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SOME NEW RESULTS ON THE FEJÉR AND HERMITE-HADAMARD INEQUALITIES

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ABSTRACT. The Hermite-Hadamard inequality and its generalization, the Fejér inequality, have many applications. A simple application is to approximate the definite integral $\int_a^b f(x)\,dx$ if the function f is convex. In this short note, we show how to relax the convexity property of the function f, and thus we obtain inequalities that involve a larger class of functions. This new study also raises some open questions.

1. Introduction. The Hermite-Hadamard inequality [**5, 6**] says that

$$f\left(\frac{a+b}{2}\right) \leqq \frac{1}{b-a}\int_a^b f(t)\,dt \leqq \frac{f(a)+f(b)}{2} \tag{1}$$

holds for any convex function $f : I \to \mathbf{R}$ and $a, b \in I$.

As a generalization of (1), the Fejér inequality [**4**] says that

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx \leqq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \tag{2}$$

holds for any convex function $f : I \to \mathbf{R}$, where $a, b \in I$ and $p : [a, b] \to \mathbf{R}$ is non-negative integrable and symmetric about $x = (a + b)/2$.

Apparently, inequality (2) goes back to inequality (1) if we put $p \equiv 1/(b - a)$. Inequalities (1) and (2) provide a simple way to evaluate the integral $\int_a^b f(x)\,dx$. These inequalities have many extensions and generalizations, see [**1, 2, 7–9**]. In this paper we present some new refinements of inequalities (1) and (2).

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Obviously, inequality (2) can be rewritten as

$$(3)\quad \begin{aligned} &\left(f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right)\int_a^b p(x)\,dx \\ &\qquad \leqq \int_a^b f(x)p(x)\,dx-\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \\ &\qquad \leqq 0 \end{aligned}$$

and

$$(4)\quad \begin{aligned} 0 &\leqq \int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \\ &\leqq \left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\int_a^b p(x)\,dx \end{aligned}$$

which says that

$$\begin{aligned} &\int_a^b f(x)p(x)\,dx-\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \\ &\qquad \leqq 0 \\ &\qquad \leqq \int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx. \end{aligned}$$

We observe that, under certain conditions, we can relax the convexity property of function f. This is the aim of the present paper.

Precisely, both inequalities (1) and (2) require function f to be convex; as a consequence, it is natural to assume that f is twice-differentiable. Consequently, $f'' \geqq 0$. Our first result concerns the case when f'' is bounded in $[a,b]$. Note that, we do not require f'' to be non-negative. Precisely, we first prove the following result:

Theorem 1. *Suppose* $p(x) \geqq 0$ *is symmetric about* $(a+b)/2$ *and* $f:[a,b]\to \mathbf{R}$ *is a twice-differentiable function such that* f'' *is bounded* *in* $[a,b]$. *Then*

$$(5)\quad m\int_a^{(a+b)/2}\left(\frac{a+b}{2}-x\right)^2 p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqq M\int_a^{(a+b)/2}\left(\frac{a+b}{2}-x\right)^2 p(x)\,dx$$

and

$$(6)\quad -M\int_a^{(a+b)/2}(x-a)(b-x)p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \leqq -m\int_a^{(a+b)/2}(x-a)(b-x)p(x)\,dx$$

where

$$m = \inf_{t\in[a,b]} f''(t), \quad M = \sup_{t\in[a,b]} f''(t).$$

Remark 1. If $f'' \geqq 0$, then we obtain an improvement of the Fejér inequality (2).

Next we consider the case when f'' is of class $L^p([a,b])$; we also obtain the following estimates:

Theorem 2. *Let* $1 < p < \infty$ *and* $p(x) \geqq 0$ *be symmetric about* $(a+b)/2$ *and* $f : [a,b] \to \mathbf{R}$ *be a twice-differentiable function such that* $f'' \in L^p([a,b])$. *Then*

$$(7)\quad \left|\int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\right| \leqq \frac{q}{2(q+1)}\,\|f''\|_p \int_a^{(a+b)/2}(a+b-2x)^{(1/q)+1}p(x)\,dx$$

and

$$(8)\quad \left|\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx-\int_a^b f(x)p(x)\,dx\right|$$
$$\leqq \frac{q}{2(q+1)}\,\|f''\|_p \int_a^{(a+b)/2}\Big((b-a)^{(1/q)+1}-(a+b-2x)^{(1/q)+1}\Big)p(x)\,dx,$$

where q is defined to be $p/(p-1)$.

Finally, it is clear to see that inequality $f'' \geqq 0$ implies that f' is non-decreasing. Therefore, in the next result, we assume that

$$(9)\qquad f'(a+b-x) \geqq f'(x), \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right].$$

Clearly, if f' is non-decreasing, then inequality (9) holds. However, it is obvious to see that the reverse statement is not true.

