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Multiple solutions for a class of quasilinear elliptic equations of $p(x)$-Laplacian type with nonlinear boundary conditions

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Using variational methods we study the non-existence and multiplicity of non-negative solutions for a class of quasilinear elliptic equations of $p(x)$-Laplacian type with nonlinear boundary conditions of the form

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = 0 \qquad & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda g(x,u) \quad & \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain with smooth boundary, n is the outer unit normal to $\partial\Omega$ and λ is a parameter. Furthermore, we want to emphasize that $g : \partial\Omega \times [0,\infty) \to \mathbb{R}$ is a continuous function that may or may not satisfy the Ambrosetti–Rabinowitz-type condition.

1. Introduction

The study of partial differential equations with $p(x)$ growth conditions has received an increasing amount of research interest in recent decades. The specific attention accorded to such problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomena that arise in elastomechanics or electrorheological fluids. For a general account of the underlying physics, and for some technical applications, we refer the reader to [11, 15, 17] and the references therein.

A typical model of an elliptic equation with $p(x)$ growth conditions is

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = g(x,u).$$

The operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplace operator and it is a natural generalization of the p-Laplace operator in which $p(x) = p > 1$ is a constant.

For this reason the equations studied in the case in which the $p(x)$-Laplace operator is involved are, in general, extensions of p-Laplacian problems. However, we point out that such generalizations are not trivial, since the $p(x)$-Laplace operator possesses more complicated nonlinearity: for example, it is inhomogeneous.

Let Ω be an open domain in $\mathbb{R}^N$ and let $N \geqslant 3$ with a bounded Lipschitz boundary $\partial\Omega$. In [5], Fan studied the problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = g(x,u) \quad \text{on } \partial\Omega, \tag{1.2}$$

where $p(\cdot)$ is a measurable real function defined on Ω, $g \in C^0(\partial\Omega\times\mathbb{R})$, and satisfies the following conditions:

(P1) $1 < p^- := \inf_{x\in\Omega} p(x) \leqslant p^+ := \sup_{x\in\Omega} p(x) < +\infty$;

(P2) there exist $\delta > 0$ and $\gamma > N$ such that $p \in W^{1,\gamma}(\Omega_\delta)$, where

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\};$$

(G1) there exist a positive constant C_1 and a function $q \in C^0(\Omega)$ satisfying $1 \leqslant q(x) < p^*(x)$ for $x \in \partial\Omega$ such that

$$|g(x,t)| \leqslant C_1(1 + |t|^{q(x)-1}) \quad \text{for } x \in \partial\Omega,\ t \in \mathbb{R}.$$

The main results of that paper can be formulated as follows.

THEOREM 1.1 (Fan [5, theorem 3.5]). *Let Ω be an unbounded domain in $\mathbb{R}^N$ with bounded Lipschitz boundary $\partial\Omega$. Suppose that conditions (P1), (P2) and (G1) are satisfied.*

(i) *If $q^+ < p^-$, then problem (1.1), (1.2) has a solution that is a global minimizer of a integral functional on $W^{1,p(x)}(\Omega)$. If, in addition, there exists a positive constant $\alpha < p^-$ such that*

$$\liminf_{t\to 0} \frac{G(x,t)}{|t|^\alpha} > 0 \quad \textit{uniformly for } x \in \partial\Omega,$$

then problem (1.1), (1.2) has a non-trivial solution u that is a global minimizer of an integral functional I with $I(u) < 0$.

(ii) *If the following conditions are satisfied:*

(G2) *there exist $\beta > p^+$ and $M > 0$ such that*

$$0 < \beta G(x,t) \leqslant t g(x,t)$$

for all $x \in \partial\Omega$ and all t such that $|t| \geqslant M$; and

(G3) $$\lim_{t\to 0} \frac{G(x,t)}{|t|^{p^+}} = 0 \quad \textit{uniformly in } x \in \partial\Omega,$$

where

$$G(x,t) = \int_0^t f(x,s)\,\mathrm{d}s,$$

then problem (1.1), (1.2) has a non-trivial solution u which is a mountain-pass-type critical point of I with $I(u) > 0$.

