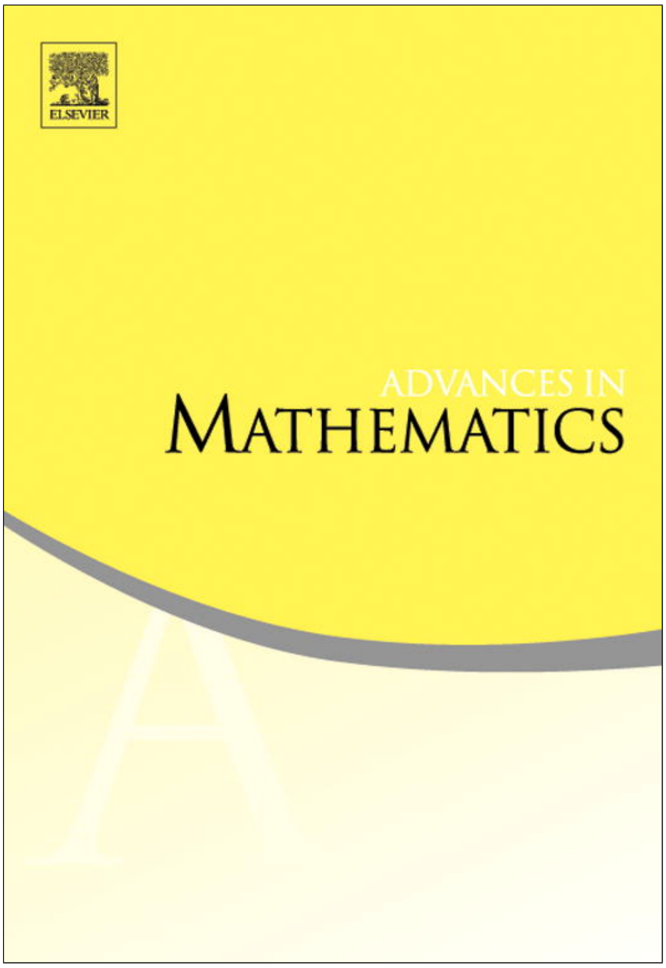
ELSEVIER
ADVANCES IN
MATHEMATICS

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 230 (2012) 2378–2415

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds

Quốc Anh Ngô*, Xingwang Xu

Department of Mathematics, National University of Singapore, Block S17 (SOC1), 10 Lower Kent Ridge Road, Singapore 119076, Singapore

Received 12 October 2011; accepted 9 April 2012

Communicated by Gang Tian

Abstract

This article mainly concerns with the non-existence, existence, and multiplicity results for positive solutions to the Einstein-scalar field Lichnerowicz equation on closed manifolds with a negative conformal-scalar field invariant. This equation arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity. Our analysis introduces variational techniques to the analysis of the Hamiltonian constraint equation, especially those cases when the prescribed scalar curvature-scalar field function may change sign. To our knowledge, such a problem remains open.

MSC: 35J60; 53C21; 53C80; 58E05; 58J45; 83C05

Keywords: Einstein-scalar field equation; Lichnerowicz equation; Picone type identity; Critical exponent; Negative exponent; Mountain pass theorem; Palais–Smale condition; Sign-changing nonlinearity

Contents

* Corresponding author.
E-mail addresses: nqanh@nus.edu.sg, bookworm_vn@yahoo.com (Q.A. Ngô), matxuxw@nus.edu.sg (X. Xu).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.04.007

1. Introduction

Along with the rapid development in general relativity, physicists pose many challenging problems to mathematicians, for example, the initial value problems, the well-posedness problems, the global stability problems, etc. Among these problems, the initial value problem turns out to be the most interesting problem from the mathematical point of view. When solving the initial value problems, one needs to solve the so-called constraint equations which can be formulated via the following system of equations defined on a Riemannian manifold $(M, \overline{g})$ without the boundary of dimension $n \geqslant 3$,

$$\begin{aligned} &\mathrm{Scal}_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + (\mathrm{trace}_{\overline{g}}\overline{K})^2 - 2\rho = 0, \\ &\nabla_{\overline{g}} \cdot \overline{K} - \nabla_{\overline{g}}\mathrm{trace}_{\overline{g}}\overline{K} - J = 0, \end{aligned} \tag{1.1}$$

where all quantities of (1.1) involving a metric are computed with respect to $\overline{g}$, an induced metric of $\mathbf{g}$ when embedded in a spacetime $(V, \mathbf{g})$, $\overline{K}$ the second fundamental form, $\mathrm{Scal}_{\overline{g}}$ the scalar curvature of $\overline{g}$, ρ a scalar, J a vector field on M, and T a tensor of the sources; see [6,7,9].

Since the constraint equations form an under-determined system, they are in general hard to solve. However, it was remarked in [6] that the conformal method can be effectively applied in the constant mean curvature setting, that is to look for the metric $\overline{g}$ of the form $u^{\frac{4}{n-2}} g$ where g is fixed. To be precise, when the conformal method is applied in this setting, the constraint equations (1.1) are easily transformed to the so-called Hamiltonian and momentum constraints. In the literature, the momentum constraint is a second-order semilinear elliptic equation that can be easily solved if we are in the constant mean curvature setting. The most difficult part is to solve the Hamiltonian constraint which can be formulated by a simple partial differential equation,

$$\Delta_g u + hu = f u^{2^\star - 1} + \frac{a}{u^{2^\star + 1}}, \quad u > 0, \tag{1.2}$$

where $\Delta_g = -\mathrm{div}_g(\nabla_g)$ is the Laplace–Beltrami operator, $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and $h, f, a \geqslant 0$ are smooth functions. Throughout this paper, equations of the form (1.2) are called the Einstein-scalar field Lichnerowicz equations.

While, as we have noted, the conformal method can be effectively applied for solving the Einstein constraint equations (1.1) in most cases, it should be pointed out that there are several cases for which either partial result or no result was achieved, especially when gravity is coupled to field sources. To see this more precise, we assume the presence of a real scalar field ψ on the space time $(V, \mathbf{g})$ with a potential U being a function of ψ, the Hamiltonian constraint equation then take the form of (1.2) with[1]

$$h = c_n\left(\mathrm{Scal}_g - |\nabla\psi|^2\right), \qquad a = c_n\left(|\sigma + \mathcal{D}W|^2 + \pi^2\right), \tag{1.3}$$

and

$$f = -c_n\left(\frac{n-1}{n}\tau^2 - 2U(\psi)\right), \tag{1.4}$$

where $c_n = \frac{n-2}{4(n-1)}$, τ is the mean curvature of M computed with respect to $\overline{g}$, σ is a transverse and traceless tensor, and the operator $\mathcal{D}$ is the conformal Killing operator relative to g. Based on the division in [8], one can observe that there are two cases corresponding to either $h < 0$ or $h \equiv 0$ with sign-changing f, for which no result was achieved. This is basically due to the fact that the method of sub- and super-solutions does not work, thus forcing us to develop a new approach.

In view of the discussion above, it is worth understanding the solvability of the constraint equations in those cases left in [8]. As a step toward achieving the full answer, the main purpose of this study was to search for some sufficient conditions for the solvability of the Einstein-scalar field constraint equations in those cases left in [8]. As such, in the current study, we face not only the presence of both the critical exponent and the negative exponent as mentioned above, but also the sign-changing problem. In order to overcome those difficulties, in our study, a careful and deep analysis of the constraint equations was developed to suit for the analysis. Besides, due to the limit of the length, this work was limited to the case of negative Yamabe-scalar field conformal invariant, namely, $h < 0$, and when f may change sign. The case $h = 0$, which creates some new phenomena and needs some new ingredients, will be treated in a separated paper [17]. We assume hereafter that $a \geqslant 0$ and $\int_M a\,dv_g > 0$ in M. This assumption implies no physical restrictions since we always have $a \geqslant 0$ in the original Einstein-scalar field theory.

Concerning the solvability of (1.2), by using the variational method [2,21], Hebey et al. in [12] recently proved some non-existence and existence results for the case of positive Yamabe-scalar field conformal invariant, namely $h > 0$. The advantage of their setting is that the first eigenvalue of the operator $\Delta_g + h$ is strictly positive, and thus, various good properties of the theory of weighted Sobolev spaces can be applied.

According to [8, Proposition 1], if we consider a new metric, say $\widetilde{g} = v^{\frac{4}{n-2}} g$, the function $h_{\widetilde{g}}$ with respect to the metric $\widetilde{g}$ verifies

$$\Delta_g v + h_g v = h_{\widetilde{g}} v^{\frac{n+2}{n-2}}.$$

[1] It is worth noticing that the coefficient of $U(\psi)$ in the expression of f is 2 instead of 4 in the original paper, we would like to thank Prof. Pollack for confirming this in [18].

The method of sub- and super-solutions [13] applied to the above equation says that we can select some function $v > 0$ such that $h_{\widetilde{g}}$ is a negative constant. By a well-known regularity result [13, Lemma 2.6], we see that v is smooth. Therefore, and thanks to the conformally covariance property of the Einstein-scalar field Lichnerowicz equations [8, Proposition 2], we can freely choose a background metric g such that h is a negative constant and by normalization we are still able to assume that the manifold M has unit volume.

In the first part of the present paper, we mainly consider the case when the function f takes both positive and negative values. The first main theorem can be stated as follows.

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that f and $a \geqslant 0$ are smooth functions on M such that $\int_M f\,dv_g < 0$, $\sup f > 0$, $\int_M a\,dv_g > 0$, and $|h| < \lambda_f$ where λ_f is given in (2.1) below. Let us also suppose that the integral of a satisfies*

$$\int_M a\,dv_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-|\,dv_g}\right)^n \int_M |f^-|\,dv_g, \tag{1.5}$$

where f^- is the negative part of f. Then there exists a number $\mathcal{C} > 0$ to be specified such that if

$$\frac{\sup f}{\int_M |f^-|\,dv_g} < \mathcal{C}, \tag{1.6}$$

Eq. (1.2) possesses at least two smooth positive solutions.

To be precise, the constant $\mathcal{C}$ appearing in (1.6) is given in (4.22) below; see also Remark 4.5. Roughly speaking, for the existence part, the constant $\mathcal{C}$ depends only on the negative part of f. However, for the multiplicity part, $\mathcal{C}$ also depends on the positive part of f. The question of whether we can find an explicit formula for $\mathcal{C}$ turns out to be difficult, even for the prescribed scalar curvature equation, for interested readers, we refer to [4].

If we assume that f does not change sign in the sense that $f \leqslant 0$ in M, we obtain necessary and sufficient solvability conditions as pointed out by Choquet-Bruhat et al. [8] in the case of (1.3) and (1.4). That is the content of our next result.

Theorem 1.2. *Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Let $h < 0$ be a constant, f and a be smooth functions on M with $a \geqslant 0$ in M, $f \leqslant 0$ but not strictly negative. Then Eq. (1.2) possesses one positive solution if and only if $|h| < \lambda_f$.*

One can easily observe that Eq. (1.2) is closely related to the Yamabe problem which was completely solved through [23,22,20,3] and the prescribing scalar curvature problem which has been studied for years by many great mathematicians. Let us mention, among others, several typical works such as [13,20,11,19,3,5]. Recently, several aspects of solutions of the Einstein-scalar field Lichnerowicz equations have been studied and achieved, we list here some works such as [10,15,14,16,17]. We should also point out that the idea of our approach was based on Rauzy [19]. However, the analysis in this work is much more involved than that used in [19]. To see the difference, let us mention the strategy used in this work. As a first step to tackle (1.2), we look for solutions of the following subcritical problem:

$$\Delta_g u + hu = f|u|^{q-2}u + \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}}. \tag{1.7}$$

Our main procedure is to show that the limit exists as first $\varepsilon \to 0$ and then $q \to 2^\star$ under various assumptions.

Before closing this section, let us briefly mention the organization of the paper and highlight some techniques used. Section 2 consists of two parts. First, we set up some notations and prove basic properties, including a non-existence result and regularity, for positive solutions of (1.2). Then, we derive two necessary conditions for the solvability of (1.2): $\int_M f dv_g < 0$ and $\lambda_f > |h|$. It is worth noticing that those conditions were first observed in the case of prescribing scalar curvature. In our study, thanks to $a \geqslant 0$, while the first condition can be proved by a simple use of integration by parts, the second condition needs some new observation. The proof we provided here is new and simple which can also be applied to the prescribing scalar curvature case. Then in Section 3, a careful analysis of the energy functional is presented by proving the various properties involving the asymptotic behavior of the energy functional that is needed in later parts. Again, it should be mentioned that the basic idea underlying the presented analysis was borrowed from [19]. The last part of the section is devoted to the proof of the Palais–Smale condition. To the best of our knowledge, there is no such a result in the literature since our energy functional contains both critical and negative exponents that cause a lot of difficulty. Having these preparation, we spend Section 4 to prove Theorem 1.1 and Section 5 to prove Theorem 1.2.

2. Notations and basic properties for positive solutions

We now set up notations. First, as they have already appeared in the previous section, throughout this paper, we use f^- and f^+ to indicate the negative and positive parts of f, respectively, that is, $f^- = \min(f, 0)$ and $f^+ = \max(f, 0)$. Following in [19], we define

$$\lambda_f = \begin{cases} \inf\limits_{u \in \mathscr{A}} \dfrac{\int_M |\nabla u|^2 dv_g}{\int_M |u|^2 dv_g}, & \text{if } \mathscr{A} \neq \emptyset, \\ +\infty, & \text{if } \mathscr{A} = \emptyset, \end{cases} \tag{2.1}$$

where

$$\mathscr{A} = \left\{ u \in H^1(M) : u \geqslant 0, u \not\equiv 0, \int_M |f^-| u dv_g = 0 \right\}. \tag{2.2}$$

Functions in $\mathscr{A}$ are to be thought of as functions that vanish on the support of f^-. Obviously, $\lambda_f \geqslant 0$. Let $H^p(M)$ be the usual Sobolev space equipped with the standard norm. By $\mathcal{K}_1$ and $\mathcal{A}_1$, we mean the best positive constants for the Sobolev embedding of $H^1(M)$ into $L^{2^\star}(M)$, that is, for all $u \in H^1(M)$, there holds

$$\|u\|_{L^{2^\star}}^2 \leqslant \mathcal{K}_1 \|\nabla u\|_{L^2}^2 + \mathcal{A}_1 \|u\|_{L^2}^2 .$$

We also denote by $2^\flat$ the average of 2 and $2^\star$, that is, $2^\flat = \frac{2n-2}{n-2}$. Throughout this paper, we always assume $q \in (2^\flat, 2^\star)$.

2.1. A lower bound for positive solutions

Our purpose here was to derive a lower bound for a positive C^2 solution u of Eq. (1.7).

Lemma 2.1. *Let u be a positive C^2 solution of (1.7) with h a negative constant. Then, there holds*

$$\min_M u \geqslant \min\left\{\left(\frac{h}{\inf f}\right)^{\frac{1}{2^\flat-2}}, 1\right\} > 0 \tag{2.3}$$

for any $q \in (2^\flat, 2^\star)$ and any $\varepsilon > 0$.

Proof. Let us assume that u achieves its minimum value at x_0. For the sake of simplicity, we denote $u(x_0)$, $f(x_0)$, and $a(x_0)$ by u_0, f_0, and a_0 respectively. Notice that $u_0 > 0$ since u is a positive solution. We then have $\Delta_g u|_{x_0} \leqslant 0$; in particular,

$$hu_0 \geqslant f_0(u_0)^{q-1} + \frac{a_0 u_0}{((u_0)^2+\varepsilon)^{\frac{q}{2}+1}} \geqslant f_0(u_0)^{q-1}.$$

Consequently, we get $f_0 < 0$ and thus $0 < \frac{h}{f_0} \leqslant (u_0)^{q-2}$ which immediately implies

$$\min_M u \geqslant \left(\frac{h}{\inf f}\right)^{\frac{1}{q-2}} \geqslant \min\left\{\left(\frac{h}{\inf f}\right)^{\frac{1}{2^\flat-2}}, 1\right\}$$

for any $q \in (2^\flat, 2^\star)$ and any $\varepsilon > 0$. This proves our lemma. □

2.2. *Regularity for weak solutions*

This subsection is devoted to the regularity of weak solutions of (1.7). We continue to assume that $h < 0$ is constant, $\varepsilon \geqslant 0$ is fixed, f and $a \geqslant 0$ are smooth.

