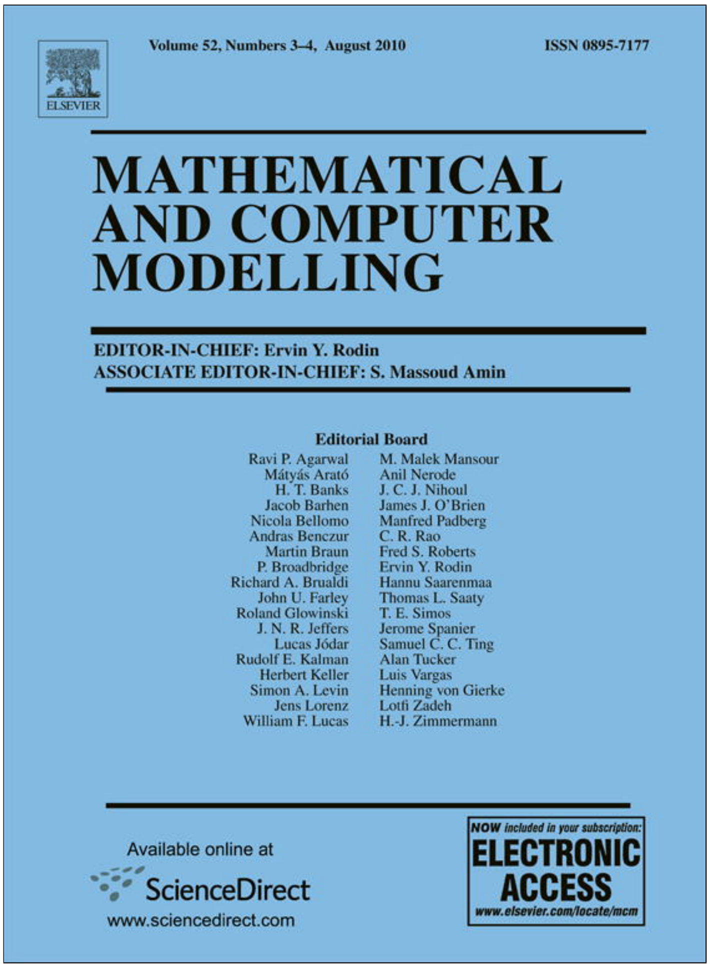

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New inequalities of Simpson-like type involving n knots and the mth derivative

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ABSTRACT

Based on recent results due to Nenad Ujević, we obtain some new inequalities of Simpson-like type involving n knots and the mth derivative where n, m are arbitrary numbers. Our method is also elementary.

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1. Introduction

In recent years, a number of authors have considered error inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of these formulas, for example, the Simpson inequality (which gives an error bound for the well-known Simpson rule) is considered in [1–10]. In [5], we can find the inequality,

$$\left|\int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| \leqq \frac{\Gamma-\gamma}{12}(b-a)^2, \tag{1}$$

where Γ, γ are real numbers, such that $\gamma < f'(t) < \Gamma$, $t \in [a, b]$. We define the Chebyshev functional,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)\,\mathrm{d}t - \frac{1}{(b-a)^2}\int_a^b f(t)\,\mathrm{d}t\int_a^b g(t)\,\mathrm{d}t.$$

Then

$$T(f,f) = \frac{1}{b-a}\|f\|_{L^2}^2 - \frac{1}{(b-a)^2}\left(\int_a^b f(t)\,\mathrm{d}t\right)^2.$$

We also define

$$\sigma(f) = (b-a)T(f,f). \tag{2}$$

In [10], the author proved the following result.

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Theorem 1 (*See [10], Theorem 1*). *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function, whose derivative $f' \in L^2[(a, b)]$. Then*

$$\left| \int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leqq \frac{(b-a)^{\frac{3}{2}}}{6}\sqrt{\sigma(f')}, \tag{3}$$

where $\sigma(\cdot)$ is defined by (2). Inequality (3) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

Since the constant $\frac{1}{6}$ in (3) is sharp, in order to strengthen (3) we have to replace the exponent $\frac{3}{2}$ on the right-hand side of (3). This leads us to strengthen (3) by enlarging the number of knots (6 knots in (3)) and replacing f' in (3) (see [11] for more details). Before stating our main result, let us introduce the following notation.