Motivated by the ideas introduced in [14] and [16], in the first instance we study the non-existence and multiplicity of solutions for the following problem:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=0 \quad \text{in } \Omega, \tag{1.3}$$

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial n}=\lambda g(x,u) \quad \text{on } \partial\Omega, \tag{1.4}$$

when Ω is a bounded domain and n is the outer unit normal to $\partial\Omega$, when $\lambda>0$ is given, when the function $g:\partial\Omega\times[0,+\infty)\to\mathbb{R}$ is continuous and the following hypotheses are satisfied:

(G1′) $g(x,0)=0$, $-C_2t^{r(x)-1}\leqslant g(x,t)\leqslant C_3t^{p(x)-1}$ for all $t\in[0,+\infty)$ and almost every $x\in\Omega$, with some constants $C_2,C_3>0$, $1\leqslant r(x)\leqslant p(x)$ for almost every $x\in\Omega$;

(G2′) there exist two positive constants t_0 and $t_1>0$ such that $G(x,t)\leqslant 0$ for $0\leqslant t\leqslant t_0$ and $G(x,t_1)>0$;

(G3′)

$$\limsup_{t\to+\infty}\frac{G(x,t)}{t^{p^+}}\leqslant 0 \quad \text{uniformly in } x.$$

It is worth recalling that in [16] Perera deals with quasilinear elliptic equations of p-Laplacian type, while in [14] Mihăilescu and Rădulescu deal with the corresponding Dirichlet problem of $p(x)$-Laplacian. It turns out that essentially similar techniques on boundary trace embedding theorems for variable exponent Sobolev spaces [5] can help us to obtain some results on the non-existence and multiplicity of solutions for (1.3), (1.4). The first results of this paper are given by the following theorems.

THEOREM 1.2. *Under hypotheses (P1), (P2) and (G1′), there exists a positive constant $\underline{\lambda}$ such that, for all $\lambda\in(0,\underline{\lambda})$, problem (1.3)–(1.4) has no positive solution.*

THEOREM 1.3. *Under hypotheses (P1), (P2), (G1′) and (G3′), there exists a positive constant $\bar{\lambda}$ such that, for all $\lambda\geqslant\bar{\lambda}$, problem (1.3)–(1.4) has at least two distinct non-negative, non-trivial weak solutions provided that*

$$p^+<\min\left\{N,\frac{(N-1)p^-}{N-p^-}\right\}.$$

One can easily see that theorem 1.2 is new and that theorem 1.3 is different from theorem 1.1: in theorem 1.3, Ω is a bounded domain and in theorem 1.1 Ω is unbounded. We also do not require the Ambrosetti–Rabinowitz-type condition as in (G2). Moreover, we obtain at least two distinct non-negative, non-trivial weak solutions instead of one, as is the case in theorem 1.1(ii).

Next, we study problem (1.3), (1.4) in the case when

$$\lambda g(x,t)=A|t|^{a-2}t+B|t|^{b-2}t$$

with $A,B>0$ and

$$1<a<p^-<p^+<b<\min\left\{N,\frac{(N-1)p^-}{N-p^-}\right\}.$$

More specifically, we consider the degenerate boundary-value problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = 0 \quad \text{in } \Omega, \tag{1.5}$$

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = A|u|^{a-2}u + B|u|^{b-2}u \quad \text{on } \partial\Omega. \tag{1.6}$$

We then conclude with the following result.

THEOREM 1.4. *There exists $\lambda^\star > 0$ such that, for any $A \in (0, \lambda^\star)$ and any $B \in (0, \lambda^\star)$, problem (1.5), (1.6) has at least two distinct non-trivial solutions.*

The above problems will be studied in the framework of variable Lebesgue and Sobolev spaces, which will be briefly described in the following section. For a good survey of related problems, see [1, 3, 6, 7, 10, 13, 15, 19] and the references therein.

2. Preliminaries

In what follows, we recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer the reader to [6, 8, 9, 12, 15].

Set

$$L_+^\infty(\Omega) = \Big\{h; h \in L^\infty(\Omega),\ \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 1\Big\}.$$

For any $h \in L_+^\infty(\Omega)$, we define

$$h^+ = \operatorname*{ess\,sup}_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \operatorname*{ess\,inf}_{x\in\Omega} h(x).$$

For any $p(x) \in L_+^\infty(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \Big\{u : \text{a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\Big\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\bigg\{\mu > 0;\ \int_\Omega \bigg|\frac{u(x)}{\mu}\bigg|^{p(x)}\,\mathrm{d}x \leqslant 1\bigg\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds and they are reflexive if and only if $1 < p^- \leqslant p^+ < \infty$. An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold:

$$|u|_{p(x)}^{p^-} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^+}$$

provided that $|u|_{p(x)} > 1$, while

$$|u|_{p(x)}^{p^+} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^-},$$

provided that $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0.$$

We also define the variable Sobolev space

$$X := W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On X we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