Lemma 2.2. *Assume that $u \in H^1(M)$ is almost everywhere non-negative weak solution of Eq. (1.7). Then we have the following.*

(a) *If $\varepsilon > 0$, then $u \in C^\infty(M)$.*
(b) *If $\varepsilon = 0$ and $u^{-1} \in L^p(M)$ for all $p \geqslant 1$, then $u \in C^\infty(M)$.*

Proof. We first rewrite (1.7) as

$$\Delta_g u + b(x)(1+u) = 0$$

with

$$b(x) = \frac{u(x)}{1+u(x)}\left(h - \frac{a(x)}{(u(x)^2+\varepsilon)^{\frac{q}{2}+1}} - f(x)|u(x)|^{q-2}\right). \tag{2.4}$$

By the Sobolev embedding, we know that $u \in L^q(M)$ for any $q \in (2^\flat, 2^\star]$. This and the conditions in both cases (a) and (b) imply

$$h - \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}+1}} - f|u|^{q-2} \in L^{\frac{q}{q-2}}(M).$$

Notice that, from $q \leqslant 2^\star$, there holds $\frac{q}{q-2} \geqslant \frac{n}{2}$. We now use the Brezis–Kato estimate [21, Lemma B.3] to conclude that $u \in L^s(M)$ for any $s > 0$. Thus, the Caldéron–Zygmund inequality implies that $u \in H^p(M)$ for any $p > 1$. The Sobolev embedding again implies that u is in $C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$. Thus, by (2.4) we know from the Schauder theory that $u \in C^{2,\alpha}(M)$ for some $\alpha \in (0,1)$. In particular, u has a strictly positive lower bound by means of Lemma 2.1. Since u stays away from zero, we can iterate this process to conclude $u \in C^\infty(M)$. □

2.3. A necessary condition for f

The purpose of this subsection was to derive a necessary condition for $\int_M f\,dv_g$ so that (1.7) admits positive smooth solution. Our argument was motivated from the well-known prescribing scalar curvature problem.

Proposition 2.3. *The necessary condition for f so that Eq. (1.7) admits positive smooth solution is $\int_M f\,dv_g < 0$. In particular, the necessary condition for (1.2) to have positive smooth solution is $\int_M f\,dv_g < 0$.*

Proof. We assume that $u > 0$ is a smooth solution of (1.7). By multiplying both sides of (1.7) by u^{1-q} and integrating over M with a notice that $h < 0$, we arrive at

$$\int_M (\Delta_g u)u^{1-q}\,dv_g > \int_M f\,dv_g + \int_M \frac{au^{2-q}}{(u^2+\varepsilon)^{\frac{q}{2}+1}}\,dv_g.$$

By the divergence theorem, one obtains

$$\int_M (\Delta_g u)u^{1-q}\,dv_g = \int_M \nabla u\cdot\nabla(u^{1-q})\,dv_g = (1-q)\int_M u^{-q}|\nabla u|^2\,dv_g.$$

This and the fact that $q > 2$ deduce that

$$\int_M f\,dv_g + \int_M \frac{au^{2-q}}{(u^2+\varepsilon)^{\frac{q}{2}+1}}\,dv_g < 0.$$

Obviously, $\int_M f\,dv_g < 0$ as claimed. □

2.4. A necessary condition for h

In this subsection, we show that the condition $|h| < \lambda_f$ is necessary if $\lambda_f < +\infty$ in order for (1.2) to have a positive smooth solution. In the light of the condition $a \geqslant 0$, one may go through [19, Section III.3] to conclude this necessary condition. Here we provide a different proof which is shorter than the proof in [19, Section III.3]. Our argument depends on a Picone type identity for integrals [1] whose proof makes use of the density. We believe that such an identity has its own interest.

Lemma 2.4. *Assume $v \in H^1(M)$ with $v \geqslant 0$ and $v \not\equiv 0$. Suppose that $u > 0$ is a smooth function. Then we have*

$$\int_M |\nabla v|^2\,dv_g = \int_M \frac{\Delta u}{u}v^2\,dv_g + \int_M u^2\left|\nabla\left(\frac{v}{u}\right)\right|^2\,dv_g.$$

We now provide a different proof for the necessary condition $|h| < \lambda_f$.

Proposition 2.5. *If Eq. (1.2) has a positive smooth solution, it is necessary to have $|h| < \lambda_f$.*

Proof. We only need to consider the case $\lambda_f < \infty$ since otherwise it is trivial. We let $v \in \mathscr{A}$ arbitrary and assume that u is a positive smooth solution to (1.2). Using Lemma 2.4 and (1.2), we find that

$$\begin{aligned}\int_M |\nabla v|^2\,dv_g &= \int_M \frac{\Delta u}{u} v^2 dv_g + \int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g \\ &= |h|\int_M v^2 dv_g + \int_M f u^{2^\star-2} v^2 dv_g \\ &\quad + \int_M \frac{a v^2}{u^{2^\star+2}} dv_g + \int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g \\ &\geqslant |h|\int_M v^2 dv_g + \int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g.\end{aligned}$$

In other words, there holds

$$\frac{\int_M |\nabla v|^2\,dv_g}{\int_M v^2 dv_g} \geqslant |h| + \frac{\int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g}{\int_M v^2 dv_g}. \tag{2.5}$$

In particular, $\lambda_f \geqslant |h| > 0$ by taking the infimum with respect to v. Notice that

$$\begin{aligned}\frac{\int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g}{\int_M v^2 dv_g} &= \frac{\int_M u^2 \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g}{\int_M u^2 \left(\frac{v}{u}\right)^2 dv_g} \\ &\geqslant \left(\frac{\inf u}{\sup u}\right)^2 \frac{\int_M \left|\nabla\left(\frac{v}{u}\right)\right|^2 dv_g}{\int_M \left(\frac{v}{u}\right)^2 dv_g} \geqslant \lambda_f \left(\frac{\inf u}{\sup u}\right)^2\end{aligned}$$

since $\frac{v}{u} \in \mathscr{A}$. Having this, we can check from (2.5) that

$$\frac{\int_M |\nabla v|^2\,dv_g}{\int_M v^2 dv_g} \geqslant |h| + \lambda_f \left(\frac{\inf u}{\sup u}\right)^2.$$

By taking the infimum with respect to v, we obtain

$$\lambda_f \geqslant |h| + \lambda_f \left(\frac{\inf u}{\sup u}\right)^2.$$

This and the fact that $\lambda_f > 0$ give us the desired result. □

2.5. *The non-existence of smooth positive solutions of finite H^1-norm*

Let u be a smooth positive solution of (1.2). The main aim of this subsection was to derive a necessary condition for a such that $\|u\|_{H^1}$ is bounded by a given constant. Such a result is basically due to Hebey–Pacard–Pollack [12]. By integrating (1.2) over M, we get

$$\int_M hu\,dv_g = \int_M f u^{2^\star-1} dv_g + \int_M \frac{a}{u^{2^\star+1}} dv_g. \tag{2.6}$$

Let $\beta = \frac{2^\star}{22^\star+1}$. With an easy computation, we obtain through the Hölder inequality the following

$$\int_M a^\beta dv_g \leqslant \left(\int_M \frac{a}{u^{2^\star+1}} dv_g\right)^\beta \left(\int_M u^{2^\star} dv_g\right)^{1-\beta}. \tag{2.7}$$

The second term on the right hand side of (2.6) can be bounded as

$$\int_M \frac{a}{u^{2^\star+1}} dv_g = \int_M h u \, dv_g - \int_M f u^{2^\star-1} dv_g \leqslant \int_M |f^-| u^{2^\star-1} dv_g, \tag{2.8}$$

while the first term can be controlled again by the Hölder inequality as

$$\int_M |f^-| u^{2^\star-1} dv_g \leqslant \left(\int_M |f^-|^{2^\star} dv_g\right)^{\frac{1}{2^\star}} \left(\int_M u^{2^\star} dv_g\right)^{\frac{2^\star-1}{2^\star}}. \tag{2.9}$$

Combining (2.6)–(2.9), we get

$$\int_M a^\beta dv_g \leqslant \left(\int_M |f^-|^{2^\star} dv_g\right)^{\frac{\beta}{2^\star}} \left(\int_M u^{2^\star} dv_g\right)^{1-\frac{\beta}{2^\star}}. \tag{2.10}$$

Summarizing those estimates, we can state our main result of this subsection.

Proposition 2.6. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. Let also a, f be smooth functions on M with $a \geqslant 0$ in M and h a negative constant. If*

$$\int_M a^{\frac{2n}{5n-2}} dv_g > (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{4n^2}{(n-2)(5n-2)}} \Lambda^{\frac{8n^2}{(n-2)(5n-2)}} \left(\int_M |f^-|^{2^\star} dv_g\right)^{\frac{n-2}{5n-2}}$$

for some $\Lambda > 0$, then the Einstein-scalar field Lichnerowicz equation (1.2) does not possess smooth positive solutions with $\|u\|_{H^1} \leqslant \Lambda$.

Proof. Let u be a smooth positive solution of (1.2) such that $\|u\|_{H^1} \leqslant \Lambda$. By the Sobolev inequality and the fact that $1 - \frac{\beta}{2^\star} = \frac{22^\star}{22^\star+1}$, we have

$$\left(\int_M u^{2^\star} dv_g\right)^{1-\frac{\beta}{2^\star}} \leqslant (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{(2^\star)^2}{22^\star+1}} \|u\|_{H^1}^{\frac{2(2^\star)^2}{22^\star+1}}.$$

This and (2.10) imply

$$\int_M a^{\frac{2^\star}{22^\star+1}} dv_g \leqslant (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{(2^\star)^2}{22^\star+1}} \Lambda^{\frac{2(2^\star)^2}{22^\star+1}} \left(\int_M |f|^{2^\star} dv_g\right)^{\frac{1}{22^\star+1}}. \tag{2.11}$$

Thus, (2.11) and the fact that $\frac{1}{22^\star+1} = \frac{n-2}{5n-2}$ prove the proposition. □

Remark 2.7. This Proposition implies that it is reasonable and necessary to have some control on the integral $\int_M a \, dv_g$ as what we did in Theorem 1.1.

3. The analysis of the energy functionals

3.1. Functional setting

For each $q \in (2^\flat, 2^\star)$ and $k > 0$, we introduce $\mathscr{B}_{k,q}$ a hyper-surface of $H^1(M)$ which is defined by

$$\mathscr{B}_{k,q} = \left\{u \in H^1(M) : \|u\|_{L^q} = k^{\frac{1}{q}}\right\}. \tag{3.1}$$

Notice that for any $k > 0$, our set $\mathscr{B}_{k,q}$ is non-empty since it always contains $k^{\frac{1}{q}}$. Now we construct the energy functional associated to problem (1.7). For each $\varepsilon > 0$, we consider the functional $F_q^\varepsilon : H^1(M) \to \mathbb{R}$ defined by

$$F_q^\varepsilon(u) = \frac{1}{2}\int_M |\nabla u|^2 dv_g + \frac{h}{2}\int_M u^2 dv_g - \frac{1}{q}\int_M f|u|^q dv_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} dv_g.$$

By a standard argument, F_q^ε is continuously differentiable on $H^1(M)$ and thus weak solutions of (1.7) correspond to critical points of the functional F_q^ε. Now we set

$$\mu_{k,q}^\varepsilon = \inf_{u \in \mathscr{B}_{k,q}} F_q^\varepsilon(u).$$

By the Hölder inequality, it is not hard to see that $F_q^\varepsilon|_{\mathscr{B}_{k,q}}$ is bounded from below by $-k \sup f + \frac{h}{2}k^{\frac{2}{q}}$ and thus $\mu_{k,q}^\varepsilon > -\infty$ if k is finite. On the other hand, using the test function $u = k^{\frac{1}{q}}$, we get

$$\mu_{k,q}^\varepsilon \leqslant \frac{h}{2}k^{\frac{2}{q}} - \frac{k}{q}\int_M f dv_g + \frac{1}{q}\int_M \frac{a}{(k^{\frac{2}{q}}+\varepsilon)^2} dv_g \tag{3.2}$$

which concludes $\mu_{k,q}^\varepsilon < +\infty$. Our aim was to find critical points of functional F_q^ε.

3.2. $\mu_{k,q}^\varepsilon$ *is achieved*

The purpose of this subsection was to show that, if k, q, and ε are fixed, then $\mu_{k,q}^\varepsilon$ is achieved by a smooth positive function, say u_ε. The proof is standard and is based on the so-called direct methods in the calculus of variations. Let $\{u_j\}_j \subset \mathscr{B}_{k,q}$ be a minimizing sequence for $\mu_{k,q}^\varepsilon$. Since $F_q^\varepsilon(u_j) = F_q^\varepsilon(|u_j|)$, we may assume from the beginning that $u_j \geqslant 0$ for all j. By the Hölder inequality, we easily get $\|u_j\|_{L^2} \leqslant k^{\frac{1}{q}}$. Now for j sufficiently large such that $F_q^\varepsilon(u_j) < \mu_{k,q}^\varepsilon + 1$, one can obtain

$$\frac{1}{2}\int_M |\nabla u_j|^2 dv_g < \mu_{k,q}^\varepsilon - \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}\sup f + 1.$$

These estimates tell us that $\{u_j\}_j$ is bounded in $H^1(M)$. Being bounded, we can assume that, up to subsequences, there exists $u_\varepsilon \in H^1(M)$ such that

$$u_j \rightharpoonup u_\varepsilon \quad \text{in } H^1(M), \qquad u_j \to u_\varepsilon \quad \text{strongly in } L^q(M), \qquad u_j \to u_\varepsilon \quad \text{a.e. in } M.$$

This shows that $u_\varepsilon \geqslant 0$ almost everywhere, and $\|u_\varepsilon\|_{L^q} = k^{\frac{1}{q}}$. In particular, $u_\varepsilon \in \mathscr{B}_{k,q}$. Now we notice that the function $a\varepsilon^{-\frac{q}{2}}$ is of class $L^q(M)$; making use of the Lebesgue Dominated Convergence Theorem, we obtain $\int_M ((u_j)^2+\varepsilon)^{-\frac{q}{2}} a dv_g \to \int_M ((u_\varepsilon)^2+\varepsilon)^{-\frac{q}{2}} a dv_g$ as $j \to +\infty$. Since the part

$$\frac{1}{2}\int_M |\nabla u_j|^2 dv_g + \frac{h}{2}\int_M u_j^2 dv_g - \frac{1}{q}\int_M f|u_j|^q dv_g$$

is weakly lower semi-continuous, we get $\mu_{k,q}^\varepsilon = \lim_{j\to+\infty} F_q^\varepsilon(u_j) \geqslant F_q^\varepsilon(u_\varepsilon)$. This and the fact that $u_\varepsilon \in \mathscr{B}_{k,q}$ immediately give $\mu_{k,q}^\varepsilon = F_q^\varepsilon(u_\varepsilon)$.

It leaves out to prove the smoothness and positivity of u_ε. Using the Euler–Lagrange equation for functional F_q^ε with the constraint (3.1), we know that u_ε solves

$$\Delta_g u_\varepsilon + h u_\varepsilon = (f+\lambda)|u_\varepsilon|^{q-2}u_\varepsilon + \frac{a u_\varepsilon}{((u_\varepsilon)^2+\varepsilon)^{\frac{q}{2}+1}} \tag{3.3}$$

in the weak sense for some constant λ. The regularity result, Lemma 2.2(a), developed in Section 2 can be applied to (3.3). It follows that $u_\varepsilon \in C^\infty(M)$ and $u_\varepsilon \geqslant 0$ in M. The Strong Maximum Principle [3, Proposition 3.75] can be applied to conclude that either $u_\varepsilon \equiv 0$ or $u_\varepsilon > 0$ in M. Since $\int_M (u_\varepsilon)^q dv_g = k \neq 0$, we know that $u_\varepsilon \not\equiv 0$. Thus, u_ε is a smooth positive solution of (3.3) and the claim follows.

3.3. Asymptotic behavior of $\mu^\varepsilon_{k,q}$

In this subsection, we investigate the behavior of $\mu^\varepsilon_{k,q}$ when both k and ε vary. In contrast to Rauzy [19], we prove the following.

