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NEW BOUNDS FOR THE OSTROWSKI-LIKE TYPE INEQUALITIES

Vu Nhat Huy and Quốc-Anh Ngô

Abstract. We improve some inequalities of Ostrowski-like type and further generalize them.

1. Introduction

In 1938, Ostrowski [8] proved the following interesting integral inequality which has received considerable attention from many researchers.

Theorem 1 (See [8]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative function $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leqq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty \tag{1}$$

for all $x \in [a,b]$.

This inequality gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$ by the value $f(x)$ at point $x \in [a,b]$. The first generalization of Ostrowski inequality was given by G. V. Milovanović and J. E. Pečarić in [7]. However, note that estimate (1) can be applied only if f' is bounded. In the first part of this paper, we will improve (1) by assuming $f' \in L^p(a,b)$ for some $1 \leqq p < \infty$. More precisely, we obtain the following theorem.

Theorem 2. *Assume that $1 \leqq p$. Let $I \subset \mathbb{R}$ be an open interval such that $[a,b] \subset I$ and let $f : I \to \mathbb{R}$ be a differentiable function such that $f' \in L^p(a,b)$. Then we have*

$$\left| f\left(x\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leqq A(x,q)\|f'\|_p \tag{2}$$

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for all $x \in [a,b]$ *where*

$$A(x,q) = \left(\frac{1}{b-a} \left(\frac{1}{q+1} \left(\frac{b-a}{2} \right)^{q+1} \right)^{\frac{1}{q}} + \left| x - \frac{a+b}{2} \right|^{\frac{1}{q}} \right)$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. $\lim_{q\to+\infty} A(x,q) = \frac{3}{2}$ for each $x \in [a,b]$.

Example 1. Let us consider the integral

$$\int_0^1 \sqrt[3]{\sin(t^2)}\,\mathrm{d}t.$$

Then we have

$$f(t) = \sqrt[3]{\sin(t^2)} \quad \text{and} \quad f'(t) = \frac{2t\cos(t^2)}{3\sqrt[3]{\sin^2(t^2)}}$$

such that $f'(t) \to \infty$ as $t \to 0$. On the other hand, we have

$$\int_0^1 |f'(t)|^2 \mathrm{d}t \leqq \frac{4}{9} \max_{0\leqq t\leqq 1} \left| \frac{t^2\cos(t^2)}{\sin(t^2)} \right| \int_0^1 \frac{\mathrm{d}t}{\sqrt[3]{\sin(t^2)}} \leqq \frac{16}{9},$$

i.e., $\|f'\|_{L^2} \leqq \frac{4}{3}$. It follows that

$$\left| \sqrt[3]{\sin(x^2)} - \int_0^1 \sqrt[3]{\sin(t^2)}\mathrm{d}t \right| \leqq \frac{4}{3} \left(\frac{1}{\sqrt{24}} + \sqrt{x - \frac{1}{2}} \right)$$

for all $x \in [0,1]$.

In recent years, a number of authors have written about generalizations of Ostrowski inequality. For example, this topic is considered in [1, 3, 4, 6, 11, 5]. In this way, some new types of inequalities are formed, such as inequalities of Ostrowski-Griiss type, inequalities of Ostrowski-Chebyshev type, etc. The first inequality of Ostrowski-Grüss type was given by Dragomir and Wang in [4]. It was generalized and improved by Matić, Pečarić, and Ujević in [6]. Cheng gave a sharp version of the mentioned inequality in [3]. Recently in [11], Ujević proved the following result which gives much better results than estimations based on [3].

Theorem 3 (See [11, Theorem 4])**.** *Let* $f : I \to \mathbb{R}$, *where* $I \subset \mathbb{R}$ *is an interval, be a twice continuously differentiable mapping in the interior* $\mathring{I}$ *of* I *with* $f'' \in L^2(a,b)$ *and let* $a, b \in \mathring{I}$, $a < b$. *Then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \leqq \frac{(b-a)^{\frac{3}{2}}}{2\pi\sqrt{3}} \|f''\|_2 \tag{3}$$

for all $x \in [a,b]$.