A simple calculation shows that the above norm is equivalent to

$$\|u\| = \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) \mathrm{d}x \leqslant 1\right\}.$$

Proposition 2.1 (Fan and Zhang [7, proposition 2.5]). *There is a constant $C > 0$ such that*

$$|u|_{p(x)} \leqslant C|\nabla u|_{p(x)} \quad \text{for all } u \in W^{1,p(x)}(\Omega).$$

By the result of the above proposition, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on X. For all $u \in X$, the following well-known inequalities are important for our argument:

$$\|u\|^{p^-} \leqslant \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, \mathrm{d}x \leqslant \|u\|^{p^+}$$

provided that $\|u\| > 1$, while

$$\|u\|^{p^+} \leqslant \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, \mathrm{d}x \leqslant \|u\|^{p^-}$$

provided that $\|u\| < 1$. We write

$$p^\star(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

Finally, we recall some embedding results regarding variable exponent Lebesgue–Sobolev spaces. For the continuous embedding between variable exponent Lebesgue–Sobolev spaces, we refer the reader to [9].

Proposition 2.2 (Fan *et al*. [9, theorem 1.1]). *If $p : \Omega \to \mathbb{R}$ is Lipschitz continuous and $p^+ < N$ then, for any $q \in L^\infty_+(\Omega)$ with $p(x) \leqslant q(x) \leqslant p^*(x)$, there is a continuous embedding $X \hookrightarrow L^{q(x)}(\Omega)$.*

For issues regarding the compact trace embedding we refer to [5].

Proposition 2.3. (Fan [5, corollary 2.1, theorem 2.2]). *Suppose that conditions (P1) and (P2) are satisfied. Then there is a continuous boundary trace embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$ for $q \in L^\infty(\partial\Omega)$ satisfying the condition*

$$1 \leqslant q(x) \leqslant \frac{(N-1)p(x)}{N-p(x)} \quad \text{for all } x \in \partial\Omega.$$

Moreover, the embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact if $q \in L^\infty(\partial\Omega)$ satisfies the condition

$$1 \leqslant q(x) + \varepsilon \leqslant \frac{(N-1)p(x)}{N-p(x)} \quad \text{for all } x \in \partial\Omega,$$

where ε is a positive constant.

3. Proofs

Proof of theorem 1.2. We observe that in [5, p. 1408], Fan has studied the following eigenvalue problem:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = 0 \qquad \text{in } \Omega, \tag{3.1}$$

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda |u|^{p(x)-2}u \quad \text{on } \partial\Omega. \tag{3.2}$$

Fan then obtains that problem (3.1), (3.2) has a first positive eigenvalue λ_1, given by

$$\lambda_1 = \min_{u \in X \setminus W_0^{1,p(x)}(\Omega)} \frac{\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x}{\int_{\partial\Omega} |u|^{p(x)}\,\mathrm{d}\sigma}, \tag{3.3}$$

where $\mathrm{d}\sigma$ is the boundary measure. So, if u is a positive solution of problem (1.3)–(1.4), then multiplying (1.3)–(1.4) by u, integrating by parts and using (G1′) gives

$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x = \lambda \int_{\partial\Omega} g(x,u)u\,\mathrm{d}\sigma \leqslant C_3\lambda \int_{\partial\Omega} |u|^{p(x)}\,\mathrm{d}\sigma,$$

and hence we can choose $\underline{\lambda} = \lambda_1/C_3$. The proof is complete. □

We consider the functional $\Phi_\lambda : X \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = I(u) - \lambda J(u), \tag{3.4}$$

where

$$I(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\right)\mathrm{d}x, \tag{3.5}$$

$$J(u) = \int_{\partial\Omega} G(x,u)\,\mathrm{d}\sigma. \tag{3.6}$$

By (P1), the Banach space X is reflexive and the functional $I \in C^1(X,\mathbb{R})$. By (P2), (G1) and proposition 2.3, we know that there is a compact trace embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$. Furthermore, the functional J is of $C^1(X,\mathbb{R})$ with

$$\langle J'(u), v\rangle = \int_\Omega g(x,u)u\,\mathrm{d}\sigma \quad \text{for all } u, v \in X.$$

DEFINITION 3.1. We say that $u \in X$ is a weak solution of problem (1.3)–(1.4) if and only if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \,\mathrm{d}x + \int_\Omega |u|^{p(x)-2}uv\,\mathrm{d}x - \lambda \int_{\partial\Omega} g(x,u)v\,\mathrm{d}\sigma = 0$$

for all $v \in X$.

