



A new Ostrowski–Grüss inequality involving $3n$ knots



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ARTICLE INFO

Keywords:

Integral inequality
Taylor expansion
Ostrowski
Ostrowski–Grüss
Simpson
Iyengar
Bernoulli polynomial

ABSTRACT

This is the fifth and last in our series of notes concerning some classical inequalities such as the Ostrowski, Simpson, Iyengar, and Ostrowski–Grüss inequalities in \mathbb{R} . In the last note, we propose an improvement of the Ostrowski–Grüss inequality which involves $3n$ knots where $n \geq 1$ is an arbitrary numbers. More precisely, suppose that $\{x_k\}_{k=1}^n \subset [0, 1]$, $\{y_k\}_{k=1}^n \subset [0, 1]$, and $\{\alpha_k\}_{k=1}^n \subset [0, n]$ are arbitrary sequences with $\sum_{k=1}^n \alpha_k = n$ and $\sum_{k=1}^n \alpha_k x_k = n/2$. The main result of the present paper is to estimate

$$\frac{1}{n} \sum_{k=1}^n \alpha_k f(a + (b-a)y_k) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k)$$

in terms of either f' or f'' . Unlike the standard Ostrowski–Grüss inequality and its known variants which basically estimate $f(x) - \left(\int_a^b f(t) dt\right)/(b-a)$ in terms of a correction term as a linear polynomial of x and some derivatives of f , our estimate allows us to freely replace $f(x)$ and the correction term by using $3n$ knots $\{x_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$ and $\{\alpha_k\}_{k=1}^n$. As far as we know, this is the first result involving the Ostrowski–Grüss inequality with three sequences of parameters.

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1. Introduction

It is no doubt that one of the most fundamental concepts in mathematics is inequality. However, as mentioned in a recent notes by Qi [23], the development of mathematical inequality theory before 1930 are scattered, dispersive, and unsystematic. Loosely speaking, the theory of mathematical inequalities has just formally started since the presence of a book by Hardy et al. [7]. Since then, the theory of mathematical inequalities has been pushed forward rapidly as a lot of books for inequalities were published worldwide.

Although the set of mathematical inequalities nowadays is huge, inequalities involving integrals and derivatives for real functions always have their own interest. Within this kind of inequalities, the one involving estimates of $\int_a^b f(t) dt$ by bounds of the derivative of its integrand turns out to be fundamental as it has a long history and has received considerable attention from many mathematicians.

Not long before 1934, at the very beginning of the history of mathematical inequalities, in 1921, Pólya derived an inequality which can be used to estimate the integral $\int_a^b f(t) dt$ by bounds of the first order derivative f' . His inequality basically says that the following holds

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$$\left| \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{4} \|f'\|_\infty, \tag{1.1}$$

for any differentiable function f having $f(a) = f(b) = 0$ and $\|f'\|_\infty = \sup_{x \in [a,b]} |f'|$. Later on, in 1938, Iyengar [15] generalized (1.1) by showing that

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{b-a}{4} \|f'\|_\infty - \frac{(f(b) - f(a))^2}{4(b-a)\|f'\|_\infty} \tag{1.2}$$

for any differentiable function f . Here the only difference is that the condition $f(a) = f(b) = 0$ is no longer assumed in (1.2). Apparently, (1.2) provides a simple error estimate for the so-called trapezoidal rule.

Also in this year, Ostrowski [21, page 226] proved another type of the Pólya–Iyengar inequality (1.2) which tells us how to approximate the difference $f(x) - \left(\int_a^b f(t) dt\right)/(b-a)$ for $x \in [a, b]$. More precisely, he proved that

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty \tag{1.3}$$

for all $x \in [a, b]$. As we have just mentioned, unlike (1.1), the inequality (1.3) provides a bound for the approximation of the integral average $\left(\int_a^b f(t) dt\right)/(b-a)$ by the value $f(x)$ at the point $x \in [a, b]$.

Similar to the inequality (1.2), the Simpson inequality, which gives an error bound for the well-known Simpson rule, has been considered widely which is given as follows

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leq \frac{\Gamma - \gamma}{12} (b-a), \tag{1.4}$$

where Γ and γ are real numbers such that $\gamma < f'(x) < \Gamma$ for all $x \in [a, b]$.

In recent years, a number of authors have written about generalizations of (1.1)–(1.4). For example, this topic is considered in [2,3,5,14,16,17,20,19,22,26,29]. In this way, some new types of inequalities are formed, such as inequalities of Ostrowski–Grüss type, inequalities of Ostrowski–Chebyshev type, etc.