If we assume f is such that f'' is of class L^p for some $1 \leqq p < \infty$, then we obtain:

Theorem 4. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior $\overset{\circ}{I}$ of I with $f'' \in L^p(a,b)$, $1 \leqq p < \infty$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqq B(q)\|f''\|_p \tag{4}$$

for all $x \in [a,b]$ where

$$B(q) = \left[\frac{3}{2}\left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}} + \frac{1}{2(b-a)}\left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\right]$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. $\lim_{q \to +\infty} B(q) = 2(b-a)$.

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 5 (See [2]). *Let $f : [a,b] \to \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a,b]$, $x_0 \in (a,b)$, then for all $x \in (a,b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x),$$

where $T_{r-1}(f, x_0, \cdot)$ is a Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

and the remainder can be given by

$$R_{r-1}(f, x_0, x) = \int_{x_0}^{x} \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!}\mathrm{d}t. \tag{5}$$

By a simple calculation, the remainder in (5) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x-x_0-t)^{r-1} f^{(r)}(x_0+t)}{(r-1)!}\mathrm{d}t$$

which helps us to deduce a similar representation of f as following

$$f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t)\,\mathrm{d}t. \tag{6}$$

Proof of Theorem 2. Denote

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

By Fundamental Theorem of Calculus

$$I(f) = F(b) - F(a).$$

Applying Lemma 5 gives

$$F(b) = F\left(\frac{a+b}{2}\right) + \frac{b-a}{2}F'\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^{b} (b-t)\, F''(t)\mathrm{d}t$$

which implies that

$$F(b) - F\left(\frac{a+b}{2}\right) = \frac{b-a}{2}f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^{b} (b-t)\, f'(t)\mathrm{d}t.$$

We see that

$$F(a) = F\left(\frac{a+b}{2}\right) + \frac{a-b}{2}F'\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^{a} (a-t)\, F''(t)\mathrm{d}t$$

which yields

$$F(a) - F\left(\frac{a+b}{2}\right) = \frac{a-b}{2}f\left(\frac{a+b}{2}\right) + \int_{a}^{\frac{a+b}{2}} (t-a)\, f'(t)\mathrm{d}t.$$

Therefore,

$$F(b) - F(a) = (b-a)f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^{b} (b-t)\, f'(t)\mathrm{d}t - \int_{a}^{\frac{a+b}{2}} (t-a)\, f'(t)\mathrm{d}t.$$

By changing $t = a + b - x$, we get

$$\int_{\frac{a+b}{2}}^{b} (b-t)\, f'(t)\mathrm{d}t = \int_{a}^{\frac{a+b}{2}} (t-a)\, f'(a+b-t)\, \mathrm{d}t$$

which helps us to deduce that

$$\int_a^b f(t)\mathrm{d}t = (b-a)f\left(\frac{a+b}{2}\right) + \int_{a}^{\frac{a+b}{2}} (t-a)\left(f'(a+b-t) - f'(t)\right)\mathrm{d}t.$$

On the other hand,

$$f(x) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{x} f'(t)\mathrm{d}t.$$

Then

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$= \int_{\frac{a+b}{2}}^{x} f'(t)\mathrm{d}t - \frac{1}{b-a}\int_{a}^{\frac{a+b}{2}} (t-a)\left(f'(a+b-t) - f'(t)\right)\mathrm{d}t.$$

Next we consider the case $1 < p < \infty$. We first have the following estimates

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} (t-a)\left(f'(a+b-t) - f'(t)\right) \mathrm{d}t\right| \\
&\leqq \left|\int_a^{\frac{a+b}{2}} (t-a) f'(a+b-t) \mathrm{d}t\right| + \left|\int_a^{\frac{a+b}{2}} (t-a) f'(t) \mathrm{d}t\right| \\
&\leqq \left(\int_a^{\frac{a+b}{2}} |f'(a+b-t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |t-a|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_a^{\frac{a+b}{2}} |f'(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |t-a|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{q+1}\left(\frac{b-a}{2}\right)^{q+1}\right)^{\frac{1}{q}} \|f'\|_p.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\left|\int_{\frac{a+b}{2}}^x f'(t) \mathrm{d}t\right| &\leqq \left|\int_{\frac{a+b}{2}}^x |f'(t)|^p \mathrm{d}t\right|^{\frac{1}{p}} \left|\int_{\frac{a+b}{2}}^x 1^q \mathrm{d}t\right|^{\frac{1}{q}} \\
&\leqq \left|\int_a^b |f'(t)|^p \mathrm{d}t\right|^{\frac{1}{p}} \left|\int_{\frac{a+b}{2}}^x 1^q \mathrm{d}t\right|^{\frac{1}{q}} \\
&= \|f'\|_p \left|x - \frac{a+b}{2}\right|^{\frac{1}{q}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(t) \mathrm{d}t\right| \\
&\leqq \left(\frac{1}{b-a}\left(\frac{1}{q+1}\left(\frac{b-a}{2}\right)^{q+1}\right)^{\frac{1}{q}} + \left|x - \frac{a+b}{2}\right|^{\frac{1}{q}}\right) \|f'\|_p.
\end{aligned}$$