Theorem 3. *Suppose that $p(x) \geqq 0$ is symmetric about $(a+b)/2$ and $f : [a,b] \to \mathbf{R}$ is a differentiable function satisfying $f'(a+b-x) \geqq f'(x)$, for all $x \in [a,(a+b)/2]$. Then*
(10)
$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx \leqq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx$$

holds.

Remark 2. It is worth noticing that the assumption f is a differentiable function which has been used in the literature; for example, in [**3**] the authors assumed f' is convex on $[a,b]$. They then obtained some refinements of the Hermite-Hadamard inequality (1).

By using Theorems 1–3, it turns out that the question of deriving a sharp version becomes open. We hope that we will soon see some responses on this problem.

2. Proofs.

Proof of Theorem 1. We firstly prove (5). Since $p(x) \geqq 0$ is symmetric about $(a+b)/2$, we have

$$\begin{aligned}\int_a^b f(x)p(x)\,dx &= \int_a^b f(a+b-x)p(a+b-x)\,dx \\ &= \int_a^b f(a+b-x)p(x)\,dx.\end{aligned}$$

So

$$\int_a^b f(x)p(x)\,dx = \frac{1}{2}\int_a^b \big(f(x)+f(a+b-x)\big)p(x)\,dx, \tag{11}$$

which gives

$$\begin{aligned}&\int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \\ &\qquad = \frac{1}{2}\left(\int_a^b \left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx\right).\end{aligned}$$

Since

$$\left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)$$

is symmetric about $(a+b)/2$, one has

$$\begin{aligned}&\int_a^b \left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx \\ &\qquad = 2\int_a^{(a+b)/2} \left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx,\end{aligned}$$

which implies

$$\begin{aligned}&\int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \\ &\qquad = \int_a^{(a+b)/2} \left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx.\end{aligned} \tag{12}$$

Since

$$f(a+b-x) - f\left(\frac{a+b}{2}\right) = \int_{(a+b)/2}^{a+b-x} f'(t)\,dt$$

and

$$f\left(\frac{a+b}{2}\right) - f(x) = \int_{x}^{(a+b)/2} f'(t)\,dt,$$

then

$$\begin{aligned} f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \\ &= \int_{(a+b)/2}^{a+b-x} f'(t)\,dt - \int_{x}^{(a+b)/2} f'(t)\,dt \\ &= \int_{x}^{(a+b)/2} f'(a+b-t)\,dt - \int_{x}^{(a+b)/2} f'(t)\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \\ &= \int_{x}^{(a+b)/2} \big(f'(a+b-t) - f'(t)\big)\,dt. \end{aligned} \tag{13}$$

Since

$$f'(a+b-t) - f'(t) = \int_{t}^{a+b-t} f''(y)\,dy \tag{14}$$

then for $t \in [a, (a+b)/2]$, one has

$$m(a+b-2t) \leqq f'(a+b-t) - f'(t) \leqq M(a+b-2t).$$

Thus,

$$\begin{aligned} \int_{x}^{(a+b)/2} m(a+b-2t)\,dt &\leqq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \\ &\leqq \int_{x}^{(a+b)/2} M(a+b-2t)\,dt. \end{aligned}$$

A simple calculation shows us that

$$m\left(\frac{a+b}{2}-x\right)^2 \leqq f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) \leqq M\left(\frac{a+b}{2}-x\right)^2.$$

Then

$$m\int_a^{(a+b)/2}\left(\frac{a+b}{2}-x\right)^2 p(x)\,dx \leqq \int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqq M\int_a^{(a+b)/2}\left(\frac{a+b}{2}-x\right)^2 p(x)\,dx.$$

This completes the proof of (5). We now prove (6). By using (11), one has

$$\int_a^b f(x)p(x)\,dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx = \frac{1}{2}\left(\int_a^b \Big(f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\Big)p(x)\,dx\right).$$

Since the following function

$$\Big(f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\Big)p(x)$$

is symmetric about $(a+b)/2$, one gets

$$\int_a^b f(x)p(x)\,dx - \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx = \int_a^{(a+b)/2}\Big(f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\Big)p(x)\,dx. \tag{15}$$

Since

$$f(b)-f(a+b-x)=\int_{a+b-x}^b f'(t)\,dt$$

and

$$f(x) - f(a) = \int_a^x f'(t)\,dt,$$

then we have

$$\begin{aligned} f(x) + f(a+b-x) - \big(f(a)+f(b)\big) \\ &= \int_a^x f'(t)\,dt - \int_{a+b-x}^b f'(t)\,dt \\ &= \int_a^x f'(t)\,dt - \int_a^x f'(a+b-t)\,dt. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) + f(a+b-x) - \big(f(a)+f(b)\big) \\ = -\int_a^x \big(f'(a+b-t) - f'(t)\big)\,dt. \end{aligned} \tag{16}$$