Next we set $g(x,t) = 0$ for $t < 0$ and consider the C^1-functional $\Phi_\lambda : X \to \mathbb{R}$ given by (3.4).

LEMMA 3.2. *If u is a critical point of Φ_λ then u is non-negative in Ω.*

Proof. Observe that if u is a critical point of Φ_λ, denoting by u_- the negative part of u, i.e. $u_-(x) = \min\{u(x), 0\}$, we have

$$\begin{aligned} 0 &= \langle \Phi'_\lambda(u), u_-\rangle \\ &= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \cdot \nabla u_- + |u|^{p(x)-2}u \cdot u_-)\,\mathrm{d}x - \lambda \int_{\partial\Omega} g(x,u)u_-\,\mathrm{d}x \\ &= \|u_-\|_X. \end{aligned} \tag{3.7}$$

It is easy to see that if $u \in X$, then $u_+, u_- \in X$ so, from (3.7), we have $u \geqslant 0$ in Ω. Thus, non-trivial critical points of the functional Φ_λ are non-negative, non-trivial solutions of problem (1.3)–(1.4). □

The above lemma shows that we can prove theorem 1.3 by using critical point theory. More precisely, we first show that, for sufficiently large $\lambda > 0$, the functional Φ_λ has a global minimizer $u_1 \geqslant 0$ such that $\Phi_\lambda(u_1) < 0$. Next, by using the mountain-pass theorem, a second critical point u_2 with $\Phi_\lambda(u_2) > 0$ is obtained.

LEMMA 3.3. *The functional Φ_λ is bounded from below, coercive and weakly lower semi-continuous on X.*

Proof. By (G1′) and (G3′), there exists a constant $C_\lambda = C(\lambda) > 0$ such that

$$\lambda G(x,t) \leqslant \frac{\lambda_1}{2p^+}|t|^{p(x)} + C_\lambda \quad \text{for almost every } x \in \partial\Omega,\ t \in \mathbb{R}.$$

Hence,

$$\begin{aligned} \Phi_\lambda(u) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\right)\mathrm{d}x - \lambda \int_{\partial\Omega} G(x,u)\,\mathrm{d}\sigma \\ &\geqslant \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,\mathrm{d}x - \int_{\partial\Omega}\left(\frac{\lambda_1}{2p^+}|u|^{p(x)} + C_\lambda\right)\mathrm{d}\sigma \\ &\geqslant \frac{1}{2p^+}\|u\|_X - C_\lambda|\partial\Omega|_{N-1}. \end{aligned}$$

Since $\partial\Omega$ is bounded, the functional Φ_λ is bounded from below and coercive on X. On the other hand, by (P1), (P2) and (G1′)–(G3′), Φ_λ is weakly lower semi-continuous on X. □

Lemma 3.3 implies, by applying the minimum principle in [18], that Φ_λ has a global minimizer u_1 and, by lemma 3.2, u_1 is a non-negative solution of problem (1.3)–(1.4). The following lemma shows that the solution u_1 is not trivial provided that λ is sufficiently large.

LEMMA 3.4. *There exists a constant $\bar\lambda > 0$ such that, for all $\lambda \geqslant \bar\lambda$, we have $\inf_{u\in X} \Phi_\lambda(u) < 0$. Hence, $u_1 \not\equiv 0$, i.e. solution u_1 is not trivial.*

Proof. Let u_0 be a constant function in X such that $u_0 = t_0$, where t_0 is as in (G2′). We have

$$\Phi_\lambda(u_0) = \int_\Omega \left(\frac{1}{p(x)}|t_0|^{p(x)}\right) \mathrm{d}x - \lambda \int_{\partial\Omega} G(x,t_0)\,\mathrm{d}\sigma < 0$$

for all sufficiently large $\lambda \geqslant \bar\lambda$. This completes the proof. □

The main difference in the arguments occurs at this point. As mentioned before, we can prove by a truncation argument that these two solutions are ordered. To this end, we first fix $\lambda \geqslant \bar\lambda$ and set

$$\hat g(x,t) = \begin{cases} 0 & \text{for } t < 0, \\ g(x,t) & \text{for } 0 \leqslant t \leqslant u_1(x), \\ g(x,u_1(x)) & \text{for } t > u_1(x), \end{cases} \tag{3.8}$$

and

$$\hat G(x,t) = \int_0^t \hat g(x,s)\,\mathrm{d}s.$$

Define the functional $\hat\Phi_\lambda : X \to \mathbb{R}$ by

$$\hat\Phi_\lambda(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\right) \mathrm{d}x - \lambda \int_{\partial\Omega} \hat G(x,u)\,\mathrm{d}\sigma. \tag{3.9}$$

With the same arguments as those used for functional Φ_λ, we can show that $\hat\Phi_\lambda$ is continuously differentiable on X and that

$$\langle \hat\Phi'_\lambda(u), \varphi\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,\mathrm{d}x + \int_\Omega |u|^{p(x)-2}u\varphi\,\mathrm{d}x - \lambda\int_{\partial\Omega} \hat g(x,u)\varphi\,\mathrm{d}\sigma$$

for all $u, \varphi \in X$.