$$I(f) = \int_a^b f(x)\,\mathrm{d}x.$$

Let $1 \leqq m, n < \infty$ and $1 \leqq p \leqq \infty$. For each $i = \overline{1, n}$, we assume $0 < x_i < 1$ such that

$$\begin{cases} x_1 + x_2 + \cdots + x_n = \dfrac{n}{2}, \\ \cdots \\ x_1^j + x_2^j + \cdots + x_n^j = \dfrac{n}{j+1}, \\ \cdots \\ x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1} = \dfrac{n}{m}, \\ x_1^m + x_2^m + \cdots + x_n^m = \dfrac{n}{m+1}. \end{cases}$$

Put

$$Q(f, n, m, x_1, \ldots, x_n) = \frac{b-a}{n}\sum_{i=1}^{n} f(a + x_i(b-a)).$$

Remark 2. With the above notations, inequality (3) reads as follows

$$\left| I(f) - Q\left(f, 6, 1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right) \right| \leqq \frac{(b-a)^{\frac{3}{2}}}{6}\sqrt{\sigma(f')}. \tag{4}$$

We are now in a position to state our main result. Precisely, we shall apply the Fundamental Theorem of Calculus, Taylor's formula and the Hölder inequality to establish the following result.

Theorem 3. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be an m-times differentiable function such that $f^{(m)} \in L^2(a, b)$. Then we have*

$$|I(f) - Q(f, n, m, x_1, .., x_n)| \leqq \left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m-1}} \right) \frac{(b-a)^{m+\frac{1}{2}}}{m!}\sqrt{\sigma\left(f^{(m)}\right)}. \tag{5}$$

This work can be considered as a continued and complementary part to our recent papers [11–13].

Remark 4. It is worth noticing that the right-hand side of (5) does not involve $x_i, i = \overline{1, n}$ and that m can be chosen arbitrarily. This means that our inequality (5) is better in some sense, especially when $b - a \ll 1$. However, the constant

$$\left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m-1}} \right) \frac{1}{m!}$$

in the inequality (5) is not sharp. This is because of the restriction of the technique that we use. It is better if we leave these to be solved by the interested reader.

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

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Lemma 5 *(See [14]). Let $f : [a, b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a, b]$, $x_0 \in (a, b)$ then for all $x \in (a, b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x)$$

where $T_{r-1}(f, x_0, \cdot)$ is Taylor's polynomial of degree $r - 1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!} \mathrm{d}t. \tag{6}$$

By a simple calculation, the remainder in (6) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x - x_0 - t)^{r-1} f^{(r)}(x_0 + t)}{(r-1)!} \mathrm{d}t$$

which helps us to deduce a similar representation of f as follows

$$f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t)\, \mathrm{d}t. \tag{7}$$

Proof of Theorem 3. Define

$$F(x) = \int_a^x f(t)\, \mathrm{d}t.$$

By Fundamental Theorem of Calculus

$$I(f) = F(b) - F(a).$$

Applying Lemma 5 to $F(x)$ with $x = a$ and $u = b - a$, we get

$$F(b) = F(a) + \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)\, \mathrm{d}t$$

which yields

$$I(f) = \sum_{k=1}^{m} \frac{(b-a)^k}{k!} F^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} F^{(m+1)}(a+t)\, \mathrm{d}t.$$

Equivalently,

$$I(f) = \sum_{k=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)\, \mathrm{d}t.$$

For each $1 \leqq i \leqq n$, applying Lemma 5 to $f(x)$ with $x = a$ and $u = x_i(b - a)$, we get

$$\begin{aligned} f(a + x_i(b-a)) &= \sum_{k=0}^{m-1} \frac{x_i^k (b-a)^k}{k!} f^{(k)}(a) + \int_0^{x_i(b-a)} \frac{(x_i(b-a) - t)^{m-1}}{(m-1)!} f^{(m)}(a+t)\, \mathrm{d}t \\ &= \sum_{k=0}^{m-1} \frac{x_i^k (b-a)^k}{k!} f^{(k)}(a) + \int_0^{b-a} \frac{x_i^m (b-a-u)^{m-1}}{(m-1)!} f^{(m)}(a + x_i u)\, \mathrm{d}u. \end{aligned} \tag{8}$$