The present paper is organized as the following. First, still in Section 1, let us use some space of the paper to mention several typical generalizations of (1.1)–(1.4). Later on, we shall review our recent works considering as generalizations of (1.1)–(1.4) which aims to propose a completely new idea in order to generalize these inequalities. In the final part of this section, we state our main result of the present paper whose proof is in Section 2.

1.1. Generalization of the Ostrowski inequality (1.3)

In the literature, there are several ways to generalize the Ostrowski inequality (1.3).

The first and most standard way is to replace the term $\|f'\|_\infty$ on the right hand side of (1.3) by $\|f'\|_q$ for any $q \geq 1$ where, throughout the paper, we denote

$$\|g\|_q = \left(\int_a^b |g(t)|^q dt \right)^{1/q},$$

for any function g . Within this direction, Theorem 1.2 in a monograph by Dragomir and Rassias [4] is the best as they were able to derive the best constant, see also [12, Theorem 2]. To be completed, let us recall the inequality that they proved

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^{1/p}}{(p+1)^{1/p}} \left(\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1} \right)^{1/p} \|f'\|_q$$

with $1/p + 1/q = 1$.

The second way to generalize the Ostrowski inequality (1.3) is to consider the so-called Ostrowski–Grüss type inequality. The only difference is that the term $(x - \frac{a+b}{2}) \frac{f(b)-f(a)}{b-a}$ will be added to control $f(x) - \left(\int_a^b f(t) dt\right)/(b-a)$. Within this type of generalization, let us recall a result due to Dragomir and Wang in [5, Theorem 2.1]. More precisely, they proved the following

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma) \tag{1.5}$$

for all $x \in [a, b]$ where f' is integrable on $[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$, for all $x \in [a, b]$ and for some constants $\gamma, \Gamma \in \mathbb{R}$.

Recently in [26], by using f'' instead of f' and replacing $\Gamma - \gamma$ by $\|f''\|_2$, Ujević proved that the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \leq \frac{(b-a)^{3/2}}{2\pi\sqrt{3}} \|f''\|_2, \tag{1.6}$$

for all $x \in [a, b]$ provided $f'' \in L^2(a, b)$.

1.2. Generalization of the Iyengar inequality (1.2)

Concerning the Iyengar inequality (1.2), by adding the term $(f'(b) - f'(a))(b - a)/8$ to the left hand side of (1.2), in [6, Corollary 1], the following Iyengar type inequality was obtained

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} + \frac{b-a}{8} (f'(b) - f'(a)) \right| \leq \frac{M}{24} \left((b-a)^2 - \frac{1}{b-a} \left(\frac{\Delta}{M}\right)^3 \right), \tag{1.7}$$

for any $f \in C^2[a, b]$ with $|f''(x)| \leq M$ and $\Delta = f'(a) - 2f'((a+b)/2) + f'(b)$. Other generalizations for (1.2) can also be found in the literature, for example, in [1].

1.3. Generalization of the Simpson inequality (1.4)

Regarding to the Simpson inequality (1.4), there are three types of generalization.

First, using higher order derivatives of f as in [18, Corollary 3], the following Simpson–Grüss type inequalities for $n = 1, 2, 3$ have been proved

$$\left| \int_a^b f(t)dt - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leq C_n (\Gamma_n - \gamma_n) (b-a)^{n+1}, \tag{1.8}$$

for any function $f : [a, b] \rightarrow \mathbb{R}$ such that $f^{(n-1)}$ is an absolutely continuous function and $\gamma_n \leq f^{(n)}(t) \leq \Gamma_n$ for some real constants γ_n and Γ_n and where $C_1 = 5/72, C_2 = 1/62$, and $C_3 = 1/1152$.

Second, we can estimate the left hand side of (1.4) by using the Chebyshev functional associated to f . To be exact, the following inequality holds

$$\left| \int_a^b f(t)dt - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leq \frac{(b-a)^{3/2}}{6} \sqrt{\sigma(f')}, \tag{1.9}$$

where the operator σ is given by $\sigma(f) = \|f\|_2^2 - \|f\|_1^2/(b-a)$.