LEMMA 3.5. *If $u \in X$ is a critical point of $\hat\Phi_\lambda$ then $u \leqslant u_1$. So u is a solution of problem (1.3)–(1.4) in the order interval $[0, u_1]$.*

Proof. If u is a critical point of $\hat\Phi'_\lambda$, then $u \geqslant 0$ as before. Moreover,

$$\begin{aligned} 0 &= \langle \hat\Phi'_\lambda(u) - \hat\Phi'_\lambda(u_1), (u-u_1)_+\rangle \\ &= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla u_1|^{p(x)-2}\nabla u_1)\nabla(u-u_1)\,\mathrm{d}x \\ &\quad + \int_\Omega (|u|^{p(x)-2}u - |u_1|^{p(x)-2}u_1)(u-u_1)^+\,\mathrm{d}x \\ &\quad - \lambda\int_{\partial\Omega} (\hat g(x,u) - g(x,u_1))(u-u_1)_+\,\mathrm{d}\sigma \end{aligned}$$

$$= \int_{u>u_1} (|\nabla u|^{p(x)-2}\nabla u - |\nabla u_1|^{p(x)-2}\nabla u_1)\nabla(u-u_1)\,\mathrm{d}x$$
$$+ \int_{u>u_1} (|u|^{p(x)-2}u - |u_1|^{p(x)-2}u_1)(u-u_1)^+\,\mathrm{d}x$$
$$\geqslant \int_{u>u_1} (|\nabla u|^{p(x)-1} - |\nabla u_1|^{p(x)-1})(|\nabla u| - |\nabla u_1|)\,\mathrm{d}x$$
$$+ \int_{u>u_1} (|u|^{p(x)-1} - |u_1|^{p(x)-1})(|u| - |u_1|)\,\mathrm{d}x,$$

which implies $u \leqslant u_1$. □

LEMMA 3.6. *There exist a constant $\rho \in (0, \|u_1\|)$ and a constant $\alpha > 0$ such that $\hat{\Phi}_\lambda(u) \geqslant \alpha$ for all $u \in X$ with $\|u\| = \rho$.*

Proof. Let $u \in X$ be fixed, such that $\|u\| < 1$ and set

$$\Gamma_u = \{x \in \partial\Omega : u(x) > \min\{u_1(x), t_0\}\}.$$

By (G2′) and (3.8) we have $\hat{G}(x, u(x)) \leqslant 0$ on $\partial\Omega \setminus \Gamma_u$. Then

$$\hat{\Phi}_\lambda(u) \geqslant \frac{1}{p^+}\|u\|^p - \lambda \int_{\Gamma_u} \hat{G}(x,u)\,\mathrm{d}\sigma.$$

Since $p^+ < \min\{N, (N-1)p^-/(N-p^-)\}$, it follows that $p^+ < p_\star(x)$ for all $x \in \bar{\Omega}$. Then there exists $q \in (p^+, (N-1)p^-/(N-p^-))$ such that X is continuously embedded in $L^q(\Omega)$. Thus, there exists a positive constant $C > 0$ such that $|u|_q \leqslant C\|u\|$ for all $u \in X$. By (G1′), Hölder's inequality and proposition 2.3,

$$\int_{\Gamma_u} \hat{G}(x,u)\,\mathrm{d}\sigma \leqslant C_3 \int_{\Gamma_u} |u|^{p(x)}\,\mathrm{d}\sigma \leqslant C_3 |\Gamma_u|_{N-1}^{1-p^+/q} \|u\|^{p^+}.$$

Hence,

$$\hat{\Phi}_\lambda(u) \geqslant \|u\|^{p^+}\left(\frac{1}{p^+} - \lambda C_3 |\Gamma_u|_{N-1}^{1-p^+/q}\right).$$