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By applying (8) to $i = \overline{1, n}$ and then summing, we deduce that

$$\begin{aligned}\sum_{i=1}^{n} f\left(a + x_i\left(b - a\right)\right) &= \sum_{i=1}^{n}\sum_{k=0}^{m-1} \frac{x_i^k\left(b-a\right)^k}{k!} f^{(k)}\left(a\right) + \sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m\left(b-a-u\right)^{m-1}}{(m-1)!} f^{(m)}\left(a + x_i u\right) \mathrm{d}u \\ &= \sum_{k=0}^{m-1} \frac{\sum\limits_{i=1}^{n} x_i^k\left(b-a\right)^k}{k!} f^{(k)}\left(a\right) + \sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m\left(b-a-u\right)^{m-1}}{(m-1)!} f^{(m)}\left(a + x_i u\right) \mathrm{d}u \\ &= \sum_{k=0}^{m-1} \frac{n\left(b-a\right)^k}{(k+1)!} f^{(k)}\left(a\right) + \sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m\left(b-a-u\right)^{m-1}}{(m-1)!} f^{(m)}\left(a + x_i u\right) \mathrm{d}u. \qquad (9)\end{aligned}$$

Thus,

$$Q\left(f, n, m, x_1, \ldots, x_n\right) = \sum_{k=0}^{m-1} \frac{\left(b-a\right)^{k+1}}{(k+1)!} f^{(k)}\left(a\right) + \frac{b-a}{n}\sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m\left(b-a-u\right)^{m-1}}{(m-1)!} f^{(m)}\left(a + x_i u\right) \mathrm{d}u.$$

Therefore,

$$\begin{aligned}\left|I\left(f\right) - Q\left(f, n, m, x_1, \ldots, x_n\right)\right| &= \Bigg| \int_0^{b-a} \frac{\left(b-a-t\right)^m}{m!} f^{(m)}\left(a+t\right) \mathrm{d}t - \frac{b-a}{n} \\ &\quad \times \sum_{i=1}^{n}\int_0^{b-a} \frac{x_i^m\left(b-a-u\right)^{m-1}}{(m-1)!} f^{(m)}\left(a + x_i u\right) \mathrm{d}u \Bigg| \\ &= \Bigg| \int_a^b \frac{\left(b-x\right)^m}{m!} f^{(m)}\left(x\right) \mathrm{d}x - \frac{b-a}{n} \\ &\quad \times \sum_{i=1}^{n}\int_a^b \frac{x_i^m\left(b-x\right)^{m-1}}{(m-1)!} f^{(m)}\left(\left(1-x_i\right)a + x_i x\right) \mathrm{d}x \Bigg|,\end{aligned}$$

which yields

$$\begin{aligned}\left|I\left(f\right) - Q\left(f, n, m, x_1, \ldots, x_n\right)\right| &= \Bigg| \int_a^b \frac{\left(b-x\right)^m}{m!}\left[f^{(m)}\left(x\right) - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\right]\mathrm{d}x \\ &\quad - \frac{b-a}{n}\sum_{i=1}^{n}\int_a^b \frac{x_i^m\left(b-x\right)^{m-1}}{(m-1)!}\Bigg[f^{(m)}\left(\left(1-x_i\right)a + x_i x\right) \\ &\quad - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\Bigg]\mathrm{d}x\Bigg| \\ &\leqq \left| \int_a^b \frac{\left(b-x\right)^m}{m!}\left[f^{(m)}\left(x\right) - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\right]\mathrm{d}x \right| \\ &\quad + \frac{b-a}{n}\sum_{i=1}^{n}\Bigg|\int_a^b \frac{x_i^m\left(b-x\right)^{m-1}}{(m-1)!}\Bigg[f^{(m)}\left(\left(1-x_i\right)a + x_i x\right) \\ &\quad - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\Bigg]\mathrm{d}x\Bigg|.\end{aligned}$$

We note by the Hölder inequality that

$$\begin{aligned}&\left| \int_a^b \frac{\left(b-x\right)^m}{m!}\left[f^{(m)}\left(x\right) - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\right]\mathrm{d}x \right| \\ &\quad \leqq \left(\int_a^b \left[\frac{\left(b-x\right)^m}{m!}\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b \left[f^{(m)}\left(x\right) - \frac{1}{b-a}\int_a^b f^{(m)}\left(t\right)\mathrm{d}t\right]^2 \mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