If $p = 1$, then

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}} (t-a)\left(f'(a+b-t) - f'(t)\right) \mathrm{d}t\right| \\
&\leqq \frac{b-a}{2} \int_a^{\frac{a+b}{2}} \left(|f'(a+b-t)| + |f'(t)|\right) \mathrm{d}t \\
&= \frac{b-a}{2} \|f'\|_1
\end{aligned}$$

and

$$\left|\int_{\frac{a+b}{2}}^{x} f'(t)\mathrm{d}t\right| \leqq \|f'\|_1$$

which helps us to claim that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqq \frac{3}{2}\|f'\|_1.$$

□

Corollary 1. *If we put $x = \frac{a+b}{2}$, then under the assumptions of Theorem 1 and $1 \leqq p < \infty$, we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqq \frac{1}{b-a}\left(\frac{1}{q+1}\left(\frac{b-a}{2}\right)^{q+1}\right)^{\frac{1}{q}}\|f'\|_p.$$

Note that

$$\frac{1}{b-a}\left(\frac{1}{q+1}\left(\frac{b-a}{2}\right)^{q+1}\right)^{\frac{1}{q}} = \frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^{\frac{1}{q}}.$$

Proof of Theorem 4. Clearly, by Lemma 5 one has

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t &= \frac{1}{b-a}\left(F(b) - F(a)\right)\\ &= \frac{1}{b-a}\left((b-a)F'(a) + \frac{(b-a)^2}{2}F''(a) + \int_a^b \frac{(b-t)^2}{2}F'''(t)\mathrm{d}t\right)\\ &= f(a) + \frac{b-a}{2}f'(a) + \frac{1}{b-a}\int_a^b \frac{(b-x)^2}{2}f''(t)\mathrm{d}t.\end{aligned}$$

Similarly,

$$f(x) = f(a) + (x-a)f'(a) + \int_a^b (b-t)f''(t)\mathrm{d}t$$

and

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= \frac{1}{b-a}\left((b-a)f'(a) + \int_a^b (b-t)f''(t)\mathrm{d}t\right)\\ &= f'(a) + \frac{1}{b-a}\int_a^b (b-t)f''(t)\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right|\\ = &\left|\int_a^b (b-x)f''(x)\,\mathrm{d}t - \frac{1}{b-a}\int_a^b \frac{(b-t)^2}{2}f''(t)\mathrm{d}t - \frac{x-\frac{a+b}{2}}{b-a}\int_a^b (b-t)f''(t)\mathrm{d}t\right|.\end{aligned}$$

If $1 < p < \infty$, then by the Hölder inequality, one has

$$\left|\int_a^b (b-t)\, f''(t)\mathrm{d}t\right| \leqq \|f''\|_p \left(\int_a^b (b-t)^q \mathrm{d}t\right)^{\frac{1}{q}} = \left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}} \|f''\|_p,$$

and

$$\frac{1}{b-a}\left|\int_a^b \frac{(b-t)^2}{2} f''(t)\mathrm{d}t\right| \leqq \frac{1}{2(b-a)}\|f''\|_p \left(\int_a^b (b-t)^{2q}\mathrm{d}t\right)^{\frac{1}{q}}$$
$$= \frac{1}{2(b-a)}\left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}} \|f''\|_p,$$

and

$$\left|\frac{x-\frac{a+b}{2}}{b-a}\int_a^b (b-t)\, f''(t)\mathrm{d}t\right| \leqq \frac{1}{2}\left|\int_a^b (b-t)\, f''(t)\mathrm{d}t\right|$$
$$\leqq \frac{1}{2}\left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}} \|f''\|_p.$$