It is sufficient to show that $|\Gamma_u| \to 0$ as $\|u\| \to 0$. Indeed, let $k = \min\{\min_{\partial\Omega} u_1, t_0\}$, where t_0 as in (G2′). Then

$$\|u\|^{p^+} \geqslant C \int_{\partial\Omega} |u|^{p(x)}\,\mathrm{d}\sigma \geqslant \int_{\Gamma_u} |u|^{p(x)}\,\mathrm{d}\sigma \geqslant C k^{p^+} |\Gamma_u|_{N-1}.$$

This ends the proof of the lemma. □

Proof of theorem 1.3. The argument used for Φ_λ shows that $\hat{\Phi}_\lambda$ is also coercive, so every Palais–Smale sequence of $\hat{\Phi}_\lambda$ is bounded and hence contains a convergent subsequence. Then all assumptions of the mountain-pass theorem in [2] are satisfied. We set

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} \hat{\Phi}_\lambda(u) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = u_1\}$ is a class of paths joining the origin to u_1. We obtain the second solution u_2 and $u_2 \not\equiv u_1$, since

$$\Phi_\lambda(u_1) < 0 < \hat{\Phi}_\lambda(u_2) = \Phi_\lambda(u_2).$$

□

To prove theorem 1.4, we consider the energy functional $\Psi_\lambda : X \to \mathbb{R}$ corresponding to problem (1.5), (1.6) as follows:

$$\Psi_\lambda(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\right) \mathrm{d}x - \frac{A}{a}\int_{\partial\Omega} |u|^a \,\mathrm{d}\sigma - \frac{B}{b}\int_{\partial\Omega} |u|^b \,\mathrm{d}\sigma.$$

Similar arguments as those used above assure us that $\Psi_\lambda \in C^1(X, \mathbb{R})$ with

$$\begin{aligned}\langle \Psi_\lambda'(u), \varphi\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla\varphi \,\mathrm{d}x + |u|^{p(x)-2}u\varphi)\,\mathrm{d}x \\ - A\int_{\partial\Omega} |u|^{a-2}u\varphi \,\mathrm{d}\sigma - B\int_{\partial\Omega} |u|^{b-2}u\varphi\,\mathrm{d}\sigma\end{aligned}$$

for all $u, \varphi \in X$. Thus, the weak solutions of problem (1.5)–(1.6) are exactly the critical points of Ψ_λ. Therefore, our idea is to prove that the functional Ψ_λ possesses two distinct critical points using the mountain-pass theorem in [2] and Ekeland's variational principle in [4].

Lemma 3.7. *The following assertions hold:*

(i) *there exist three positive constants ρ, $\lambda^\star$ and α such that $\Psi_\lambda(u) \geqslant \alpha$ for all $u \in X$ with $\|u\|_X = \rho$ and all $A, B \in (0, \lambda^\star)$;*

(ii) *there exists $\psi \in X$ such that $\lim_{t\to+\infty} \Psi_\lambda(t\psi) = -\infty$;*

(iii) *there exists $\varphi \in X$ such that $\varphi \geqslant 0$, $\varphi \not\equiv 0$ and $\Psi_\lambda(t\varphi) < 0$ for all sufficiently small $t > 0$.*

Proof. (i) Since

$$1 < a < p^- < p^+ < b < \min\left\{N, \frac{(N-1)p^-}{N - p^-}\right\},$$

using proposition 2.3 we find that X is continuously embedded in $L^a(\partial\Omega)$ and in $L^b(\partial\Omega)$. Thus, there exist two positive constants c_1 and c_2 such that

$$\int_{\partial\Omega} |u|^a \,\mathrm{d}x \leqslant c_1\|u\|^a \quad \text{and} \quad \int_{\partial\Omega} |u|^b \,\mathrm{d}x \leqslant c_1\|u\|^b$$

for all $u \in X$. Therefore, for any $u \in X$ with $\|u\| = 1$ we have

$$\begin{aligned}\Psi_\lambda(u) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}|u|^{p(x)}\right) \mathrm{d}x - \frac{A}{a}\int_{\partial\Omega} |u|^a \,\mathrm{d}\sigma - \frac{B}{b}\int_{\partial\Omega} |u|^b \,\mathrm{d}\sigma \\ &\geqslant \frac{1}{p^+} - \frac{A}{a}c_1 - \frac{B}{b}c_2.\end{aligned}$$

Then, taking

$$\lambda^* = \min\left\{\frac{a}{4p^+c_1}, \frac{b}{4p^+c_2}\right\}$$

we obtain, for all $A, B \in (0, \lambda^*)$, that

$$\Psi_\lambda(u) \geqslant \frac{1}{2p^+} = \alpha$$

for all $u \in X$ with $\|u\| = 1$.