Lemma 3.1. *There holds* $\lim_{k\to 0+} \mu^{k^{\frac{2}{q}}}_{k,q} = +\infty$. *In particular, there is some $k_\star$ sufficiently small and independent of both q and ε such that $\mu^\varepsilon_{k_\star,q} > 0$ for any $\varepsilon \leqslant k_\star$.*

Proof. The way that ε comes and plays immediately shows us that $\mu^\varepsilon_{k,q}$ is strictly monotone decreasing in ε for fixed k and q. For any $\varepsilon \leqslant k^{\frac{2}{q}}$ and $1 < \frac{q}{2} < \frac{2^\star}{2}$, and similar to (2.7), we can estimate the integral involving a. To be precise, for any $u \in \mathscr{B}_{k,q}$, we have

$$\begin{aligned}\int_M \sqrt{a}\,dv_g &\leqslant \left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}dv_g\right)^{\frac{1}{2}} \left(\int_M (u^2+\varepsilon)^{\frac{q}{2}}dv_g\right)^{\frac{1}{2}} \\ &\leqslant 2^{\frac{q}{4}}\sqrt{k}\left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}dv_g\right)^{\frac{1}{2}},\end{aligned}$$

where we have used the fact $(u^2+\varepsilon)^{\frac{q}{2}} \leqslant 2^{\frac{q}{2}-1}(|u|^q + k)$. Squaring both sides, we get

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}dv_g \geqslant \frac{1}{2^{\frac{2^\star}{2}}k}\left(\int_M \sqrt{a}\,dv_g\right)^2.$$

This helps us to conclude

$$F^\varepsilon_q(u) \geqslant k^{\frac{2}{q}}\frac{h}{2} - \frac{k}{q}\sup f + \frac{1}{2^{\frac{2^\star}{2}}qk}\left(\int_M \sqrt{a}\,dv_g\right)^2$$

which proves that $\mu^{k^{\frac{2}{q}}}_{k,q} \to +\infty$ as $k \to 0+$. It is a simple task to find some small $k_\star < 1$ independent of both q and ε such that

$$k_\star^{\frac{2}{q}}\frac{h}{2} - \frac{k_\star}{q}\sup f + \frac{1}{2^{\frac{2^\star}{2}}qk_\star}\left(\int_M \sqrt{a}\,dv_g\right)^2 > 0,$$

for example, one can choose $k_\star$ as

$$k_\star = \min\left\{\frac{1}{2^{\frac{2^\star}{2}-1}2^\star}\frac{\left(\int_M \sqrt{a}\,dv_g\right)^2}{(\sup f + |h|)}, \left(\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{n-1}, 1\right\}.$$

For such a choice of $k_\star$, we notice that $k_\star < k_\star^{\frac{2}{q}}$. The proof follows. $\square$

We now investigate the behavior of $\mu^{\varepsilon}_{k,q}$ as $k \to +\infty$. A direct use of constant functions as in (3.2) gives us nothing since f changes its sign. To avoid this difficulty we need to construct a new suitable test function, to this end we have to control f^- by using a suitable cut-off function which is supported in the positive part of f.

Lemma 3.2. *There holds* $\mu^{\varepsilon}_{k,q} \to -\infty$ *as* $k \to +\infty$ *if* $\sup f > 0$.

Proof. We first choose a point, say $x_0 \in M$, such that $f(x_0) > 0$. For example, one can choose x_0 such that $f(x_0) = \max_M f$. By the continuity of f, there exists some $r_0 > 0$ sufficiently small such that $f(x) > 0$, for any $x \in \overline{B}_{r_0}(x_0)$ and $f(x) \geqslant 0$ for any $x \in \overline{B}_{2r_0}(x_0)$. Let $\varphi : [0, +\infty) \to [0, 1]$ be a smooth non-negative function such that

$$\varphi(t) = \begin{cases} 1, & 0 \leqslant t \leqslant r_0^2, \\ 0, & t \geqslant 4r_0^2. \end{cases}$$

For small r_0, the function φ is clearly smooth. We then define

$$w(x) = \varphi(\operatorname{dist}(x, x_0)^2), \quad x \in M$$

and set

$$g(t) = \int_M f e^{tw} dv_g, \quad t \in \mathbb{R}.$$

Obviously, g is continuous and $g(0) < 0$ by the assumption $\int_M f dv_g < 0$. For arbitrary t, we have

$$\begin{aligned} g(t) &\geqslant \Big(\min_{\overline{B}_{r_0}(x_0)} f^+\Big) \int_{B_{r_0}(x_0)} e^{tw} dv_g + \int_M f^- e^{tw} dv_g \\ &\geqslant \Big(\min_{\overline{B}_{r_0}(x_0)} f^+\Big) \operatorname{vol}(B_{r_0}(x_0)) e^t - \int_{M \setminus B_{2r_0}(x_0)} |f^-| dv_g. \end{aligned}$$

Thus, there exists some t_0 sufficiently large such that $g(t_0) \geqslant 1$. The monotonicity property of g, that can be seen from

$$g'(t) = \int_M f w e^{tw} dv_g = \int_{B_{2r_0}(x_0)} f^+ w e^{tw} dv_g > 0,$$

allows us to conclude that $g(t) \geqslant 1$ for any $t \geqslant t_0$. We now take a positive function $v \in C^1(M)$ of the form $ce^{t_0 w(x)}$, $x \in M$, where c is a positive constant chosen in such a way that $\int_M v^q dv_g = 1$. By our construction above, the function $e^{t_0 w(x)}$ is independent of both q and ε. Therefore,

$$\int_M f v^q dv_g = c^q g(q t_0) > c^q g(t_0) > 0. \tag{3.4}$$

Since $k^{\frac{1}{q}} v \in \mathscr{B}_{k,q}$, a direct computation leads us to

$$F^{\varepsilon}_q(k^{\frac{1}{q}} v) \leqslant \frac{1}{2} k^{\frac{2}{q}} \left(\|\nabla v\|^2_{L^2} + h \|v\|^2_{L^2} - \frac{2}{q} k^{1-\frac{2}{q}} \int_M f v^q dv_g \right) + \frac{1}{q} \varepsilon^{-\frac{q}{2}} \int_M a dv_g.$$

With the help of (3.4) we deduce $F^{\varepsilon}_q(k^{\frac{1}{q}} v) \to -\infty$ by sending $k \to +\infty$ in the preceding inequality, thus proving our claim. □

We are going to show that there exists k_0 such that $\mu^\varepsilon_{k_0,q} < 0$ and $\mu^\varepsilon_{k,q} > 0$ for some $k > k_0$. These results together with Lemmas 3.1 and 3.2 give us a full description of the asymptotic behavior of $\mu^\varepsilon_{k,q}$. First we prove the existence of such a k_0.

Lemma 3.3. *There exists $k_0 > 0$ independent of ε such that $\mu^\varepsilon_{k_0,q} \leqslant 0$ for any $\varepsilon > 0$ provided*

$$\int_M a\,dv_g \leqslant \left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{\frac{q+2}{q-2}} \frac{|h|}{4}(q-2). \tag{3.5}$$

In particular, $k_0 > k_\star$.

Proof. By removing the negative term involving f^+, we know from (3.2) that

$$F^\varepsilon_q\big(k^{\frac{1}{q}}\big) \leqslant \frac{h}{2}k^{\frac{2}{q}} + \frac{k}{q}\int_M |f^-|\,dv_g + \frac{1}{qk}\int_M a\,dv_g.$$

Clearly, the non-positivity of the right hand side of this inequality is equivalent to

$$\int_M a\,dv_g \leqslant \frac{|h|q}{2}k^{\frac{q+2}{q}} - k^2\int_M |f^-|\,dv_g. \tag{3.6}$$

By a simple calculation, at

$$k_0 = \left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{\frac{q}{q-2}}$$

the right hand side of (3.6) is equal to

$$\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{\frac{q+2}{q-2}} \frac{|h|}{4}(q-2).$$

Thus, by definition, we claim that $\mu^\varepsilon_{k_0,q} \leqslant 0$ provided $\int_M a\,dv_g$ satisfies (3.5). The fact that $k_0 > k_\star$ can be seen from Lemma 3.1. □

Now we have the following remark which also plays some role in our argument.

Remark 3.4. It follows from $q \in (2^\flat, 2^\star)$ that

$$\min\left\{\left(\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{n-1}, 1\right\} \leqslant k_0$$

since $\frac{2+q}{4} > 1$ and the function $\frac{q}{q-2}$ is monotone decreasing. Moreover, if we keep the term involving f^+ in the proof of Lemma 3.3, we immediately see that

$$F^\varepsilon_q\left(k_0^{\frac{1}{q}}\right) \leqslant -\frac{k_0}{q}\int_M f^+\,dv_g.$$

Thus, we can easily control the growth of $\mu^\varepsilon_{k_0,q}$ as below

$$\mu^\varepsilon_{k_0,q} \leqslant -\frac{1}{2^\star}\min\left\{\left(\frac{|h|}{\int_M |f^-|\,dv_g}\right)^{n-1}, 1\right\}\int_M f^+\,dv_g \tag{3.7}$$

for any $\varepsilon \geqslant 0$. Keep in mind that the right hand side of (3.7) is strictly negative and is independent of both q and ε provided $\sup f > 0$ which is always the case in this section. Furthermore, from the choice of $k_\star$ as in the proof of Lemma 3.1 we have $k_\star < k_0$.

Since the right hand side of (3.5) depends on q, its behavior for q near $2^\star$ is needed in future argument. In fact, under the condition (3.8) below, we show that it is monotone increasing.

Lemma 3.5. *As a function of* q,

$$\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|dv_g}\right)^{\frac{q+2}{q-2}}\frac{|h|}{4}(q-2)$$

is monotone increasing in $(2^\flat, 2^\star)$ *provided*

$$\frac{2^\star|h|}{2} \leqslant \int_M |f^-|dv_g. \tag{3.8}$$

Proof. This is elementary. Let

$$\beta(q) = \frac{q+2}{q-2}\log\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|dv_g}\right) + \log\left(\frac{|h|}{4}(q-2)\right).$$

Our condition (3.8) implies that

$$\beta'(q) = -\frac{4}{(q-2)^2}\log\left(\frac{2+q}{4}\frac{|h|}{\int_M |f^-|dv_g}\right) + \frac{2}{q-2} > 0,$$

if $q > 2$. The conclusion follows. □

Remark 3.6. The preceding proof shows that $\beta'(q)$ is non-negative for any $q \in (2^\flat, 2^\star)$. Also, a simple calculation shows that the term on the right hand side of (1.5) equals $\lim_{q\to 2^\star} e^{\beta(q)}$ since $\frac{2^\star+2}{2^\star-2} = n-1$. This suggests that a good condition for $\int_M a dv_g$ could be (1.5).

Notice that, so far our estimate on $\mu^\varepsilon_{k,q}$ is still not enough for our purpose. We need finer estimates. We prove that, as a function of k where $k \geqslant k_0$, $\mu^\varepsilon_{k,q}$ is bounded from above by a constant independent of $q \in (2^\flat, 2^\star)$ and $\varepsilon > 0$.

Lemma 3.7. *Assume that* (1.5) *holds. Then there exists some constant* μ *independent of* q *and* ε *such that* $\mu^\varepsilon_{k,q} \leqslant \mu$ *for any* $\varepsilon > 0$, $q \in (2^\flat, 2^\star)$ *and* $k \geqslant k_0$. *In other words,* $\mu^\varepsilon_{k,q}$ *has an upper bound when* k *is large.*

Proof. Thanks to the proof of Lemma 3.2, we can conclude our lemma by taking a positive function v of the following form $v(x) = ce^{t_0 w(x)}$, $x \in M$ where c is a positive constant chosen so that $\int_M v^q dv_g = 1$. Since $h < 0$, we first have

$$F^\varepsilon_q(k^{\frac{1}{q}}v) \leqslant \frac{1}{2}k^{\frac{2}{q}}\int_M |\nabla v|^2 dv_g - \frac{k}{q}\int_M f v^q dv_g + \frac{1}{qk}\int_M a v^{-q} dv_g.$$

Observe that

$$\int_M f v^q dv_g = c^q g(qt_0) \geqslant \left(\int_M e^{2^\star t_0 w} dv_g\right)^{-1}.$$

For the term $\frac{1}{qk}\int_M av^{-q}dv_g$, using the fact that $v^q \geqslant c^q$, we get

$$\frac{1}{qk}\int_M av^{-q}dv_g \leqslant \frac{1}{2k}\left(\int_M e^{2^\star t_0 w}dv_g\right)\left(\int_M a dv_g\right).$$

We still have to analyze the last integral but thanks to $c \leqslant 1$ this is trivial. Putting all the estimates together, we conclude

$$F_q^\varepsilon(k^{\frac{1}{q}}v) \leqslant \frac{1}{2}(k+1)^{\frac{2}{2^\flat}}\left\|\nabla(e^{t_0 w})\right\|_{L^2}^2 - \frac{k}{2^\star}\left(\int_M e^{2^\star t_0 w}dv_g\right)^{-1} + \frac{1}{2k}\left(\int_M e^{2^\star t_0 w}dv_g\right)\left(\int_M a dv_g\right). \tag{3.9}$$

As a function of k and with $k \geqslant k_0$, it is clear that the right hand side of (3.9) achieves its maximum, say μ due to $\frac{2}{2^\flat} < 1$. This helps us to complete the proof. $\square$

In order to take the limit as $q \to 2^\star$, we still need to control L^q-norm of the mountain pass solutions. Since our mountain pass solutions have non-negative energy, what we really need is to show that there is an upper bound $k_{\star\star} > \max\{k_0, 1\}$ independent of ε and q such that $\mu_{k,q}^\varepsilon < 0$ for any $k \geqslant k_{\star\star}$. This is done by the following lemma.

Lemma 3.8. *There is some $k_{\star\star}$ sufficiently large and independent of both q and ε such that $\mu_{k,q}^\varepsilon < 0$ for any $k \geqslant k_{\star\star}$.*

Proof. From the proof of Lemma 3.7, it is easy to see that the right hand side of (3.9), being considered as a function of k, is continuous and independent of q and ε. Again, thanks to $\frac{2}{2^\flat} < 1$, we know that the function on the right hand side of (3.9) goes to $-\infty$ as $k \to +\infty$. Consequently, there is some $k_{\star\star} > \max\{k_0, 1\}$ sufficiently large and independent of both q and ε such that $\mu_{k,q}^\varepsilon < 0$ for any $k \geqslant k_{\star\star}$ and any $\varepsilon > 0$. $\square$

Before completing this subsection, we prove another interesting property of $\mu_{k,q}^\varepsilon$ saying that $\mu_{k,q}^\varepsilon$ is continuous with respect to k for each ε fixed. The idea of the proof given here followed the same lines as in [19].

Proposition 3.9. *For $\varepsilon > 0$ fixed, $\mu_{k,q}^\varepsilon$ is continuous with respect to k.*

Proof. Since $\mu_{k,q}^\varepsilon$ is well-defined at any point k, we have to verify that for each k fixed and for any sequence $k_j \to k$ there holds $\mu_{k_j,q}^\varepsilon \to \mu_{k,q}^\varepsilon$ as $j \to +\infty$. This is equivalent to showing that there exists a subsequence of $\{k_j\}_j$, still denoted by k_j, such that $\mu_{k_j,q}^\varepsilon \to \mu_{k,q}^\varepsilon$ as $j \to +\infty$. We suppose that $\mu_{k,q}^\varepsilon$ and $\mu_{k_j,q}^\varepsilon$ are achieved by $u \in \mathscr{B}_{k,q}$ and $u_j \in \mathscr{B}_{k_j,q}$ respectively. Keep in mind that u and u_j are positive smooth functions on M. Our aim was to prove the boundedness of $\{u_j\}_j$ in $H^1(M)$. It then suffices to control $\|\nabla u_j\|_{L^2}$. As in (3.4), we have

$$\int_M |\nabla u_j|^2 dvg < 2\left(\mu_{k_j,q}^\varepsilon - \frac{h}{2}k_j^{\frac{2}{q}} + \frac{k_j}{q}\sup f\right). \tag{3.10}$$

Thus, we have to control $\mu_{k_j,q}^\varepsilon$. By the homogeneity we can find a sequence of positive numbers $\{t_j\}_j$ such that $t_j u \in \mathscr{B}_{k_j,q}$. Since $k_j \to k$ as $j \to +\infty$ and $k_j^{\frac{2}{q}} = \|t_j u\|_{L^q} = t_j k^{\frac{2}{q}}$, we immediately see that $t_j \to 1$ as $j \to +\infty$. Now we can use $t_j u$ to control $\mu^{\varepsilon}_{k_j,q}$. Indeed, using the function $t_j u$ we know that

$$\mu^{\varepsilon}_{k_j,q} \leqslant t_j^2\left(\frac{1}{2}\int_M |\nabla u|^2 dvg + \frac{h}{2}\int_M u^2 dvg\right) - \frac{1}{q}t_j^q\int_M f u^q dvg + \frac{1}{q}\int_M \frac{a}{((t_j u)^2+\varepsilon)^{\frac{q}{2}}} dvg. \tag{3.11}$$

Notice that u is fixed and t_j belongs to a neighborhood of 1 for large j. Thus, $\{\mu^{\varepsilon}_{k_j,q}\}_j$ is bounded which also implies by (3.11) that $\{\|\nabla u_j\|_{L^2}\}_j$ is bounded. Hence $\{u_j\}_j$ is bounded in $H^1(M)$. Being bounded, there exists $\overline{u} \in H^1(M)$ such that, up to subsequences, $u_j \to \overline{u}$ strongly in $L^p(M)$ for any $p \in [1, 2^\star)$. Consequently, $\lim_{j\to+\infty} \|u_j\|_{L^q} = \|\overline{u}\|_{L^q} = k^{\frac{2}{q}}$, that is, $\overline{u} \in \mathscr{B}_{k,q}$. In particular, $F^{\varepsilon}_q(u) \leqslant F^{\varepsilon}_q(\overline{u})$. We now use weak lower semi-continuity property of F^{ε}_q to deduce that

$$F^{\varepsilon}_q(u) \leqslant F^{\varepsilon}_q(\overline{u}) \leqslant \liminf_{j\to+\infty} F^{\varepsilon}_q(u_j).$$

We now use our estimate for $\mu^{\varepsilon}_{k_j,q}$ above to see that $\limsup_{j\to+\infty} \mu^{\varepsilon}_{k_j,q} \leqslant F^{\varepsilon}_q(u)$. This is due to the Lebesgue Dominated Convergence Theorem and the fact that $t_j \to 1$ as $j \to +\infty$. Therefore, $\lim_{j\to+\infty} \mu^{\varepsilon}_{k_j,q} = \mu^{\varepsilon}_{k,q}$ which proves the continuity of $\mu^{\varepsilon}_{k,q}$. □

The following subsection is basically due to Rauzy [19]. Here we just relax some conditions in the Rauzy arguments for future benefit. It is worth to reproduce several parts in order to make the paper to be self-contained.