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We now compare the last integral on the right-hand side of the above inequality with $\sqrt{\sigma(f^{(m)})}$. More precisely, one has

$$\begin{aligned}
&\left(\int_a^b \left[f^{(m)}(x) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[f^{(m)}(x) - \frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[\left(f^{(m)}(x)\right)^2 - 2f^{(m)}(x)\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a} + \left(\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\right)^2\right]\mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[f^{(m)}(x)\right]^2 \mathrm{d}x - 2\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\int_a^b f^{(m)}(x)\,\mathrm{d}x + \int_a^b \left[\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[f^{(m)}(x)\right]^2 \mathrm{d}x - 2\frac{\left[f^{(m-1)}(b) - f^{(m-1)}(a)\right]^2}{b-a} + \left[\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\right]^2 (b-a)\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[f^{(m)}(x)\right]^2 \mathrm{d}x - \frac{\left[f^{(m-1)}(b) - f^{(m-1)}(a)\right]^2}{b-a}\right)^{\frac{1}{2}} \\
&= \sqrt{b-a}\left(\frac{1}{b-a}\int_a^b \left[f^{(m)}(x)\right]^2 \mathrm{d}x - \left[\frac{f^{(m-1)}(b) - f^{(m-1)}(a)}{b-a}\right]^2\right)^{\frac{1}{2}} \\
&= \sqrt{b-a}\left(\frac{1}{b-a}\int_a^b \left[f^{(m)}(x)\right]^2 \mathrm{d}x - \frac{1}{(b-a)^2}\left[\int_a^b f^{(m)}(x)\,\mathrm{d}x\right]^2\right)^{\frac{1}{2}} \\
&= \sqrt{\sigma\left(f^{(m)}\right)}.
\end{aligned}$$

Hence,

$$\left|\int_a^b \frac{(b-x)^m}{m!}\left[f^{(m)}(x) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]\mathrm{d}x\right| \leqq \frac{(b-a)^{m+\frac{1}{2}}}{m!\sqrt{2m+1}}\sqrt{\sigma\left(f^{(m)}\right)}. \tag{10}$$

Again by the Hölder inequality, one obtains

$$\begin{aligned}
&\left|\int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!}\left[f^{(m)}((1-x_i)a + x_i x) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]\mathrm{d}x\right| \\
&\leqq \left(\int_a^b \left[\frac{x_i^m (b-x)^{m-1}}{(m-1)!}\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_a^b \left[f^{(m)}((1-x_i)a + x_i x) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]^2 \mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

Clearly,

$$\left(\int_a^b \left[\frac{x_i^m (b-x)^{m-1}}{(m-1)!}\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} = \frac{x_i^m}{(m-1)!}\left(\int_a^b (b-x)^{2m-2}\,\mathrm{d}x\right)^{\frac{1}{2}} = \frac{x_i^m}{(m-1)!}\frac{(b-a)^{m-\frac{1}{2}}}{\sqrt{2m-1}},$$

and

$$\begin{aligned}
&\left(\int_a^b \left[f^{(m)}((1-x_i)a + x_i x) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{x_i}\int_a^{(1-x_i)a + x_i b} \left[f^{(m)}(y) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqq \left(\frac{1}{x_i}\int_a^b \left[f^{(m)}(y) - \frac{1}{b-a}\int_a^b f^{(m)}(t)\,\mathrm{d}t\right]^2 \mathrm{d}y\right)^{\frac{1}{2}}
\end{aligned}$$

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$$= \frac{1}{\sqrt{x_i}} \left(\int_a^b \left[f^{(m)}(y) - \frac{1}{b-a} \int_a^b f^{(m)}(t)\,\mathrm{d}t \right]^2 \mathrm{d}y \right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{x_i}} \sqrt{\sigma\left(f^{(m)}\right)} \leqq \frac{1}{x_i} \sqrt{\sigma\left(f^{(m)}\right)}.$$

Therefore,

$$\left| \int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} \left[f^{(m)}\left((1-x_i)a + x_i x\right) - \frac{1}{b-a} \int_a^b f^{(m)}(t)\,\mathrm{d}t \right] \mathrm{d}x \right|$$
$$\leqq x_i^{m-1} \frac{(b-a)^{m-\frac{1}{2}}}{(m-1)!\sqrt{2m-1}} \sqrt{\sigma\left(f^{(m)}\right)}.$$