Thus,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right|$$
$$\leqq \left[\frac{3}{2}\left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}} + \frac{1}{2(b-a)}\left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\right]\|f''\|_p.$$

If $1 = p$, then again by the Hölder inequality, one has

$$\left|\int_a^b (b-t)\, f''(t)\mathrm{d}t\right| \leqq (b-a)\int_a^b |f''(t)|\,\mathrm{d}t = (b-a)\|f''\|_1,$$

and

$$\frac{1}{b-a}\left|\int_a^b \frac{(b-t)^2}{2} f''(t)\mathrm{d}t\right| \leqq \frac{1}{b-a}\frac{(b-a)^2}{2}\int_a^b |f''(t)|\,\mathrm{d}t = \frac{b-a}{2}\|f''\|_1,$$

and

$$\left|\frac{x-\frac{a+b}{2}}{b-a}\int_a^b (b-t)\, f''(t)\mathrm{d}t\right| \leqq \frac{1}{2}\left|\int_a^b (b-t)\, f''(t)\mathrm{d}t\right| \leqq \frac{1}{2}(b-a)\|f''\|_1.$$

Hence,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqq 2(b-a)\|f''\|_1.$$

Therefore,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right|$$
$$\leqq\left[\frac{3}{2}\left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}}+\frac{1}{2(b-a)}\left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\right]\|f''\|_p.$$

□

Corollary 2. *If we put $x=\frac{a+b}{2}$, then under the assumptions of Theorem 3 and $1\leqq p<\infty$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|$$
$$\leqq\left[\frac{3}{2}\left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}}+\frac{1}{2(b-a)}\left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}}\right]\|f''\|_p.$$

3. Applications in numerical integral

Let $\Gamma=\{x_0=a<x_1<\cdots<x_n=b\}$ be a given subdivision of the interval $[a,b]$ such that $h=x_{i+1}-x_i=\frac{b-a}{n}$. Then we obtain the following theorem by using Corollary 1.

Theorem 6. *Under the assumptions of Theorem 2 and $1\leqq p<\infty$, we have*

$$\left|\frac{1}{n}\sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leqq\frac{1}{2n}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2}\right)^{\frac{1}{q}}\|f'\|_p.$$

Proof. We have

$$\left|f\left(\frac{x_{i-1}+x_i}{2}\right)-\frac{n}{b-a}\int_{x_{i-1}}^{x_i} f(t)\mathrm{d}t\right|\leqq\frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2n}\right)^{\frac{1}{q}}\|f'\|_{p,[x_{i-1},x_i]}$$

where

$$\|f'\|_{p,[x_{i-1},x_i]}=\left(\int_{x_{i-1}}^{x_i}|f'(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}.$$

Then,

$$\left|\frac{1}{n}\sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|$$
$$\leqq\frac{1}{2n}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{b-a}{2n}\right)^{\frac{1}{q}}\sum_{i=1}^n\|f'\|_{p,[x_{i-1},x_i]}.$$

Put

$$\alpha_i=\int_{x_{i-1}}^{x_i}|f'(t)|^p\mathrm{d}t.$$

Then

$$\sum_{i=1}^{n} \|f'\|_{p,[x_{i-1},x_i]} = \sum_{i=1}^{n} \alpha_i^{\frac{1}{p}} \leqq n^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} \alpha_i \right)^{\frac{1}{p}} = n^{1-\frac{1}{p}} \|f'\|_p.$$

Therefore,

$$\left| \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right|$$
$$\leqq \frac{1}{2n} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{b-a}{2n} \right)^{\frac{1}{q}} n^{1-\frac{1}{p}} \|f'\|_p$$
$$= \frac{1}{2n} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{b-a}{2} \right)^{\frac{1}{q}} \|f'\|_p. \qquad \square$$

If we use Corollary 2, we then obtain the following theorem whose proof will be omitted.

Theorem 7. *Under the assumptions of Theorem 4 and* $1 \leqq p < \infty$*, we have*

$$\left| \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2} \right) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right|$$
$$\leqq \frac{1}{n^2} \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \right] \|f''\|_p.$$

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