3.4. The study of $\lambda_{f,\eta,q}$

At the beginning of the section we temporarily leave our equation to study another minimizing problem. The proof of our main result depends on $\lambda_{f,\eta,q}$ which will be defined below. This quantity was first introduced by Rauzy [19]. To be precise, we introduce $\mathscr{A}(\eta, q)$, another subspace of $H^1(M)$, which is defined as the following

$$\mathscr{A}(\eta, q) = \left\{u \in H^1(M) : \|u\|_{L^q} = 1, \int_M |f^-|u|^q dv_g = \eta\int_M |f^-| dv_g\right\}. \tag{3.12}$$

We assume for a moment that $\mathscr{A}(\eta, q)$ is not empty which will be mentioned later after proving Lemma 3.10 below. We define the number

$$\lambda_{f,\eta,q} = \inf_{u\in\mathscr{A}(\eta,q)} \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^2}}. \tag{3.13}$$

We are going to prove the following result.

Lemma 3.10. *As a function of η, $\lambda_{f,\eta,q}$ is monotone decreasing.*

In the present case, it is hard to consider the equality sign, nevertheless we study the following problem first

$$\lambda'_{f,\eta,q} = \inf_{u\in\mathscr{A}'(\eta,q)} \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^2}},$$

where

$$\mathscr{A}'(\eta, q) = \left\{ u \in H^1(M) : \|u\|_{L^q} = 1, \int_M |f^-||u|^q dv_g \leqslant \eta \int_M |f^-| dv_g \right\}.$$

With q and η being fixed, the set $\mathscr{A}'(\eta, q)$ is not empty since it includes the set of functions $u \in H^1(M)$ such that $\|u\|_{L^q} = 1$ and with supports in the set

$$\{x \in M : f(x) > 0\} \subset \left\{ x \in M, |f^-|(x) < \eta \int_M |f^-| dv_g \right\}.$$

As can be seen form the definition, if $\eta_1 \leqslant \eta_2$ then $\mathscr{A}'(\eta_1, q) \subset \mathscr{A}'(\eta_2, q)$; thus proving $\lambda'_{f,\eta_2,q} \leqslant \lambda'_{f,\eta_1,q}$. This amounts to saying that $\lambda'_{f,\eta,q}$ is monotone decreasing. We are going to prove $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. For that reason, it suffices to prove $\lambda'_{f,\eta,q} \geqslant \lambda_{f,\eta,q}$ since the reverse is trivial. The fact that $\mathscr{A}'(\eta, q)$ is not empty implies that $\lambda'_{f,\eta,q}$ is finite. We are now in a position to prove Lemma 3.10.

Proof of Lemma 3.10. We first prove that $\lambda'_{f,\eta,q}$ is achieved. Let $\{v_j\}_j \subset \mathscr{A}'(\eta, q)$ be a minimizing sequence for $\lambda'_{f,\eta,q}$. Obviously the sequence $\{|v_j|\}_j$ is still a minimizing sequence in $\mathscr{A}'(\eta, q)$ and therefore we can assume from the beginning that $v_j \geqslant 0$ in M. We can prove with arguments already used many times that $\{v_j\}_j$ is bounded in $H^1(M)$. Then up to subsequences, there exists $v \in H^1(M)$ such that

$$v_j \rightharpoonup v \quad \text{in } H^1(M), \qquad v_j \to v \quad \text{strongly in } L^q(M), \qquad v_j \to v \quad \text{a.e. in } M.$$

With arguments that already used before, it is not hard to show that $v \in \mathscr{A}'(\eta, q)$. Then by weak lower semi-continuity of the norm, one can show that $\|\nabla v\|_{L^2}^2 \|v\|_{L^2}^{-2} \leqslant \lambda'_{f,\eta,q}$. Thus, $\lambda'_{f,\eta,q}$ is achieved by v. Using [3, Proposition 3.49], we may assume $v \geqslant 0$, otherwise, we just replace v by $|v|$. Now we assume by contradiction that $v \notin \mathscr{A}(\eta, q)$, then there exists a positive constant κ such that

$$\int_M |f^-| (v + \kappa)^q \, dv_g = \eta \int_M |f^-| dv_g.$$

Now we notice that from $(v + \kappa)\|v + \kappa\|_{L^q}^{-1} \in \mathscr{A}'(\eta, q)$ we have

$$\left\| \nabla \left(\frac{v + \kappa}{\|v + \kappa\|_{L^q}} \right) \right\|_{L^2}^2 \left\| \frac{v + \kappa}{\|v + \kappa\|_{L^q}} \right\|_{L^2}^{-2} = \frac{\|\nabla (v + \kappa)\|_{L^2}^2}{\|v + \kappa\|_{L^2}^2} < \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^2}^2},$$

which gives us a contradiction. Hence, $v \in \mathscr{A}(\eta, q)$ which also proves $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. Consequently, $\lambda_{f,\eta,q}$ is decreasing as a function of η. □

Remark 3.11. The fact that $\mathscr{A}(\eta, q)$ is not empty is a direct consequence of the proof of Lemma 3.10.

Our next lemma describes a comparison between $\lambda_{f,\eta,q}$ and λ_f. Intuitively, $\mathscr{A}$ is smaller than $\mathscr{A}'(\eta, q)$, thus making $\lambda_{f,\eta,q} \leqslant \lambda_f$. We now prove this affirmatively.

Lemma 3.12. *For each $q \in (2^\flat, 2^\star)$ and $\eta > 0$ fixed, if* $\sup f > 0$, *then* $\lambda_{f,\eta,q} \leqslant \lambda_f$.

Proof. We pick $u \in \mathscr{A}$ arbitrarily. From the definition of $\mathscr{A}$ and the fact that $\sup f > 0$ we must have

$$\int_M u^q dv_g > 0, \qquad \int_M |f^-| u^q dv_g = 0.$$

We now choose $\varepsilon > 0$ such that $\int_M (\varepsilon u)^q \, dv_g = 1$. This amounts to saying that $\varepsilon u \in \mathscr{A}'(\eta, q)$ which helps us to write

$$\lambda'_{f,\eta,q} \leqslant \|\nabla(\varepsilon u)\|^2_{L^2} \, \|\varepsilon u\|^{-2}_{L^2} = \|\nabla u\|^2_{L^2} \, \|u\|^{-2}_{L^2}.$$

Since the preceding inequality holds for any $u \in \mathscr{A}$, we may take the infimum on both sides with respect to u to arrive at $\lambda'_{f,\eta,q} \leqslant \lambda_f$. The proof follows easily since we have seen that $\lambda'_{f,\eta,q} = \lambda_{f,\eta,q}$. $\square$

It is worth noticing that for each q fixed one can show that $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$. However, since we are interested in the critical case, that is equivalent to sending q to $2^\star$, we do not need that result. Instead, we prove the following.

Lemma 3.13. *For each $\delta > 0$ fixed, there exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, there exists $q_\eta \in (2^\flat, 2^\star)$ so that $\lambda_{f,\eta,q} \geqslant \lambda_f - \delta$ for every $q \in (q_\eta, 2^\star)$.*

Proof. We assume by contradiction that there is some $\delta_0 > 0$ such that for every $\eta_0 > 0$, there exist $\eta < \eta_0$ and a monotone sequence $\{q_j\}_j$ converging to $2^\star$ so that $\lambda_{f,\eta,q_j} < \lambda_f - \delta_0$ for every j. We can furthermore assume that λ_{f,η,q_j} is achieved by some $v_{\eta,q_j} \in \mathscr{A}(\eta, q_j)$. We then immediately have

$$\left\|\nabla v_{\eta,q_j}\right\|^2_{L^2} \left\|v_{\eta,q_j}\right\|^{-2}_{L^2} < \lambda_f - \delta_0$$

for any j. With arguments already used many times we can prove that there exists $v_{\eta,2^\star} \in H^1(M)$ such that

$$v_{\eta,q_j} \rightharpoonup v_{\eta,2^\star} \quad \text{in } H^1(M), \qquad v_{\eta,q_j} \to v_{\eta,2^\star} \quad \text{strongly in } L^2(M).$$

From the preceding inequality, the following estimate

$$\left\|\nabla v_{\eta,2^\star}\right\|^2_{L^2} \left\|v_{\eta,2^\star}\right\|^{-2}_{L^2} \leqslant \lambda_f - \delta_0$$

holds by sending $j \to \infty$. Besides, the Hölder inequality implies $1 \leqslant \|v_{\eta,q_j}\|_{L^{2^\star}}$ for each j. Using this and the Sobolev inequality applied to v_{η,q_j}, we get

$$1 \leqslant \left(\mathcal{K}_1 \frac{\left\|\nabla v_{\eta,q_j}\right\|^2}{\left\|v_{\eta,q_j}\right\|^2} + \mathcal{A}_1\right) \left\|v_{\eta,q_j}\right\|^2 \leqslant \left(\mathcal{K}_1(\lambda_f - \delta_0) + \mathcal{A}_1\right) \left\|v_{\eta,q_j}\right\|^2$$

which yields $(\mathcal{K}_1\lambda_f + \mathcal{A}_1)^{-1} \leqslant \|v_{\eta,q_j}\|^2_{L^2}$ for each j. Passing to the limit as $j \to \infty$, we obtain $(\mathcal{K}_1\lambda_f + \mathcal{A}_1)^{-1} \leqslant \|v_{\eta,2^\star}\|^2_{L^2}$. For every $q_j \geqslant 2^\flat$, by the Hölder inequality and the fact that $v_{\eta,q_j} \in \mathscr{A}(\eta, q_j)$ one has $\int_M |v_{\eta,q_j}|^{2^\flat} dv_g \leqslant 1$ and

$$\begin{aligned}\int_M |f^-||v_{\eta,q_j}|^{2^\flat} dv_g &\leqslant \left(\int_M |f^-||v_{\eta,q_j}|^{q_{\eta_j}} dv_g\right)^{\frac{2^\flat}{q_j}} \left(\int_M |f^-| dv_g\right)^{1-\frac{2^\flat}{q_j}} \\ &= \eta^{\frac{2^\flat}{q_j}} \int_M |f^-| dv_g.\end{aligned}$$

By the Fatou lemma and the fact that $\operatorname{vol}(M) = 1$, we deduce that

$$\int_M |v_{\eta,2^\star}|^{2^\flat} dv_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,2^\star}|^{2^\flat}\,dv_g \leqslant \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-|\,dv_g.$$

Now we let $\eta_0 \to 0$, then clearly $\eta \to 0$. The boundedness of $v_{\eta,2^\star}$ in $H^1(M)$ implies that there exists $v \in H^1(M)$ such that up to subsequences

$$v_{\eta,2^\star} \rightharpoonup v \quad \text{in } H^1(M), \qquad v_{\eta,2^\star} \to v \quad \text{strongly in } L^2(M), \qquad v_{\eta,2^\star} \to v \quad \text{a.e. in } M.$$

Before giving out contradiction, we notice that

$$\|\nabla v\|_{L^2}^2 \leqslant (\lambda_f - \delta_0)\,\|v\|_{L^2}^2\,. \tag{3.14}$$

Then it is enough to see

$$\begin{aligned} 0 \leqslant \int_M |f^-||v|^{2^\flat}\,dv_g &\leqslant \lim_{\eta\to 0}\int_M |f^-||v_{\eta,2^\star}|^{2^\flat}\,dv_g \\ &\leqslant \lim_{\eta\to 0}\left(\eta^{\frac{2^\flat}{2^\star}}\int_M |f^-|\,dv_g\right) = 0. \end{aligned}$$

In other words, we would have $\int_M |f^-||v|^{2^\flat}dv_g = 0$. In particular, $\int_M |f^-||v|dv_g = 0$. The strong convergence $v_{\eta,2^\star} \to v$ in $L^2(M)$ also implies $(\mathcal{K}_1\lambda_f + \mathcal{A}_1)^{-1} \leqslant |v|_{L^2}^2$. Therefore, $v \not\equiv 0$, and thus $|v| \in \mathscr{A}$. By the definition of λ_f, we know that

$$\lambda_f\,\|v\|_{L^2}^2 \leqslant \|\nabla|v|\|_{L^2}^2 = \|\nabla v\|_{L^2}^2 \tag{3.15}$$

The inequalities (3.14) and (3.15) obviously provide us a desired contradiction. This proves the lemma. □

With the information of $\lambda_{f,\eta,q}$ studied in previous lemmas, let us go back to our energy functional. Here we prove that, for any $\varepsilon > 0$ and for some $k > k_0$, $\mu_{k,q}^\varepsilon > 0$. A similar result was studied in [19, Proposition 2].

Proposition 3.14. *Suppose that $|h| < \lambda_f$ and $\sup f > 0$. Then there exists $\eta_0 > 0$ sufficiently small and its corresponding q_{η_0} sufficiently close to $2^\star$ such that*

$$\delta = \frac{\lambda_{f,\eta_0,q} + h}{2} > \frac{3}{8}(\lambda_f + h) \tag{3.16}$$

for any $q \in (q_{\eta_0}, 2^\star)$. For such a δ, we denote

$$C_q = \frac{\eta_0}{4|h|}\underbrace{\min\left\{\frac{\delta}{(\mathcal{A}_1 + 2\mathcal{K}_1(|h| + 2\delta))}, \frac{|h|}{2}\right\}}_{m}. \tag{3.17}$$

If

$$\frac{\sup f}{\int_M |f^-|\,dv_g} < C_q \tag{3.18}$$

then there exists an interval $I_q = [k_{1,q}, k_{2,q}]$ so that for any $k \in I_q$, any $\varepsilon > 0$, and any $u \in \mathscr{B}_{k,q}$, there holds $F_q^\varepsilon(u) > \frac{1}{2}mk^{\frac{2}{q}}$. In particular, $\mu_{k,q}^\varepsilon > 0$ for any $k \in I_q$ and any $\varepsilon > 0$.

Proof. It follows from Lemma 3.13 that there exist some $0 < \eta_0 < 2$ and its corresponding $q_{\eta_0} \in (2^\flat, 2^\star)$ such that

$$0 \leqslant \lambda_f - \lambda_{f,\eta_0,q} < \frac{1}{4}(\lambda_f - |h|)$$

for any $q \in (q_{\eta_0}, 2^\star)$. This immediately confirms (3.16). We now let

$$k_{1,q} = \left(\frac{|h|q}{\eta_0 \int_M |f^-| dv_g}\right)^{\frac{q}{q-2}}. \tag{3.19}$$

We may see without any difficulty $k_0 < k_{1,q}$. We assume from now on that $k \geqslant k_{1,q}$. We write

$$F_q^\varepsilon(u) = G_q(u) - \frac{1}{q}\int_M f^+ |u|^q dv_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}} dv_g},$$

where

$$G_q(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{h}{2}\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q dv_g.$$

Then there are two possible cases.

Case 1. Assume that

$$\int_M |f^-||u|^q dv_g \geqslant \eta_0 k \int_M |f^-| dv_g.$$

In this case, the term G_q can be estimated from below as follows

$$\begin{aligned} G_q(u) &\geqslant \frac{h}{2}\|u\|_{L^2}^2 + \frac{\eta_0 k}{q}\int_M |f^-| dv_g \\ &\geqslant \frac{|h|}{2} k^{\frac{2}{q}} \Bigg(\underbrace{\frac{2\eta_0 \int_M |f^-| dv_g}{|h|q} k^{1-\frac{2}{q}}}_{\geqslant 2} - 1\Bigg) \\ &\geqslant \frac{|h|}{2} k^{\frac{2}{q}}, \end{aligned} \tag{3.20}$$

where in the last inequality we have used the fact that $k \geqslant k_{1,q}$ and (3.19).