(ii) Let $\psi \in X$ be a constant function and let $\psi \geqslant 0$, $\psi \not\equiv 0$ and $t > 1$. We have

$$\begin{aligned}\Psi_\lambda(t\psi) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla(t\psi)|^{p(x)} + \frac{1}{p(x)}|t\psi|^{p(x)}\right) \mathrm{d}x \\ &\quad - \frac{A}{a}\int_{\partial\Omega} |t\psi|^a \,\mathrm{d}\sigma - \frac{B}{b}\int_{\partial\Omega} |t\psi|^b \,\mathrm{d}\sigma \\ &\leqslant \frac{t^{p^+}}{p^-}\int_\Omega (|\nabla\psi|^{p(x)} + |\psi|^{p(x)}) \,\mathrm{d}x - \frac{B}{b}t^b \int_{\partial\Omega} |\psi|^b \,\mathrm{d}x.\end{aligned}$$

Since $b > p^+$ we deduce that $\lim_{t\to+\infty} \Psi_\lambda(t\psi) = -\infty$.

(iii) Let $\varphi \in X$ be a constant function, let $\varphi \geqslant 0$, $\varphi \not\equiv 0$ and $t \in (0, 1)$. We have

$$\begin{aligned}\Psi_\lambda(t\varphi) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla t\varphi|^{p(x)} + \frac{1}{p(x)}|t\varphi|^{p(x)}\right) \mathrm{d}x \\ &\quad - \frac{A}{a}\int_{\partial\Omega} |t\varphi|^a \,\mathrm{d}\sigma - \frac{B}{b}\int_{\partial\Omega} |t\varphi|^b \,\mathrm{d}\sigma \\ &\leqslant \frac{t^{p^-}}{p^-}\int_\Omega (|\nabla\varphi|^{p(x)} + |\varphi|^{p(x)}) \,\mathrm{d}x - \frac{A}{a}t^a \int_{\partial\Omega} |\varphi|^a \,\mathrm{d}x < 0\end{aligned}$$

for all $t < \delta^{1/(p^- - a)}$ with

$$0 < \delta < \min\left\{1, \frac{(A/a)p^+ \int_\Omega |\varphi|^a \,\mathrm{d}x}{\int_\Omega |\nabla\varphi|^{p(x)} \,\mathrm{d}x}\right\}.$$

It follows that (iii) is proved. □

LEMMA 3.8. *Ψ_λ satisfies the Palais–Smale condition on X.*

Proof. Let λ^* be defined as above, and let $A \in (0, \lambda^*)$ and $B \in (0, \lambda^*)$ be fixed. Assume that $\{u_n\}$ is a Palais–Smale sequence in X, i.e.

$$|\Psi_\lambda(u_n)| \leqslant \bar{c} \quad \text{and} \quad \Psi_\lambda(u_n) \to 0 \quad \text{in } X^\star. \tag{3.10}$$

We first prove that $\{u_n\}$ is bounded in X. Indeed, assume by contradiction that $\{u_n\}$ is not bounded in X. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that

$\|u_n\| > 1$ for any integer n. We have, for sufficiently large n,

$$
\begin{aligned}
&\bar{c} + 1 + \|u_n\| \\
&\quad \geqslant \Psi_\lambda(u_n) - \frac{1}{b}\langle \Psi_\lambda'(u_n), u_n\rangle \\
&\quad = \int_\Omega \left(\frac{1}{p(x)}|\nabla u_n|^{p(x)} + \frac{1}{p(x)}|u_n|^{p(x)}\right) \mathrm{d}x - \frac{A}{a}\int_{\partial\Omega} |u_n|^a \,\mathrm{d}x - \frac{B}{b}\int_{\partial\Omega} |u_n|^b \,\mathrm{d}x \\
&\qquad - \frac{1}{b}\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) \,\mathrm{d}x + \frac{A}{b}\int_{\partial\Omega} |u_n|^a \,\mathrm{d}x + \frac{B}{b}\int_{\partial\Omega} |u_n|^b \,\mathrm{d}x \\
&\quad \geqslant \left(\frac{1}{p^+} - \frac{1}{b}\right)\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)}) \,\mathrm{d}x + A\left(\frac{1}{b} - \frac{1}{a}\right)\int_{\partial\Omega} |u_n|^a \,\mathrm{d}x \\
&\quad \geqslant \left(\frac{1}{p^+} - \frac{1}{b}\right)\|u_n\|^{p^-} + A\left(\frac{1}{b} - \frac{1}{a}\right) c_1 \|u_n\|^a.
\end{aligned}
$$