Thus,

$$\sum_{i=1}^n \left| \int_a^b \frac{x_i^m (b-x)^{m-1}}{(m-1)!} \left[f^{(m)}\left((1-x_i)a + x_i x\right) - \frac{1}{b-a} \int_a^b f^{(m)}(t)\,\mathrm{d}t \right] \mathrm{d}x \right|$$
$$\leqq \frac{n(b-a)^{m-\frac{1}{2}}}{m!\sqrt{2m-1}} \sqrt{\sigma\left(f^{(m)}\right)}. \tag{11}$$

Combining (10) and (11) gives

$$|I(f) - Q(f, n, m, x_1, ., x_n)| \leqq \left(\frac{(b-a)^{m+\frac{1}{2}}}{m!\sqrt{2m+1}} + \frac{b-a}{n} \frac{n(b-a)^{m-\frac{1}{2}}}{m!\sqrt{2m-1}} \right) \sqrt{\sigma\left(f^{(m)}\right)},$$

or equivalently,

$$|I(f) - Q(f, n, m, x_1, ., x_n)| \leqq \left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m-1}} \right) \frac{(b-a)^{m+\frac{1}{2}}}{m!} \sqrt{\sigma\left(f^{(m)}\right)}$$

which completes the proof. □

3. Examples

In this section, by applying our main theorem, we will obtain some new inequalities which cannot be easy obtained from [10].

Example 6. Assume $n = 6$, $m = 1, 2$, or 3. Clearly $x_1 = 0$, $x_2 = x_3 = x_4 = x_5 = \frac{1}{2}$, and $x_6 = 1$ satisfy the following linear system

$$\begin{cases} x_1 + x_2 + \cdots + x_6 = \dfrac{6}{2}, \\ \cdots \\ x_1^j + x_2^j + \cdots + x_6^j = \dfrac{6}{j+1}, \\ \cdots \\ x_1^m + x_2^m + \cdots + x_6^m = \dfrac{6}{m+1}. \end{cases}$$

Therefore, we obtain the following inequalities

$$\left| \int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leqq \left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m-1}} \right) \frac{(b-a)^{m+\frac{1}{2}}}{m!} \sqrt{\sigma\left(f^{(m)}\right)}.$$

Example 7. Assume $n = 3$, $m = 3$. By solving the following linear system

$$\begin{cases} x_1 + x_2 + x_3 = \dfrac{3}{2}, \\ x_1^2 + x_2^2 + x_3^2 = \dfrac{3}{3}, \\ x_1^3 + x_2^3 + x_3^3 = \dfrac{3}{4}, \end{cases}$$

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we obtain $\{x_1, x_2, x_3\}$ is a permutation of

$$\left\{\frac{1}{2}, 1-\frac{1}{2}\left(1\pm\frac{\sqrt{2}}{2}\right), \frac{1}{2}\left(1\pm\frac{\sqrt{2}}{2}\right)\right\}.$$

Therefore, we obtain the following inequalities

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{3}\left(f\left(a+\left(1-\frac{1}{2}\left(1\pm\frac{\sqrt{2}}{2}\right)\right)(b-a)\right)\right.\right.$$
$$\left.\left. + f\left(a+\frac{1}{2}(b-a)\right) + f\left(a+\left(1-\frac{1}{2}\left(1\pm\frac{\sqrt{2}}{2}\right)\right)(b-a)\right)\right)\right| \leqq \frac{\sqrt{7}+\sqrt{5}}{6\sqrt{35}}(b-a)^{\frac{7}{2}}\sqrt{\sigma(f''')}.$$

Example 8. If $n = 2$, $m = 2$, then by solving the following system

$$\begin{cases} x_1 + x_2 = \dfrac{2}{2}, \\ x_1^2 + x_2^2 = \dfrac{2}{3}, \end{cases}$$

we obtain

$$(x_1, x_2) = \left(\frac{1}{2}\pm\frac{\sqrt{3}}{6}, \frac{1}{2}\mp\frac{\sqrt{3}}{6}\right).$$

We then obtain

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{2}\left(f\left(a+\left(\frac{1}{2}-\frac{\sqrt{3}}{6}\right)(b-a)\right) + f\left(a+\left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right)(b-a)\right)\right)\right|$$
$$\leqq \frac{\sqrt{3}+\sqrt{5}}{2\sqrt{15}}(b-a)^{\frac{5}{2}}\sqrt{\sigma(f'')}.$$

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