Case 2. Assume that

$$\int_M |f^-||u|^q dv_g < \eta_0 k \int_M |f^-| dv_g.$$

Under this condition, it is clear that $k^{-\frac{1}{q}} u \in \mathscr{A}'(\eta_0, q)$ which implies $\|\nabla u\|_{L^2}^2 \|u\|_{L^2}^{-2} \geqslant \lambda_{f,\eta_0,q}$ by the definition of $\lambda_{f,\eta_0,q}$. Therefore, we can estimate $G_q(u)$ as follows

$$G_q(u) \geqslant \frac{1}{2}\left(\lambda_{f,\eta_0,q} + h\right)\|u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q dv_g.$$

Clearly,

$$\|u\|_{L^2}^2 = \frac{2}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q dv_g - G_q(u)\right).$$

Now if we write $\delta\|u\|_{L^2}^2$ as $\delta\|u\|_{L^2}^2 = (\alpha+\beta)\|u\|_{L^2}^2$ where $\alpha = \frac{\beta\mathcal{A}_1}{2|h|\mathcal{K}_1}$ and $\alpha+\beta=\delta$, we then get

$$\begin{aligned} G_q(u) &\geqslant \alpha\|u\|_{L^2}^2 + \frac{2\beta}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{q}\int_M |f^-||u|^q dv_g - G_q(u)\right) \\ &\quad + \frac{1}{q}\int_M |f^-||u|^q dv_g \\ &\geqslant \alpha\|u\|_{L^2}^2 + \frac{2\beta}{|h|}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 - G_q(u)\right) \end{aligned}$$

which gives

$$\left(1+\frac{2\beta}{|h|}\right)G_q(u) \geqslant \frac{\beta}{|h|}\left(\|\nabla u\|_{L^2}^2 + \frac{\alpha|h|}{\beta}\|u\|_{L^2}^2\right).$$

Using $\mathcal{K}_1\|\nabla u\|_{L^2}^2 + \mathcal{A}_1\|u\|_{L^2}^2 \geqslant k^{\frac{2}{q}}$ and the fact that $\frac{\alpha|h|}{\beta} = \frac{\mathcal{A}_1}{2\mathcal{K}_1}$, one easily obtains

$$\|\nabla u\|_{L^2}^2 + \frac{\alpha|h|}{\beta}\|u\|_{L^2}^2 \geqslant \frac{k^{\frac{2}{q}}}{2\mathcal{K}_1}.$$

Since $\beta = \frac{2\mathcal{K}_1|h|\delta}{\mathcal{A}_1+2\mathcal{K}_1|h|}$, we therefore have

$$G_q(u) \geqslant \frac{\beta}{2|h|}\frac{k^{\frac{2}{q}}}{\mathcal{K}_1}\left(1+\frac{2\beta}{|h|}\right)^{-1} = \frac{\delta}{\mathcal{A}_1+2\mathcal{K}_1(|h|+2\delta)}k^{\frac{2}{q}}. \tag{3.21}$$

It now follows from (3.17), (3.20) and (3.21) that $G_q(u) \geqslant mk^{\frac{2}{q}}$. Thus, we obtain

$$F_q^\varepsilon(u) \geqslant mk^{\frac{2}{q}} - \frac{k}{q}\sup f.$$

If we let $k < \left(\frac{mq}{2\sup f}\right)^{\frac{q}{q-2}}$ we then get $F_q^\varepsilon(u) > \frac{1}{2}mk^{\frac{2}{q}} > 0$. Since

$$\sup f \leqslant C_q\int_M |f^-|dv_g = \frac{m\eta_0}{4|h|}\int_M |f^-|dv_g,$$

one has, by (3.19), the following

$$\left(\frac{mq}{2\sup f}\right)^{\frac{q}{q-2}} \geqslant \left(\frac{2q|h|}{\eta_0\int_M |f^-|dv_g}\right)^{\frac{q}{q-2}} = 2^{\frac{q}{q-2}}k_{1,q}.$$

Hence, if we set $k_{2,q} = 2^{\frac{n}{2}}k_{1,q}$, then for arbitrary $k \in [k_{1,q}, k_{2,q}]$ we always have $F_q^\varepsilon(u) > \frac{1}{2}mk^{\frac{2}{q}}$. In other words, $\mu_{k,q}^\varepsilon > 0$ for arbitrary $k \in [k_{1,q}, k_{2,q}]$ which completes the proof. □

Remark 3.15. It is natural to ask whether $\mu_{k,q}^\varepsilon > 0$ still holds when k is large under the case when $\sup f \leqslant 0$. We shall consider this situation in the last section.

3.5. The Palais–Smale condition

This subsection is devoted to the proof of the Palais–Smale compactness condition. To our knowledge, there is no such a result in the literature since our energy functional contains both critical and negative exponents that cause a lot of difficulty. In addition, the negative constant h also raises several difficulties.

Proposition 3.16. *Suppose that the conditions (3.16)–(3.18) hold. Then for each $\varepsilon > 0$ fixed, the functional $F_q^\varepsilon(\cdot)$ satisfies the Palais–Smale condition.*

Proof. Let $\varepsilon > 0$ be fixed. Suppose that $\{v_j\}_j \subset H^1(M)$ is a Palais–Smale sequence for F_q^ε, that is, there exists a constant $\mathcal{C}$ such that

$$F_q^\varepsilon(v_j) \to \mathcal{C}, \qquad \|\delta F_q^\varepsilon(v_j)\|_{H^{-1}} \to 0 \quad \text{in } H^{-1}(M) \text{ as } j \to \infty.$$

As the first step, we prove that, up to subsequences, $\{v_j\}_j$ is bounded in $H^1(M)$. Without loss of generality, we may assume that $\|v_j\|_{H^1} \geqslant 1$ for all j. By means of the Palais–Smale sequence, one can derive

$$\frac{1}{2}\|\nabla v_j\|_{L^2}^2 + \frac{h}{2}\|v_j\|_{L^2}^2 - \frac{1}{q}\int_M f|v_j|^q\,dv_g + \frac{1}{q}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}}\,dv_g = \mathcal{C} + o(1) \tag{3.22}$$

and

$$\int_M \nabla v_j\cdot\nabla\xi\,dv_g + h\int_M v_j\xi\,dv_g - \int_M f|v_j|^{q-2}v_j\xi\,dv_g - \int_M \frac{av_j\xi}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}}\,dv_g = o(1)\|\xi\|_{H^1} \tag{3.23}$$

for any $\xi \in H^1(M)$. By letting $\xi = v_j$ in (3.23), we obtain

$$\|\nabla v_j\|_{L^2}^2 + h\|v_j\|_{L^2}^2 - \int_M f|v_j|^q\,dv_g - \int_M \frac{av_j^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}}\,dv_g = o(1)\|v_j\|_{H^1}. \tag{3.24}$$

For the sake of simplicity, let us denote

$$k_j = \int_M |v_j|^q\,dv_g.$$

There are two possible cases.

Case 1. Assume that there exists a subsequence of $\{v_j\}_j$, still denoted by $\{v_j\}_j$, such that

$$\int_M |f^-||v_j|^q\,dv_g \geqslant \eta_0 k_j \int_M |f^-|\,dv_g.$$

Using (3.17) and (3.18), we get that

$$\begin{aligned}
F_q^\varepsilon(v_j) &\geqslant \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-|\,dv_g - \frac{1}{q}\int_M f^+|v_j|^q\,dv_g \\
&\geqslant \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-|\,dv_g - \frac{k_j}{q}\sup f \\
&\geqslant \frac{h}{2}k_j^{\frac{2}{q}} + \frac{\eta_0 k_j}{q}\int_M |f^-|\,dv_g - \frac{k_j}{q}\frac{\eta_0}{8}\int_M |f^-|\,dv_g \\
&= \left(\frac{7\eta_0}{8}\int_M |f^-|\,dv_g\right)\frac{k_j}{q} - \frac{|h|}{2}k_j^{\frac{2}{q}}.
\end{aligned}$$

This and the fact that $F_q^\varepsilon(v_j) \to \mathcal{C}$ imply that $\{k_j\}_j$ is bounded. In other words, $\{v_j\}_j$ is bounded in $L^q(M)$. Hence, the Hölder inequality and (3.22) imply that $\{v_j\}_j$ is also bounded in $H^1(M)$.

Case 2. In contrast to Case 1, for all j sufficiently large, we assume that

$$\int_M |f^-||v_j|^q dv_g < \eta_0 k_j \int_M |f^-| dv_g.$$

Using (3.22) and (3.24), we obtain

$$\begin{aligned} -\frac{1}{q}\int_M f|v_j|^q dv_g = &-\frac{2}{q-2}\mathcal{C} + o(1)\|v_j\|_{H^1} + o(1) \\ &+ \frac{1}{q-2}\int_M \frac{av_j^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} dv_g \\ &+ \frac{2}{q(q-2)}\int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} dv_g. \end{aligned}$$

Therefore, we may rewrite F_q^ε as follows

$$F_q^\varepsilon(v_j) \geqslant \frac{1}{2}\|\nabla v_j\|_{L^2}^2 + \frac{h}{2}\|v_j\|_{L^2}^2 - \frac{2}{q-2}\mathcal{C} + o(1)\|v_j\|_{H^1} + o(1) + A_j, \tag{3.25}$$

where

$$A_j = \frac{1}{q-2}\left(\int_M \frac{a(v_j)^2}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}} dv_g + \int_M \frac{a}{((v_j)^2+\varepsilon)^{\frac{q}{2}}} dv_g\right).$$

Dividing (3.25) by $\|v_j\|_{L^2}$ and using the equivalent norm to $\|v_j\|_{H^1} = \|\nabla v_j\|_{L^2} + \|v_j\|_{L^2}$, one obtains

$$\begin{aligned} \frac{F_q^\varepsilon(v_j)}{\|v_j\|_{L^2}} \geqslant &\frac{\|\nabla v_j\|_{L^2}}{\|v_j\|_{L^2}}\left(\frac{1}{2}\|\nabla v_j\|_{L^2} + o(1)\right) + \frac{h}{2}\|v_j\|_{L^2} \\ &- \frac{2}{(q-2)\|v_j\|_{L^2}}\mathcal{C} + o(1) + \frac{o(1)}{\|v_j\|_{L^2}} + \frac{A_j}{\|v_j\|_{L^2}}. \end{aligned} \tag{3.26}$$

Observe that, from the definition of $\lambda_{f,\eta_0,q}$, there holds $\|\nabla v_j\|_{L^2}^2 \geqslant \lambda_{f,\eta_0,q}\|v_j\|_{L^2}^2$. Therefore, from (3.26) and for j large enough, there holds

$$\begin{aligned} \frac{F_q^\varepsilon(v_j)}{\|v_j\|_{L^2}} \geqslant &\frac{\lambda_{f,\eta_0,q}+h}{2}\|v_j\|_{L^2} + o(1)\sqrt{\lambda_{f,\eta_0,q}} \\ &- \frac{2}{(q-2)\|v_j\|_{L^2}}\mathcal{C} + o(1) + \frac{o(1)}{\|v_j\|_{L^2}} + \frac{A_j}{\|v_j\|_{L^2}}. \end{aligned}$$

If $\|v_j\|_{L^2} \to +\infty$ as $j \to \infty$, then we clearly would reach a contradiction by taking the limit in the previous equation as $j \to \infty$ since $\lambda_{f,\eta_0,q} + h > 0$ and $A_j > 0$ as we notice that $F_q^\varepsilon(v_j)\|v_j\|_{L^2}^{-1} \to 0$ as $j \to \infty$. Thus, $\{v_j\}_j$ is bounded in $L^2(M)$. This and (3.25) also imply that $\{\nabla v_j\}_j$ is bounded in $L^2(M)$. Consequently, $\{v_j\}_j$ is bounded in $H^1(M)$. Combining Cases 1 and 2 above, we conclude that there exists a bounded subsequence of $\{v_j\}_j$ in $H^1(M)$, still denoted by $\{v_j\}_j$. This completes the first step.

Being bounded, there exists $v \in H^1(M)$ such that up to subsequences

$$v_j \rightharpoonup v \quad \text{in } H^1(M), \qquad v_j \to v \quad \text{strongly in } L^2(M), \qquad v_j \to v \quad \text{a.e. in } M.$$

We now prove that $v_j \to v$ strongly in $H^1(M)$. Using (3.23) with ξ replaced by $v_j - v$, we get

$$\begin{aligned} &\int_M \nabla v_j \cdot \nabla(v_j - v)dv_g + h\int_M v_j(v_j - v)dv_g \\ &\quad - \int_M f|v_j|^{q-2}v_j(v_j - v)dv_g - \int_M \frac{av_j}{((v_j)^2+\varepsilon)^{\frac{q}{2}+1}}(v_j - v)dv_g \to 0 \end{aligned} \tag{3.27}$$

as $j \to \infty$. It is not hard to show that the limit of the second term and the fourth term vanishes as $j \to \infty$. For the third term, the limit also vanishes as one can use the Hölder inequality and the fact that $v_j \to v$ strongly in $L^{\frac{2^\star}{2^\star-(q-1)}}(M)$. Therefore, we would obtain

$$\int_M \nabla v_j \cdot \nabla(v_j - v)dv_g \to 0 \quad \text{as } j \to \infty.$$

The preceding limit, the following identity

$$\int_M |\nabla v_j - \nabla v|^2 dv_g = \int_M \nabla v_j \cdot (\nabla v_j - \nabla v)dv_g - \int_M \nabla v \cdot (\nabla v_j - \nabla v)dv_g$$

and the fact that $v_j \to v$ strongly in $L^2(M)$ and $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^2(M)$ prove that $v_j \to v$ strongly in $H^1(M)$. This completes the proof of the Palais–Smale condition. □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. This can be done through three steps. First, because of Lemma 3.5, we need to make use of the condition (3.8) in order to guarantee the existence of the first solution. This is the content of Proposition 4.1. Next we show that if, in addition, $\sup f$ can be controlled by some positive number, then (1.2) has at least two positive solutions. In the last step, we remove the condition (3.8) by using a scaling argument.

4.1. The existence of the first solution

In this subsection, we obtain the existence of the first solution of (1.2). Notice that, we require (3.8) to hold. This restriction will be removed by using a scaling argument later.

Proposition 4.1. *Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Let $h < 0$ be a constant, f and $a \geqslant 0$ be smooth functions on M with $\int_M a\,dv_g > 0$, $\int_M f\,dv_g < 0$, and $\sup f > 0$. We further assume that (3.8) holds and*

$$\int_M a\,dv_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-|dv_g}\right)^n \int_M |f^-|dv_g.$$

Then there is a positive number $\mathcal{C}_1$ given by (4.1) below such that if

$$\frac{\sup f}{\int_M |f^-|dv_g} < \mathcal{C}_1,$$

then (1.2) admits at least one smooth positive solution.

Proof. Since our proof is quite long, we divide it into several claims for the sake of clarity.