We also have

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(y)\,dy \tag{17}$$

which implies, for $t \in [a, (a+b)/2]$, that

$$m(a+b-2t) \leqq f'(a+b-t) - f'(t) \leqq M(a+b-2t).$$

Hence,

$$\begin{aligned} -\int_a^x M(a+b-2t)\,dt \leqq f(x) + f(a+b-x) - \big(f(a)+f(b)\big) \\ \leqq -\int_a^x m(a+b-2t)\,dt. \end{aligned}$$

Thus,

$$\begin{aligned} -M(x-a)(b-x) \leqq f(x) + f(a+b-x) - \big(f(a)+f(b)\big) \\ \leqq -m(x-a)(b-x). \end{aligned}$$

It follows that

$$\begin{aligned}-M\int_a^{(a+b)/2}&(x-a)(b-x)p(x)\,dx\\&\leqq \int_a^b f(x)p(x)\,dx-\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx\\&\leqq -m\int_a^{(a+b)/2}(x-a)(b-x)p(x)\,dx.\end{aligned}$$

The proof is complete. □

Proof of Theorem 2. We firstly prove (7). From (12)–(13), one has

$$\begin{aligned}\int_a^b f(x)p(x)\,dx&-f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\\&=\frac{1}{2}\left(\int_a^b\left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx\right)\\&=\int_a^{(a+b)/2}\left(f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right)p(x)\,dx\end{aligned}$$

and

$$\begin{aligned}f(x)+f(a+b-x)&-2f\left(\frac{a+b}{2}\right)\\&=\int_x^{(a+b)/2}\big(f'(a+b-t)-f'(t)\big)\,dt.\end{aligned}$$

Note that, by (14),

$$f'(a+b-t)-f'(t)=\int_t^{a+b-t}f''(y)\,dy$$

where $a\leqq t\leqq (a+b)/2$, which implies

$$\begin{aligned}|f'(a+b-t)&-f'(t)|\\&\leqq\left(\int_t^{a+b-t}dy\right)^{1/q}\left(\int_t^{a+b-t}|\,f''(y)|^p dy\right)^{1/p}\\&\leqq\left(\int_t^{a+b-t}dy\right)^{1/q}\|f''\|_p\\&=(a+b-2t)^{1/q}\|f''\|_p.\end{aligned}$$

Thus,

$$\left|\int_a^b f(x)p(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx\right| \\ \leqq \frac{q}{2(q+1)}\,\|f''\|_p \int_a^{(a+b)/2} (a+b-2x)^{(1/q)+1} p(x)\,dx.$$

The proof of (7) is complete. We now prove (8). From (15)–(16), one has

$$\left|\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - \int_a^b f(x)p(x)\,dx\right| \\ \leqq \frac{1}{2}\int_a^b \big|f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\big|p(x)\,dx \\ \leqq \int_a^{(a+b)/2} \big|f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\big|p(x)\,dx$$

and

$$\big|f(x)+f(a+b-x)-\big(f(a)+f(b)\big)\big| \leqq \int_a^x \big|f'(a+b-t)-f'(t)\big|\,dt.$$

Note that, by (17),

$$f'(a+b-t)-f'(t) = \int_t^{a+b-t} f''(y)\,dy$$

where $a \leqq t \leqq (a+b)/2$, which implies

$$|f'(a+b-t)-f'(t)| \leqq \left(\int_t^{a+b-t} dy\right)^{1/q}\left(\int_t^{a+b-t} |f''(y)|^p dy\right)^{1/p} \\ \leqq \left(\int_t^{a+b-t} dy\right)^{1/q} \|f''\|_p \\ = (a+b-2t)^{1/q}\|f''\|_p.$$

Therefore,

$$\left|\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - \int_a^b f(x)p(x)\,dx\right| \\ \leqq \frac{q}{2(q+1)}\,\|f''\|_p \int_a^{(a+b)/2} \Big((b-a)^{(1/q)+1} - (a+b-2x)^{(1/q)+1}\Big)p(x)\,dx.$$

Proof of Theorem 3. The proof of Theorem 3 comes from the proofs of Theorems 1 and 2. From (12) and (13), one has

$$\int_a^b f(x)p(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx$$
$$= \int_a^{(a+b)/2}\left(\int_x^{(a+b)/2}\big(f'(a+b-t)-f'(t)\big)\,dt\right)p(x)\,dx.$$

Similarly, from (15) and (16), one gets

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx - \int_a^b f(x)p(x)\,dx$$
$$= \int_a^{(a+b)/2}\left(\int_a^{x}\big(f'(a+b-t)-f'(t)\big)\,dt\right)p(x)\,dx.$$

Thus, the proof follows from the assumption. □

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