From the inequality above we know that $\{u_n\}_n$ is bounded in X since $p^+ < b$. Thus, there exists $u_1 \in X$ such that, passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u_1 in X. We know from proposition 2.3 that there is a compact trace embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$. It follows that $\{u_n\}_n$ converges strongly to u_1 in $L^a(\partial\Omega)$ and $L^b(\partial\Omega)$. On the other hand, relation (3.10) yields

$$\langle \Psi_\lambda'(u_n), u_n - u_1\rangle = 0.$$

Using the above information, we find that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n - u_1) \,\mathrm{d}x = 0. \tag{3.11}$$

Relation (3.11) and the fact that u_n converges weakly to u_1 in X enable us to apply [19, proposition 2.6] in order to obtain that u_n converges strongly to u_1 in X. This completes the proof. □

Proof of theorem 1.4. Following from the proof of lemma 3.8, and since Ψ is of class C^1 and relation (3.10) holds true, we conclude that

$$\Psi_\lambda'(u_1) = 0, \qquad \Psi_\lambda(u_1) = \bar{c}.$$

It follows that u_1 is a non-trivial weak solution of problem (1.5)–(1.6).

We prove now that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. By lemma 3.7(i) it follows that, on the boundary of the unit ball centred at the origin in X and denoted by $B_1(0)$, we have

$$\inf_{\partial B_1(0)} \Psi_\lambda > 0.$$

On the other hand, by lemma 3.7(iii), there exists $\varphi \in X$ such that $\Psi_\lambda(t\varphi) < 0$ for all sufficiently small $t > 0$. Moreover, for any $u \in B_1(0)$, the inequality

$$\Psi_\lambda(u) \geqslant \frac{1}{p^+}\|u\|^{p^+} - \frac{A}{a}c_1\|u\|^a - \frac{B}{b}c_2\|u\|^b$$

holds and we deduce that

$$-\infty < \underline{c} := \inf_{\overline{B_1(0)}} \Psi_\lambda < 0.$$

Now let

$$0 < \varepsilon < \inf_{\partial B_1(0)} \Psi_\lambda - \inf_{B_1(0)} \Psi_\lambda.$$

Applying Ekeland's variational principle for the functional $\Psi_\lambda : \overline{B_1(0)} \to \mathbb{R}$, there exists $u_\varepsilon \in \overline{B_1(0)}$ such that

$$\Psi_\lambda(u_\varepsilon) < \inf_{\overline{B_1(0)}} \Psi_\lambda + \varepsilon, \tag{3.12}$$

$$\Psi_\lambda(u_\varepsilon) < \Psi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon. \tag{3.13}$$

Since

$$\Psi_\lambda(u_\varepsilon) < \inf_{\overline{B_1(0)}} \Psi_\lambda + \varepsilon < \inf_{B_1(0)} \Psi_\lambda + \varepsilon < \inf_{\partial B_1(0)} \Psi_\lambda,$$

it follows that $u_\varepsilon \in B_1(0)$. Now we define $M : \overline{B_1(0)} \to \mathbb{R}$ by $M(u) = \Psi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of M and thus

$$\frac{M(u_\varepsilon + t\nu) - M(u_\varepsilon)}{t} \geqslant 0$$

for a small $t > 0$ and ν in the unit sphere of X. The above relation yields

$$\frac{\Psi_\lambda(u_\varepsilon + t\nu) - \Psi_\lambda(u_\varepsilon)}{t} + \varepsilon\|\nu\| \geqslant 0.$$

Letting $t \to 0$, it follows that

$$\langle \Psi'_\lambda(u_\varepsilon), \nu\rangle + \varepsilon\|\nu\| > 0$$

and we infer that $\|\Psi'_\lambda(u_\varepsilon)\| \leqslant \varepsilon$. We deduce that there exists $\{u_n\} \subset B_1(0)$ such that $\Psi_\lambda(u_n) \to \underline{c}$ and $\Psi'_\lambda(u_n) \to 0$. Using the fact that Ψ_λ satisfies the Palais–Smale condition on X, we deduce that $\{u_n\}$ converges strongly to u_2 in X. Thus, u_2 is a weak solution for the problem (1.5)–(1.6) and, since $0 > \underline{c} = \Psi_\lambda(u_2)$, it follows that u_2 is non-trivial. Finally, we point out the fact that $u_1 \neq u_2$ since

$$\Psi_\lambda(u_1) = \bar{c} > 0 > \underline{c} = \Psi_\lambda(u_2).$$

The proof is complete. □

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