Claim 1. *There exists a $q_0 \in (2^\flat, 2^\star)$ such that for all $q \in (q_0, 2^\star)$ and some sufficiently small ε, there will be k_0 and $k_\star$ with the following properties: $k_0 < k_\star$ and $\mu^\varepsilon_{k_0,q} \leqslant 0$ while $\mu^\varepsilon_{k_\star,q} > 0$.*

Proof of Claim 1. We observe that, from Lemma 3.5, the condition (3.8), and Remark 3.6, there is some $q_0 \in (2^\flat, 2^\star)$ such that the condition (3.5) holds for all $q \in (q_0, 2^\star)$. Hence, by Lemma 3.3, there exists a $k_0 > 0$ small enough such that $\mu^\varepsilon_{k_0,q} \leqslant 0$. Notice that $2^\flat > 2$ for any $n \geqslant 3$. The existence of such a k_0 makes it possible for us to select some $k_\star$ such that $k_\star < \min\{k_0, 1\}$ and $\mu^\varepsilon_{k_\star,q} > 0$ for any $\varepsilon \leqslant k_\star$. This settles Claim 1. □

Claim 2. *Eq. (1.7) with ε replaced by 0 has two positive solutions, say $u_{1,q}$ and $u_{2,q}$.*

Proof of Claim 2. By using Proposition 3.14, we have η_0 and its corresponding $q_{\eta_0} \in (2^\flat, 2^\star)$ such that $\delta = \frac{1}{2}(\lambda_{f,\eta_0,q} + h) > \frac{3}{8}(\lambda_f + h)$ for any $q \in (q_{\eta_0}, 2^\star)$. Thanks to Lemma 3.12, one has $\frac{3}{8}(\lambda_f + h) \leqslant \delta \leqslant \frac{1}{2}(\lambda_f + h)$. This amounts to helping us to have a lower bound for C_q given by (3.17). Indeed, a simple calculation shows that $C_q \geqslant \mathcal{C}_1$ where

$$\mathcal{C}_1 = \frac{\eta_0}{4|h|} \min\left\{\frac{3}{8}\frac{\lambda_f + h}{\mathcal{A}_1 + 2\mathcal{K}_1\lambda_f}, \frac{|h|}{2}\right\}. \tag{4.1}$$

Note that $\mathcal{C}_1$ is independent of q and thus never vanishing for any $q \in (q_{\eta_0}, 2^\star)$. Observe that

$$\lim_{q\to 2^\star} k_{1,q} = \left(\frac{2^\star|h|}{\eta_0 \int_M |f^-| dv_g}\right)^{\frac{n}{2}} = \ell, \qquad \lim_{q\to 2^\star} k_{2,q} = 2^{\frac{n}{2}}\ell.$$

By Proposition 3.14, there exists an interval $I_q = [k_{1,q}, k_{2,q}]$ such that $\mu^\varepsilon_{k,q} > 0$ for any $k \in I_q$. Recall that $k_\star < k_0 < k_{1,q}$, where $k_\star$ is given as in Claim 1. □

The existence of $u^\varepsilon_{1,q}$ with energy $\mu^\varepsilon_{k_1^\varepsilon,q}$. We first define the number

$$\mu^\varepsilon_{k_1^\varepsilon,q} = \inf_{u\in\mathscr{D}_{k,q}} F^\varepsilon_q(u),$$

where

$$\mathscr{D}_{k,q} = \left\{u \in H^1(M) : k_\star \leqslant \|u\|^q_{L^q} \leqslant k_{1,q}\right\}.$$

Due to the monotonicity of $k_{1,q}$, we know that $\|u\|^q_{L^q} < \ell$ for any $u \in \mathscr{D}_{k,q}$. It follows from Section 3.1 and Lemma 3.3 that $\mu^\varepsilon_{k_1^\varepsilon,q}$ is finite and non-positive. Similar arguments to those used in Section 3.2 show that $\mu^\varepsilon_{k_1^\varepsilon,q}$ is achieved by some positive smooth function $u^\varepsilon_{1,q}$. In particular, $\mu^\varepsilon_{k_1^\varepsilon,q}$ is the energy of $u^\varepsilon_{1,q}$. Obviously, $u^\varepsilon_{1,q}$ is a solution of (1.7). It is not hard to verify that any minimizing sequence for $\mu^\varepsilon_{k_1^\varepsilon,q}$ is bounded in $H^1(M)$. Now the lower semi-continuity of H^1-norm implies that $\|u^\varepsilon_{1,q}\|_{H^1}$ is bounded with the bound independent of q and ε. If we denote $\|u^\varepsilon_{1,q}\|^q_{L^q} = k_1^\varepsilon$ we immediately have $k_1^\varepsilon \in (k_\star, k_{\star\star})$.

The existence of $u_{1,q}$ with strictly negative energy $\mu_{k_1,q}$. In what follows, we let $\{\varepsilon_j\}_j$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each j, let $u^{\varepsilon_j}_{1,q}$ be a smooth positive function in M such that

$$\Delta_g u^{\varepsilon_j}_{1,q} + h u^{\varepsilon_j}_{1,q} = f(u^{\varepsilon_j}_{1,q})^{q-1} + \frac{a u^{\varepsilon_j}_{1,q}}{((u^{\varepsilon_j}_{1,q})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{4.2}$$

in M. Being bounded, there exists $u_{1,q} \in H^1(M)$ such that up to subsequences

$$u_{1,q}^{\varepsilon_j} \rightharpoonup u_{1,q} \text{ in } H^1(M), \quad u_{1,q}^{\varepsilon_j} \to u_{1,q} \text{ strongly in } L^2(M), \quad u_{1,q}^{\varepsilon_j} \to u_{1,q} \text{ a.e. in } M.$$

Using Lemma 2.1, the Lebesgue Dominated Convergence Theorem can be applied to conclude that $\int_M (u_{1,q})^{-p} dv_g$ is finite for all p. Now sending $j \to \infty$ in (4.2), we get that $u_{1,q}$ is a weak solution of the following subcritical equation

$$\Delta_g u_{1,q} + h u_{1,q} = f(u_{1,q})^{q-1} + \frac{a}{(u_{1,q})^{q+1}}. \tag{4.3}$$

Thus, the regularity result, Lemma 2.2(b), developed in Section 2 can be applied to (4.3). It follows that $u_{1,q} \in C^\infty(M)$. Since $u_{1,q}^{\varepsilon_j} \to u_{1,q}$ strongly in $L^q(M)$ as $j \to \infty$, if we denote $\|u_{1,q}\|_{L^q}^q = k_1$, we still have $k_1 \in (k_\star, k_{\star\star})$. Consequently, there holds $u_{1,q} \not\equiv 0$. With Lemma 2.1 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1,q}$ is strictly positive. From Remark 3.4 and the fact that $u_{1,q}^{\varepsilon_j}$ has strictly negative energy $\mu^{\varepsilon_j}_{k_1^{\varepsilon_j},q}$, by passing to the limit as $j \to \infty$, we know that $u_{1,q}$ also has strictly negative energy $\mu_{k_1,q}$. Thus, we have shown that $u_{1,q}$ is a smooth positive solution of (4.3) as claimed. Keep in mind that we still have $\|u_{1,q}\|_{L^q}^q \leqslant k_{\star\star}$ since we have a strong convergence.

The existence of $u_{2,q}^\varepsilon$ with energy $\mu^\varepsilon_{k_2^\varepsilon,q}$. Let $k^\star$ be a real number such that

$$\mu^\varepsilon_{k^\star,q} = \max\left\{\mu^\varepsilon_{k,q} : k_{1,q} \leqslant k \leqslant k_{2,q}\right\}.$$

Obviously, $\mu^\varepsilon_{k^\star,q} > 0$. Now we choose $\overline{k}_1 \in (k_0, k_{1,q})$ and $\overline{k}_2 \in (k_{2,q}, k_{\star\star})$ in such a way that $\mu^\varepsilon_{\overline{k}_1,q} = \mu^\varepsilon_{\overline{k}_2,q} = 0$. The existence of $\overline{k}_i$ is guaranteed by Proposition 3.9. Notice that $\mu^\varepsilon_{\overline{k}_1,q}$ and $\mu^\varepsilon_{\overline{k}_2,q}$ have been proved to be achieved, say by $u_{\overline{k}_1,q}$ and $u_{\overline{k}_2,q}$ respectively. We now set

$$\Gamma = \left\{\gamma \in C([0,1]; H^1(M)) : \gamma(0) = u_{\overline{k}_1,q}, \gamma(1) = u_{\overline{k}_2,q}\right\}.$$

Consider the functional $E(v) = F_q^\varepsilon(u_{\overline{k}_1,q} + v)$ for any non-negative real valued function v with

$$\|v\| = \left(\int_M |u_{\overline{k}_1,q} + v|^q dv_g\right)^{\frac{1}{q}}.$$

First we have $E(0) = 0$. Let $\rho = (k^\star)^{\frac{1}{q}}$. If $\|v\| = \rho$, then set $u = u_{\overline{k}_1,q} + v$, then $\int_M |u|^q dv_g = k^\star$. Hence

$$E(v) = F_q^\varepsilon(u) \geqslant \mu^\varepsilon_{k^\star,q} > 0.$$

Next we set $v_1 = u_{\overline{k}_2,q} - u_{\overline{k}_1,q}$, then clearly $E(v_1) = 0$ and $\|v_1\| = (\overline{k}_2)^{\frac{1}{q}} > \rho$. Notice that our functional E satisfies the Palais–Smale condition as we have shown for F_q^ε. Thus, Theorem 6.1 in [21, Chapter II] can be applied to E to conclude that the number

$$\mu^\varepsilon_{k_2^\varepsilon,q} = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} E(\gamma(t) - u_{\overline{k}_1,q})$$

is a critical value of the functional E. Clearly, $\mu^\varepsilon_{k_2^\varepsilon,q} > 0$. Thus, there exists a Palais–Smale sequence $\{u_j\}_j \subset H^1(M)$ for the functional F_q^ε at the level $\mu^\varepsilon_{k_2^\varepsilon,q}$. Since $F_q^\varepsilon(u_j) = F_q^\varepsilon(|u_j|)$ for any j, we can assume $u_j \geqslant 0$ for all j. Consequently, Proposition 3.16 implies that, up to subsequences, $u_j \to u^\varepsilon_{2,q}$ strongly in $H^1(M)$ for some $u^\varepsilon_{2,q} \in H^1(M)$ as $j \to \infty$. Therefore, the function $u^\varepsilon_{2,q}$ with positive energy $\mu_{k^\varepsilon_{2},q}$ satisfies the following equation

$$\Delta_g u^\varepsilon_{2,q} + hu^\varepsilon_{2,q} = f(u^\varepsilon_{2,q})^{q-1} + \frac{au^\varepsilon_{2,q}}{((u^\varepsilon_{2,q})^2 + \varepsilon)^{\frac{q}{2}+1}} \tag{4.4}$$

in the weak sense where we denote $\|u^\varepsilon_{2,q}\|^q_{L^q} = k^\varepsilon_2$. The non-negativity of $\{u_j\}_j$ implies that $u^\varepsilon_{2,q} \geqslant 0$ almost everywhere, and thus the regularity result, Lemma 2.2(a), can be applied to (4.4). It follows that $u^\varepsilon_{2,q} \in C^\infty(M)$ which also implies $u^\varepsilon_{2,q} \geqslant 0$ in M. To see $u^\varepsilon_{2,q}$ is not identically zero, thanks to Lemma 3.7 we first know that $\mu^\varepsilon_{2,q} \leqslant \mu < \infty$. Now, if $u^\varepsilon_{2,q} = 0$, then we have $\frac{1}{q}\varepsilon^{-\frac{q}{2}} \int_M a\,dv_g = \mu^\varepsilon_{2,q} \leqslant \mu < \infty$ which is impossible if ε is small enough. Thus, $u^\varepsilon_{2,q} > 0$ on M if ε is sufficiently small which we will always assume from now on. In view of Lemma 3.8, we know that $k^\varepsilon_2 > 0$ is bounded from above by $k_{\star\star}$ independent of both ε and q.

The existence of $u_{2,q}$ with positive energy $\mu_{k_2,q}$. We now let $\{\varepsilon_j\}_j$ be a sequence of small positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each j, let $u^{\varepsilon_j}_{2,q}$ be a smooth positive function in M such that

$$\Delta_g u^{\varepsilon_j}_{2,q} + hu^{\varepsilon_j}_{2,q} = f(u^{\varepsilon_j}_{2,q})^{q-1} + \frac{au^{\varepsilon_j}_{2,q}}{((u^{\varepsilon_j}_{2,q})^2 + \varepsilon_j)^{\frac{q}{2}+1}} \tag{4.5}$$

in M. The boundedness of $\{k^{\varepsilon_j}_2\}_j$ tells us that sequence $\{u^{\varepsilon_j}_{2,q}\}_j$ is bounded in $H^1(M)$, hence, there exists $u_{2,q} \in H^1(M)$ such that up to subsequences

$$u^{\varepsilon_j}_{2,q} \rightharpoonup u_{2,q} \text{ in } H^1(M), \quad u^{\varepsilon_j}_{2,q} \to u_{2,q} \text{ strongly in } L^2(M), \quad u^{\varepsilon_j}_{2,q} \to u_{2,q} \text{ a.e. in } M.$$

Consequently, $u_{2,q} \geqslant 0$ almost everywhere in M. We now denote $\|u_{2,q}\|^q_{L^q} = k_2$. Since the sequence $\{u^{\varepsilon_j}_{2,q}\}_j$ is bounded from below by means of Lemma 2.1, the Lebesgue Dominated Convergence theorem can be applied to conclude that $(u_{2,q})^{-1} \in L^p(M)$ for any $p > 0$. By letting $j \to \infty$ in (4.5), we get that $u_{2,q}$ is the second weak solution of the following subcritical equation

$$\Delta_g u_{2,q} + hu_{2,q} = f(u_{2,q})^{q-1} + \frac{a}{(u_{2,q})^{q+1}}. \tag{4.6}$$

Now the regularity result, Lemma 2.2(b), can be applied to (4.6). It follows that $u_{2,q} \in C^\infty(M)$ and thus $u_{2,q} > 0$ in M. Since $u^{\varepsilon_j}_{2,q}$ has positive energy $\mu_{k^{\varepsilon_j}_2,q}$, by passing to the limit as $j \to \infty$, we know that the energy of $u_{2,q}$ is still non-negative, i.e., $\mu_{k_2,q} > 0$, thus proving $u_{1,q} \not\equiv u_{2,q}$ by means of (3.7). Note that k_2 is still bounded from above by $k_{\star\star}$ independent of both ε and q. This completes the proof of Claim 2.

Claim 3. *Eq. (1.2) has at least one positive solution.*

Proof of Claim 3. Recall that $\mu_{k_i,q}$ are the energy of $u_{i,q}$ found in Claim 2, i.e.,

$$\mu_{k_i,q} = \frac{1}{2}\int_M |\nabla u_{i,q}|^2\,dv_g + \frac{h}{2}\int_M (u_{i,q})^2 dv_g - \frac{1}{q}\int_M f(u_{i,q})^q dv_g + \frac{1}{q}\int_M \frac{a}{(u_{i,q})^q} dv_g.$$

Keep in mind that by k_i we mean $\|u_{i,q}\|_{L^q}^q = k_i$. We now estimate $\mu_{k_1,q}$ and $\mu_{k_2,q}$. We have noticed that $\mu_{k_1,q} < 0 < \mu_{k_2,q} < \mu$. Since $k_1 \in (k_\star, k_{1,q})$ and $h < 0$, we obtain

$$\begin{aligned}\frac{1}{2}\|\nabla u_{1,q}\|_{L^2}^2 &\leqslant \mu_{k_1,q} + \frac{1}{q}\int_M f(u_{1,q})^q dv_g - \frac{h}{2}k_1^{\frac{2}{q}}\\ &\leqslant \frac{k_1}{q}\sup f - \frac{h}{2}k_1^{\frac{2}{q}},\end{aligned}$$

which concludes that the sequence $\{u_{1,q}\}_q$ remains bounded in $H^1(M)$. Similarly, from Lemma 3.7 and the following estimate

$$\begin{aligned}\frac{1}{2}\|\nabla u_{2,q}\|_{L^2}^2 &\leqslant \mu_{k_2,q} + \frac{1}{q}\int_M f(u_{2,q})^q dv_g - \frac{h}{2}k_2^{\frac{2}{q}}\\ &\leqslant \mu + \frac{k_2}{q}\sup f - \frac{h}{2}k_2^{\frac{2}{q}},\end{aligned}$$

we know that the sequence $\{u_{2,q}\}_q$ is also bounded in $H^1(M)$. Combining these facts, we get

$$\|u_{i,q}\|_{H^1}^2 \leqslant 2\mu + \frac{2k_i}{q}\sup f + (1-h)k_i^{\frac{2}{q}}.$$

Thanks to $k_{\star\star} > 1$ and $q > 2^\flat$, if we denote

$$\Lambda = \left(2\mu + (\sup f)k_{\star\star} + (1-h)k_{\star\star}^{\frac{2}{2^\flat}}\right)^{\frac{1}{2}}$$

we then see that $\|u_{i,q}\|_{H^1} \leqslant \Lambda$ for $i = 1, 2$. Thus, up to subsequences, there exists $u_i \in H^1(M)$ such that

$$u_{i,q} \rightharpoonup u_i \quad \text{in } H^1(M), \qquad u_{i,q} \to u_i \quad \text{strongly in } L^2(M), \qquad u_{i,q} \to u_i \quad \text{a.e. in } M$$

as $q \to 2^\star$. Notice that $u_{i,q}$ verify

$$\begin{aligned}&\int_M \nabla u_{i,q}\cdot\nabla v\, dv_g + h\int_M u_{i,q} v\, dv_g - \int_M f(u_{i,q})^{q-1} v\, dv_g\\ &\quad - \int_M \frac{a}{(u_{i,q})^{q+1}} v\, dv_g = 0\end{aligned} \tag{4.7}$$

for any $v \in H^1(M)$. We have already seen in the proof of the Palais–Smale condition that

$$\int_M \left(\nabla u_{i,q} - \nabla u_i\right)\cdot\nabla v\, dv_g \to 0, \qquad \int_M \left(u_{i,q} - u_i\right) v\, dv_g \to 0 \quad \text{as } q \to 2^\star.$$

A strictly positive lower bound for $u_{i,q}$ helps us to conclude that

$$\int_M \frac{a}{(u_{i,q})^{q+1}} v\, dv_g \to \int_M \frac{a}{(u_i)^{2^\star+1}} v\, dv_g \quad \text{as } q \to 2^\star.$$

So far, we can pass to the limit every terms on the left hand side of (4.7) except the term involving f. Since $u_{i,q} \to u_i$ almost everywhere, $(u_{i,q})^{q-1} \to (u_i)^{2^\star-1}$ almost everywhere as $q \to 2^\star$. By the Hölder inequality, one obtains

$$\left\|(u_{i,q})^{q-1}\right\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant \left(\left(\int_M (u_{i,q})^{2^\star} dv_g\right)^{\frac{q-1}{2^\star-1}}\right)^{\frac{2^\star-1}{2^\star}} = \|u_{i,q}\|_{L^{2^\star}}^{q-1}. \tag{4.8}$$

Making use of the Sobolev inequality, we further obtain

$$\left\|(u_{i,q})^{q-1}\right\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant (\mathcal{K}_1+\mathcal{A}_1)^{\frac{q-1}{2}} \|u_{i,q}\|_{H^1}^{q-1}$$

which proves the boundedness of $(u_{i,q})^{q-1}$ in $L^{\frac{2^\star}{2^\star-1}}(M)$. According to [3, Theorem 3.45], we have $(u_{i,q})^{q-1} \rightharpoonup (u_i)^{\frac{2^\star}{2^\star-1}}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$. Thanks to the embedding $H^1(M) \hookrightarrow L^{2^\star}(M)$, we have $v \in L^{2^\star}(M)$ which also implies $fv \in L^{2^\star}(M)$ since f is smooth. Therefore, by the definition of weak convergence, there holds

$$\int_M f(u_{i,q})^{q-1} v\,dv_g \to \int_M f(u_i)^{2^\star-1} v\,dv_g \quad \text{as } q \to 2^\star.$$

With this convergence in hand, we are now in a position to send $q \to 2^\star$ in (4.7) thus proving that u_i are weak solutions to (1.2). Using Lemma 2.2(b) we conclude that $u_i \in C^\infty(M)$ and $u_i > 0$ in M. □

So far we have just shown that u_i are solutions of (1.2). However, we have no information enough to guarantee that these solutions are distinct even that $\lim_{q\to 2^\star}(F_q^0(u_{1,q}) - F_q^0(u_{2,q})) \neq 0$. Therefore, we have here only the existence part. In the next subsection we show that u_i are in fact different provided $\sup f$ is sufficiently small, thus proving Theorem 1.1. □

4.2. The existence of the second solution

We wish to compare $F_{2^\star}^0(u_1)$ and $F_{2^\star}^0(u_2)$. Recall that

$$F_{2^\star}^0(u_i) = \frac{1}{2}\int_M |\nabla u_i|^2\,dv_g + \frac{h}{2}\int_M (u_i)^2 dv_g - \frac{1}{2^\star}\int_M f(u_i)^{2^\star} dv_g + \frac{1}{2^\star}\int_M \frac{a}{(u_i)^{2^\star}} dv_g.$$

Here we introduce a trick without using any concentration-compactness principle. This can be done once we can show that $\lim_{q\to 2^\star} F_q^0(u_{i,q}) = F_{2^\star}^0(u_i)$ for $i = 1, 2$. If we carefully look at the formula for $F_q^0(u_{i,q})$, the only difficult part is to show that

$$\int_M f(u_{i,q})^q dv_g \to \int_M f(u_i)^{2^\star} dv_g \quad \text{as } q \to 2^\star.$$

In contrast to the previous subsection, the bigger exponents generally make us impossible to guarantee such a convergence. To avoid this difficulty, we have to make $\sup f$ sufficiently small. Intuitively, such a small f is equivalent to saying, for example, that $f(u_{i,q})^{q-1}$ behaves exactly the same as $f(u_{i,q})^q$. We first prove that following.

Proposition 4.2. *There holds $\|\nabla u_{i,q}\|_{L^2} \to \|\nabla u_i\|_{L^2}$ as $q \to 2^\star$.*

Proof. This is elementary. It suffices to prove that $\nabla u_{i,q} \to \nabla u_i$ strongly in $L^2(M)$. Using (4.7) with v replaced by $u_{i,q} - u_i$, we arrive at

$$\begin{aligned}&\int_M \nabla u_{i,q}\cdot\nabla(u_{i,q}-u_i)dv_g + h\int_M u_{i,q}(u_{i,q}-u_i)dv_g \\ &\quad - \int_M f(u_{i,q})^{q-1}(u_{i,q}-u_i)dv_g - \int_M \frac{a}{(u_{i,q})^{q+1}}(u_{i,q}-u_i)dv_g = 0. \qquad (4.9)\end{aligned}$$

From (4.9) by a similar argument use in the proof of the Palais–Smale condition it is easy to show that

$$\int_M \nabla u_{i,q} \cdot \nabla(u_{i,q} - u_i)dv_g \to 0 \quad \text{as } q \to 2^\star.$$

Using this the fact that $\nabla u_{i,q} \rightharpoonup \nabla u_i$ weakly in $L^2(M)$, we obtain

$$\int_M |\nabla(u_{i,q} - u_i)|^2 dv_g \to 0 \quad \text{as } q \to 2^\star.$$

In other words, $\nabla u_{i,q} \to \nabla u_i$ strongly in $L^2(M)$. □

Now we conclude that $\int_M f(u_{i,q})^q dv_g \to \int_M f(u_i)^{2^\star} dv_g$ as $q \to 2^\star$. We prove the following proposition.

Proposition 4.3. *We assume that all requirements in Proposition 4.1 are fulfilled. We further assume that f verifies*

$$\sup f < \mathcal{C}_2,$$

where the number $\mathcal{C}_2 > 0$ is given in (4.14) below. Then

$$\int_M f(u_{i,q})^q dv_g \to \int_M f(u_i)^{2^\star} dv_g \quad \text{as } q \to 2^\star.$$

Proof. In (4.7), we choose $v = (u_{i,q})^{1+2\delta}$ for some $\delta > 0$ to be determined later, we arrive at

$$\frac{1+2\delta}{(1+\delta)^2}\int_M |\nabla w_{i,q}|^2 dv_g = |h|\int_M (w_{i,q})^2 dv_g + \int_M f(w_{i,q})^2(u_{i,q})^{q-2}dv_g + \int_M \frac{a}{(u_{i,q})^{q-2\delta}}dv_g,$$

where $w_{i,q} = (u_{i,q})^{1+\delta}$. This and the Sobolev inequality applied to $w_{i,q}$ tell us that

$$\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant \left(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|h| + \mathcal{A}_1\right)\|w_{i,q}\|_{L^2}^2 + \mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}\left(\int_M f^+(w_{i,q})^2(u_{i,q})^{q-2}dv_g + \int_M \frac{a}{(u_{i,q})^{q-2\delta}}dv_g\right). \tag{4.10}$$

We now use the Hölder inequality one more time

$$\int_M (w_{i,q})^2(u_{i,q})^{q-2}dv_g \leqslant \left(\int_M (w_{i,q})^{2^\star}dv_g\right)^{\frac{2}{2^\star}}\left(\int_M (u_{i,q})^{\frac{(q-2)2^\star}{2^\star-2}}dv_g\right)^{1-\frac{2}{2^\star}}.$$

Notice that $\frac{(q-2)2^\star}{2^\star-2} < q$ so long as $q < 2^\star$. Again, by the Hölder and Sobolev inequalities, one gets

$$\int_M (u_{i,q})^{\frac{(q-2)2^\star}{2^\star-2}}dv_g \leqslant \left(\int_M (u_{i,q})^{2^\star}dv_g\right)^{\frac{q-2}{2^\star-2}} \leqslant (\mathcal{K}_1+\mathcal{A}_1)^{\frac{2^\star(q-2)}{2(2^\star-2)}}\|u_{i,q}\|_{H^1}^{\frac{2^\star(q-2)}{2^\star-2}}.$$

Therefore,

$$\int_M (w_{i,q})^2 (u_{i,q})^{q-2} dv_g \leqslant \|w_{i,q}\|_{L^{2^\star}}^2 (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{q-2}{2}} \|u_{i,q}\|_{H^1}^{q-2}.$$

Using (4.10) and our calculation above, it is obvious that

$$\begin{aligned}\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant{}& \left(\mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta}|h| + \mathcal{A}_1\right) \|w_{i,q}\|_{L^2}^2 \\ &+ \mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta} (\sup f)\, (\mathcal{K}_1 + \mathcal{A}_1)^{\frac{q-2}{2}} \|u_{i,q}\|_{H^1}^{q-2} \|w_{i,q}\|_{L^{2^\star}}^2 \\ &+ \mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta} \int_M \frac{a}{(u_{i,q})^{q-2\delta}} dv_g. \end{aligned} \tag{4.11}$$

We wish to impose the condition of $\sup f$ so that

$$\mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta} (\sup f)\, (1 + \mathcal{K}_1 + \mathcal{A}_1)^{\frac{2^\star-2}{2}} \Lambda^{2^\star-2} < \frac{1}{2} \tag{4.12}$$

fulfills. This can be done for a suitable choice of small $\delta > 0$ that will be fixed provided $\sup f$ verifies

$$\mathcal{K}_1 (\sup f)\, (1 + \mathcal{K}_1 + \mathcal{A}_1)^{\frac{2^\star-2}{2}} \Lambda^{2^\star-2} < \frac{1}{2}. \tag{4.13}$$

Notice that Λ also contains $\sup f$, therefore a straightforward calculation shows us that it is enough for (4.13) to assume $\sup f < \mathcal{C}_2$ where

$$\mathcal{C}_2 = \min\left\{ \frac{1}{2\mathcal{K}_1} (1 + \mathcal{K}_1 + \mathcal{A}_1)^{-\frac{2^\star-2}{2}} \left(2\mu + k_{\star\star} + (1-h) k_{\star\star}^{\frac{2}{2^\flat}}\right)^{-\frac{2^\star-2}{2}}, 1 \right\}. \tag{4.14}$$

In view of (4.11), we get from (4.12) that

$$\|w_{i,q}\|_{L^{2^\star}}^2 \leqslant 2\left(\mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta}|h| + \mathcal{A}_1\right) \|w_{i,q}\|_{L^2}^2 + 2\mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta} \int_M \frac{a}{(u_{i,q})^{q-2\delta}} dv_g.$$

By the choice of δ satisfying (4.12) and $1 + \delta < \frac{2^\star}{2}$, we can verify that

$$\|w_{i,q}\|_{L^2} = \|(u_{i,q})^{1+\delta}\|_{L^2} = \|u_{i,q}\|_{L^{2(1+\delta)}}^{1+\delta} \leqslant \|u_{i,q}\|_{L^{2^\star}}^{1+\delta}.$$

This and the Sobolev inequality imply that $\|w_{i,q}\|_{L^2}$ can be controlled by some constant depending on Λ. On the other hand, $\int_M a (u_{i,q})^{-(q-2\delta)} dv_g$ is bounded from above since $q - 2\delta > 0$ and $u_{i,q}$ has a strictly positive constant lower bound independent of q. All discussion above shows that $\{\|w_{i,q}\|_{L^{2^\star}}\}_q$ is bounded, that is, $\{\|u_{i,q}\|_{L^{2^\star(1+\delta)}}\}_q$ is bounded. We are now in a position to make use of [3, Theorem 3.45]. First, by the Hölder inequality as in (4.8), one obtains $\|(u_{i,q})^q\|_{L^{1+\delta}} \leqslant \|u_{i,q}\|_{L^{2^\star(1+\delta)}}^q$, that means $(u_{i,q})^q$ is bounded in $L^{1+\delta}(M)$. This and the fact that $(u_{i,q})^q \to (u_i)^{2^\star}$ almost everywhere in M imply $(u_{i,q})^q \rightharpoonup (u_i)^{2^\star}$ weakly in $L^{1+\delta}(M)$. Therefore, by the definition of weak convergence and the fact that $L^{1+\frac{1}{\delta}}(M)$ is the dual space of $L^{1+\delta}(M)$, there holds

$$\int_M f(u_{i,q})^q dv_g \to \int_M f(u_i)^{2^\star} dv_g \quad \text{as } q \to 2^\star$$

since $f \in L^{1+\frac{1}{\delta}}(M)$. □

Proposition 4.4. *We assume that all requirements in Proposition 4.3 are fulfilled. Then Eq. (1.2) possesses at least two smooth positive solutions, one has strictly negative energy and the other has positive energy.*

Proof. It suffices to compare the energies of u_i. Using Propositions 4.2 and 4.3, we can send $q \to 2^\star$ in the preceding equalities to reach $\lim_{q\to 2^\star} F_q^0(u_{i,q}) = F_{2^\star}^0(u_i)$, $i = 1, 2$. In view of (3.7), there holds $F_{2^\star}^0(u_1) < 0 < F_{2^\star}^0(u_2)$. Thus, u_i have different energies. This completes the proof. □

4.3. The scaling argument

In this subsection, we use the scaling technique to complete the proof of Theorem 1.1 by removing the condition (3.8) mentioned in Proposition 4.1. We first observe that under the variable change $\widetilde{u} = \frac{u}{c}$, where c is a suitable constant to be determined later, Eq. (1.2) becomes

$$\Delta_g \widetilde{u} + h\widetilde{u} = c^{2^\star-2} f \widetilde{u}^{2^\star-1} + \frac{1}{c^{2^\star+2}} \frac{a}{\widetilde{u}^{2^\star+1}}. \tag{4.15}$$

We wish to find a suitable constant $c > 0$ such that our new coefficients $\widetilde{f}$ and $\widetilde{a}$ verify the conditions in Propositions 4.1 and 4.2 where

$$\widetilde{f} = c^{2^\star-2} f, \qquad \widetilde{a} = \frac{a}{c^{2^\star+2}}. \tag{4.16}$$

Clearly, once u is a solution of Eq. (4.16), cu will solve Eq. (1.2) accordingly. Obviously, the coefficient h remains unchanged after the scaling and we also have $\lambda_f = \lambda_{\widetilde{f}}$ since $c > 0$. Therefore, the following conditions

$$|h| < \lambda_{\widetilde{f}}, \quad \widetilde{a} > 0, \quad \int_M \widetilde{f} dv_g < 0, \quad \sup \widetilde{f}^+ > 0$$

are fulfilled. Besides, it is obvious to see that

$$\frac{\sup \widetilde{f}}{\int_M |\widetilde{f}^-| dv_g} = \frac{\sup f}{\int_M |f^-| dv_g}.$$

We now wish to remove (3.8) but still keep other conditions. In other words, we have to choose a suitable c so that the following conditions

$$\frac{2^\star |h|}{2} \leqslant \int_M |\widetilde{f}^-| dv_g, \tag{4.17}$$

and

$$\sup \widetilde{f} < \mathcal{C}_2, \tag{4.18}$$

and

$$\int_M \widetilde{a} dv_g < \frac{1}{n-2} \left(\frac{n-1}{n-2}\right)^{n-1} \left(\frac{|h|}{\int_M |\widetilde{f}^-| dv_g}\right)^n \int_M |\widetilde{f}^-| dv_g \tag{4.19}$$

hold. Indeed, (4.17) and (4.19) can be rewritten as the following

$$\frac{2^\star|h|}{2} \leqslant c^{2^\star-2}\int_M |f^-|\,dv_g \tag{4.20}$$

and

$$\frac{1}{c^{2^\star+2}}\int_M a\,dv_g < \frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{c^{2^\star-2}\int_M |f^-|\,dv_g}\right)^n c^{2^\star-2}\int_M |f^-|\,dv_g. \tag{4.21}$$

Notice that $(2^\star-2)n = 22^\star$, this is about to say that again the right hand side of (4.21) can be rewritten as

$$\frac{1}{c^{2^\star+2}}\frac{1}{n-2}\left(\frac{n-1}{n-2}\right)^{n-1}\left(\frac{|h|}{\int_M |f^-|\,dv_g}\right)^n \int_M |f^-|\,dv_g.$$

By canceling the factor $\frac{1}{c^{2^\star+2}}$, one can easily see that the condition (1.5) is invariant under the variable change. In view of (4.18), we can choose

$$c = \left(\frac{2^\star|h|}{2\int_M |f^-|\,dv_g}\right)^{\frac{1}{2^\star-2}}.$$

It suffices to prove that this particular choice of c and the condition (1.6) are enough to guarantee (4.18). Notice that

$$\sup \widetilde{f} = (\sup f)\left(\frac{2^\star|h|}{2\int_M |f^-|\,dv_g}\right) = \frac{2^\star|h|}{2}\frac{\sup f}{\int_M |f^-|\,dv_g}.$$

Therefore, if we assume

$$\frac{\sup f}{\int_M |f^-|\,dv_g} < \frac{2}{2^\star|h|}\mathcal{C}_2,$$

then the condition (4.18) holds. In conclusion, if the constant $\mathcal{C}$ in the statement of Theorem 1.1 equals

$$\min\left\{\mathcal{C}_1, \frac{2}{2^\star|h|}\mathcal{C}_2\right\} \tag{4.22}$$

we know that Eq. (1.2) has at least two positive smooth solutions. This finishes the proof of Theorem 1.1.

Remark 4.5. Before finishing the proof of Theorem 1.1, it is important to note that the existence of the constant $\mathcal{C}_1$ depends only on the negative part of f and the set $\{x \in M: f(x) \geqslant 0\}$, thus, is independent of $\sup f$. To see this, let us notice from the definition of the sets $\mathscr{A}$ and $\mathscr{A}(\eta, q)$ that λ_f and $\lambda_{f,\eta,q}$ depend only on f^-. This ensures that the existence of $\mathcal{C}_1$ given by (4.1) depends only on f^-. Now one can observe that the condition

$$(\sup f)\left(\int_M |f^-|\,dv_g\right)^{-1} < \mathcal{C}_1$$

actually makes sense and therefore we do have the existence part. However, since the constant $\mathcal{C}_2$ depends on μ and $k_{\star\star}$, it is hard to check whether or not the condition

$$(\sup f)\left(\int_M |f^-|\,dv_g\right)^{-1} < \frac{2}{2^\star|h|}\mathcal{C}_2$$

actually holds but we believe that an example for this case exists. We hope that we will soon see some responses on this issue.

5. Proof of Theorem 1.2

According to [8, Proposition 4], if we restrict ourselves to $f \leqslant 0$ but not strictly negative, the solvability of (1.2), where h, f, and a take the form (1.3) and (1.4), is equivalent to solving the so-called prescribing scalar curvature-scalar field problem

$$\Delta_g u + hu = f u^{2^\star-1}. \tag{5.1}$$

The proof of this fact depends heavily on the conformal covariance property of all these coefficients, that cannot be true for general h, f, and a. Concerning (5.1), Rauzy provided, among other things, necessary and sufficient conditions for the solvability of (5.1) in the general form, that is, for any $f \leqslant 0, a \geqslant 0$ and $h < 0$ a constant. Based on this point, in this section, we prove that there is a natural extension of the Rauzy result for the prescribing scalar curvature equation (5.1) to (1.2) which also provides for necessary and sufficient conditions for the solvability of (1.2). Notice that we have already proved necessary conditions.

5.1. *Asymptotic behavior of* $\mu^\varepsilon_{k,q}$

It is not hard to see that we can go through Lemma 3.1 without any difficulty, that is, for small ε, $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to 0$. Now we want to study the behavior of $\mu^\varepsilon_{k,q}$ for $k \to +\infty$. As can be seen from Section 3 that if f has zero value somewhere in M, then in order to control $\mu^\varepsilon_{k,q}$ for large k, we must study $\lambda_{f,\eta,q}$. Depending on how large the set $\{f = 0\}$ is, there are two possible cases.

Case 1. Suppose that $\sup f = 0$ and $\int_{\{f=0\}} 1\,dv_g > 0$. A careful study shows that all results from Section 3.5 remain hold. So we omit it here.

Case 2. Suppose that $\sup f = 0$ and $\int_{\{f=0\}} 1\,dv_g = 0$. In this context, one may still define $\lambda_{f,\eta,q}$ as in (3.13) for each $\eta \neq 0$. We notice that from the set $\mathscr{A}$ is empty we would have $\lambda_f = +\infty$. Furthermore, it remains true that $\lambda_{f,\eta,q}$ is decreasing as a function of η whose proof is exactly the same as the proof of Lemma 3.10. Our next lemma gives a full description for $\lambda_{f,\eta,q}$ similarly to those proved in Section 3.

We now prove an analogous version of Lemma 3.13.

Lemma 5.1. *There exists η_0 such that for all $\eta < \eta_0$, there exists $q_\eta \in (2^\flat, 2^\star)$ so that $\lambda_{f,\eta,q} > |h|$ for every $q \in (q_\eta, 2^\star)$.*

Proof. We assume by contradiction that for every η_0, there exist $\eta < \eta_0$ and a monotone sequence $\{q_j\}_j$ converging to $2^\star$ so that $\lambda_{f,\eta,q_j} \leqslant |h|$ for any j. We assume furthermore that λ_{f,η,q_j} is achieved by some $v_{\eta,q_j} \in \mathscr{A}(\eta, q_j)$. Then the following estimate holds

$\|\nabla v_{\eta,q_j}\|_{L^2}^2 \|v_{\eta,q_j}\|_{L^2}^{-2} \leqslant |h|$ for any j. As in the proof of Lemma 3.13, there exists $v_{\eta,2^\star} \in H^1(M)$ such that

$$v_{\eta,q_j} \to v_{\eta,2^\star} \quad \text{strongly in } L^2(M), \qquad v_{\eta,q_j} \to v_{\eta,2^\star} \quad \text{a.e. in } M$$

as $j \to \infty$. Thus $\|\nabla v_{\eta,2^\star}\|_{L^2}^2 \|v_{\eta,2^\star}\|_{L^2}^{-2} \leqslant |h|$ and $(\mathcal{K}_1|h| + \mathcal{A}_1)^{-1} \leqslant \|v_{\eta,2^\star}\|_{L^2}^2$. For every $q_j \geqslant 2^\flat$, by the Hölder inequality and the fact that $v_{\eta,q_j} \in \mathscr{A}(\eta, q_j)$ one has

$$\int_M |v_{\eta,q_j}|^{2^\flat} dv_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,q_j}|^{2^\flat} dv_g \leqslant \eta^{\frac{2^\flat}{q_j}} \int_M |f^-| dv_g.$$

Followed by the proof of Lemma 3.13, we obtain

$$\int_M |v_{\eta,2^\star}|^{2^\flat} dv_g \leqslant 1$$

and

$$\int_M |f^-||v_{\eta,2^\star}|^{2^\flat} dv_g \leqslant \eta^{\frac{2^\flat}{2^\star}} \int_M |f^-| dv_g.$$

Now let $\eta \to 0$ and being bounded, there exists $v \in H^1(M)$ such that up to subsequences

$$v_{\eta,2^\star} \rightharpoonup v \quad \text{in } H^1(M), \qquad v_{\eta,2^\star} \to v \quad \text{strongly in } L^2(M), \qquad v_{\eta,2^\star} \to v \quad \text{a.e. in } M.$$

Clearly, $\|\nabla v\|_{L^2}^2 \leqslant |h| \, \|v\|_{L^2}^2$. With a similar argument used in Lemma 3.13 we conclude $\int_M |f^-||v|^{2^\flat} dv_g = 0$. Therefore, $v = 0$ almost everywhere since $f < 0$ almost everywhere. The strong convergence $v_{\eta,2^\star} \to v$ in $L^2(M)$ also implies that $\lim_{\eta \to 2^\star} \|v_{\eta,2^\star}\|_{L^2} = 0$ which provides us a desired contradiction since $\|v_{\eta,2^\star}\|_{L^2}$ has a strictly positive lower bound. □

Remark 5.2. As can be seen from this proof, a stronger form of Lemma 5.1 can be obtained where $|h|$ is replaced by any given positive constant. However, we do not need that strong one. Besides, unlike the argument used in the proof of Lemma 3.13, in the case $\sup f > 0$ function v satisfying $\int_M |f^-||v|^{2^\star} dv_g = 0$ may not be zero as it could be concentrated in the positive part of f.

We are now in a position to study the behavior of $\mu_{k,q}^\varepsilon$ for $k \to +\infty$ when $\sup f = 0$.

Proposition 5.3. *Suppose* $\sup f = 0$. *If*

- *either* $\int_{\{f=0\}} 1 dv_g = 0$ *or*
- $\int_{\{f=0\}} 1 dv_g > 0$ *and* $\lambda_f > |h|$

holds, then $\mu_{k,q}^\varepsilon \to +\infty$ *as* $k \to +\infty$ *for any* $\varepsilon \geqslant 0$ *sufficiently small and any* q *sufficiently close to* $2^\star$ *but all are fixed.*

Proof. We begin to prove that there is some $\eta_0 > 0$ sufficiently small and its corresponding $q_{\eta_0} \in (2^\flat, 2^\star)$ sufficiently close to $2^\star$ such that $\delta_0 = \frac{1}{2}(\lambda_{f,\eta_0,q} + h) > 0$ for any $q \in (q_{\eta_0}, 2^\star)$. We consider two cases separately.

Case 1. Suppose that $\sup f = 0$ and $\int_{\{f=0\}} 1 dv_g = 0$. Under this case, there holds $f < 0$ almost everywhere which implies that the set $\mathscr{A}$ is empty, therefore $\lambda_f = +\infty$. Since h is fixed, we know from Lemma 5.1 that we can find some η_0 sufficiently small and its corresponding $q_{\eta_0} \in (2^\flat, 2^\star)$ such that $\lambda_{f,\eta_0,q} + h \gg 0$ for all $q \in (q_{\eta_0}, 2^\star)$, and thus proving the positivity of δ_0.

Case 2. Suppose that $\sup f = 0$ and $\int_{\{f=0\}} 1 dv_g > 0$. Under this case, λ_f is well-defined and finite. Notice that $\lambda_f + h > 0$. Since all results in Section 3.5 still hold, as in the proof of Proposition 3.14, there exist some $\eta_0 < 2$ and its corresponding $q_{\eta_0} \in (2^\flat, 2^\star)$ such that $0 \leqslant \lambda_f - \lambda_{f,\eta_0,q} < \frac{1}{4}(\lambda_f - |h|)$ for any $q \in (q_{\eta_0}, 2^\star)$. Therefore, $\delta_0 > \frac{3}{8}(\lambda_f + h)$.

Now having the strictly positivity of δ_0 we can easily go through the proof of Proposition 3.14; hence we get $G_q(u) \geqslant mk^{\frac{2}{q}}$ where m is given as in (3.17) which implies that $F_q^\varepsilon(u) \geqslant mk^{\frac{2}{q}}$ due to $\sup f = 0$. Since δ_0 has a strictly positive lower bound, so does m. The proof now follows easily. $\square$

5.2. *Proof of Theorem 1.2 completed*

From now on, we restrict ourselves to the case $q \in (q_{\eta_0}, 2^\star)$. Let us first do some calculation. By solving the following equation

$$\frac{h}{2}k^{\frac{2}{q}} - \frac{k}{q}\int_M f dv_g = 0,$$

we easily see that

$$\mu_{k_0,q}^\varepsilon < \frac{1}{2k_0}\int_M a dv_g,$$

where

$$k_0 = \left(\frac{q}{2}\frac{h}{\int_M f dv_g}\right)^{\frac{q}{q-2}}.$$

It is then easy to bound k_0, say $k_1 < k_0 < k_2$ with $k_2 > 1$ where k_i are independent of both ε and q. We are now in a position to prove Theorem 1.2 whose proof is similar to the proof of Theorem 1.1, therefore we just sketch it and omit in details.

Proposition 5.4. *If* $\sup f = 0$, *then Eq.* (1.2) *admits a positive solution* u.

Sketch of proof. From the study of the behavior of $\mu_{k,q}^\varepsilon$, we can prove the existence of $k_\star$ and $k_{\star\star}$ independent of ε and q with $k_\star < k_1 < k_0 < k_2 < k_{\star\star}$ such that $\mu_{k_0,q}^\varepsilon < \min\{\mu_{k_\star,q}^\varepsilon, \mu_{k_{\star\star},q}^\varepsilon\}$. Then we define

$$\mu_{k_1^\varepsilon,q}^\varepsilon = \inf_{u\in\mathscr{D}_{k,q}} F_q^\varepsilon(u)$$

for each ε and q fixed, where

$$\mathscr{D}_{k,q} = \left\{u \in H^1(M) : k_\star \leqslant \|u\|_{L^q}^q \leqslant k_{\star\star}\right\}.$$

It then turns out that $\mu_{k_1^\varepsilon,q}^\varepsilon$ is achieved by a smooth positive function u_q^ε which is exactly the smooth solution to (1.7). Since $\|u_q^\varepsilon\|_{L^q}$ is uniformly bounded, by using a sequence $\{\varepsilon_j\}_j$ of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$ we can prove, up to subsequences, that

$u_q^{\varepsilon_j} \rightharpoonup u_q$ in $H^1(M)$ as $j \to \infty$. We then show that u_q is a smooth positive solution to (1.7) with ε replaced by 0. Finally, we send $q \to 2^\star$ and do the same argument to claim that (1.2) admits a smooth positive solution. □

In order to make the paper unique, let us mention here the case $\sup f < 0$ although this has been done in [8] by using the method of sub- and super-solution. Suppose $\sup f < 0$. It suffices to study the asymptotic behavior of $\mu_{k,q}^\varepsilon$ for large k. Clearly, for any $u \in \mathscr{B}_{k,q}$,

$$F_q^\varepsilon(u) \geqslant \left(\frac{h}{2} + \frac{1}{2^\star} k^{1-\frac{2}{q}} |\sup f|\right) k^{\frac{2}{q}}.$$

It is then immediate to deduce that $\mu_{k,q}^\varepsilon \to +\infty$ as $k \to +\infty$ since $1 - \frac{2}{q} > 0$. Hence we can easily prove the existence of at least one positive smooth solution to (1.2). More precisely, we prove

Proposition 5.5. *If* $\sup f < 0$, *then Eq.* (1.2) *admits a positive smooth solution* u.

Sketch of proof. The proof of this proposition is similar to the proof of Proposition 5.4. The way to find $k_\star$ is exactly the same as in the proof of Proposition 5.4. The existence of $k_{\star\star}$ can be found as in the proof of Proposition 5.4. Having the existence of $k_\star$ and $k_{\star\star}$ independent of ε and q we can go through the proof of Proposition 5.4 to reach the existence of smooth solution to our Eq. (1.2). □

Acknowledgments

This work is supported by NUS Research Grant R-146-000-127-112. Both authors would like to thank Prof. Ma Li for his interest and enlighten discussion on the problem. Our thanks also go to the referee for his/her comments which improved the presentation of this article.

References

[1] B. Abdellaoui, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential, Ann. Mat. Pura Appl. 182 (2003) 247–270.

[2] A. Ambrosett, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, in: Springer Monographs in Mathematics, Springer, New York, 1998.

[4] T. Aubin, S. Bismuth, Courbure scalaire prescrite sur les variétés riemanniennes compactes dans le cas négatif, J. Funct. Anal. 143 (1997) 529–541.

[5] X. Chen, X. Xu, The scalar curvature flow on S^n–perturbation theorem revisited, Invent Math. 187 (2012) 395–506.

[6] Y. Choquet-Bruhat, General Relativity and the Einstein Equations, in: Oxford Mathematical Monographs, Oxford University Press, 2009.

[7] Y. Choquet-Bruhat, J. Isenberg, D. Pollack, The Einstein-scalar field constraints on asymptotically Euclidean manifolds, Chin. Ann. Math. 27B (2006) 31–52.

[8] Y. Choquet-Bruhat, J. Isenberg, D. Pollack, The constraint equations for the Einstein-scalar field system on compact manifolds, Classical Quantum Gravity 24 (2007) 808–828.

[9] P.T. Chruściel, G.J. Galloway, D. Pollack, Mathematical general relativity: a sampler, Bull. Amer. Math. Soc. 47 (2010) 567–638.

[10] O. Druet, E. Hebey, Stability and instability for Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds, Math. Z. 263 (2009) 33–67.

[11] J. Escobar, R. Schoen, Conformal metrics with prescribed scalar curvature, Invent. Math. 86 (1986) 243–254.

[12] E. Hebey, F. Pacard, D. Pollack, A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds, Comm. Math. Phys. 278 (2008) 117–132.

[13] J. Kazdan, F. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geom. 10 (1975) 113–134.
[14] L. Ma, Liouville type theorem and uniform bound for the Lichnerowicz equation and the Ginzburg–Landau equation, C. R. Math. Acad. Sci. Paris 348 (2010) 993–996.
[15] L. Ma, X. Xu, Uniform bound and a non-existence result for Lichnerowicz equation in the whole n-space, C. R. Math. Acad. Sci. Paris 347 (2009) 805–808.
[16] Q.A. Ngô, X. Xu, Liouville type result for smooth positive solutions of the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds, 2011. Preprint.
[17] Q.A. Ngô, X. Xu, Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds in the null case, 2011. Preprint.
[18] D. Pollack, Private communication.
[19] A. Rauzy, Courbures scalaires des variétés d'invariant conforme négatif, Trans. Amer. Math. Soc. 347 (1995) 4729–4745.
[20] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom. 20 (1984) 479–495.
[21] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Berlin, 2008.
[22] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Sc. Norm. Super. Pisa (3) 22 (1968) 265–274.
[23] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21–37.