Third, we can generalize (1.4) by using different points rather than $a, (a+b)/2$, and b . In fact, the following inequality was proved in [25, Theorem 3]

$$\left| \int_a^b f(t)dt - \frac{b-a}{2} \left(f\left(\frac{a+b}{2} - (2-\sqrt{3})(b-a)\right) + f\left(\frac{a+b}{2} + (2-\sqrt{3})(b-a)\right) \right) \right| \leq \frac{7-4\sqrt{3}}{8} \|f''\|_\infty (b-a)^3, \tag{1.10}$$

for any twice differentiable function f such that f'' is bounded and integrable. Another generalization that follows this idea was obtained in [24, Theorem 7] by considering $\|f''\|_2$ instead of $\|f''\|_\infty$. This leads us to the following result

$$\left| \int_a^b f(t)dt - \frac{b-a}{2} \left(f\left(\frac{a+b}{2} - \frac{3-\sqrt{6}}{2}(b-a)\right) + f\left(\frac{a+b}{2} + \frac{3-\sqrt{6}}{2}(b-a)\right) \right) \right| \leq \sqrt{\frac{49}{80} - \frac{\sqrt{6}}{4}} \|f''\|_2 (b-a)^{5/2}. \tag{1.11}$$

In the following subsection, we summarize our previous works concerning to some generalizations of all inequalities mentioned above. Our aim is to highlight the main idea that has been used through these works and that probably is the source of our inspiration to write this paper.

1.4. Our previous works

Several years ago, we initiated a new research direction which aims to propose a completely new way to treat inequalities of the type (1.1)–(1.4). Before briefly reviewing our results, let us recall some notations that we introduced in [8] for the first time.

For each $k = \overline{1, n}$, we choose a knot x_k for which $0 \leq x_k < 1$. We then put

$$Q(f, n, x_1, \dots, x_n) = \frac{b-a}{n} \sum_{k=1}^n f(a + (b-a)x_k)$$

and

$$I(f) = \int_a^b f(t)dt.$$

The basic idea of our research direction is to approximate $I(f)$ by $Q(f, n, x_1, \dots, x_n)$ under suitable choices of the knots x_k .

Our mission started in 2009 with a generalization of the inequality (1.10), see [8, Theorem 4]. In fact, by assuming further that our knots x_k satisfy the following system of algebraic equations

$$\begin{cases} x_1 + x_2 + \dots + x_n = \frac{n}{2}, \\ \dots \\ x_1^j + x_2^j + \dots + x_n^j = \frac{n}{j+1}, \\ \dots \\ x_1^{m-1} + x_2^{m-1} + \dots + x_n^{m-1} = \frac{n}{m}, \end{cases}$$

we were able to prove that

$$|I(f) - Q(f, n, x_1, \dots, x_n)| \leq \frac{1}{m!} \left(\left(\frac{1}{mq+1} \right)^{1/q} + \left(\frac{1}{(m-1)q+1} \right)^{1/q} \right) \|f^{(m)}\|_p (b-a)^{m+1/q}, \tag{1.12}$$

for any m th differentiable function f such that $f^{(m)} \in L^p(a, b)$ and where q is chosen in such a way that $1/p + 1/q = 1$.

Surprisingly, except for the constant appearing on the right hand side of (1.12) which is not optimal, however, as far as we know, all generalizations of either (1.4) and (1.10) or (1.11) always take the form of (1.12) by selecting suitable x_k , see [8] for some examples. Moreover, our inequality (1.12) provides a new way to generate new inequalities of the form (1.10) and (1.11).

Following this research direction, in 2010, we found a new generalization for (1.8) which basically gives us the following estimate

$$|I(f) - Q(f, n, x_1, \dots, x_n)| \leq \frac{2m+5}{4} \frac{(b-a)^{m+1}}{(m+1)!} (S-s) \tag{1.13}$$

for any m th differentiable function $f : [a, b] \rightarrow \mathbb{R}$ where $S := \sup_{a \leq x \leq b} f^{(m)}(x)$ and $s := \inf_{a \leq x \leq b} f^{(m)}(x)$, see [9, Theorem 2]. Here the sequence $\{x_k\}_k$ is assumed to satisfy a new system of equations given by

$$\begin{cases} x_1 + x_2 + \dots + x_n = \frac{n}{2}, \\ \dots \\ x_1^j + x_2^j + \dots + x_n^j = \frac{n}{j+1}, \\ \dots \\ x_1^{m-1} + x_2^{m-1} + \dots + x_n^{m-1} = \frac{n}{m}, \\ x_1^m + x_2^m + \dots + x_n^m = \frac{n}{m+1}. \end{cases}$$

As can be seen, the estimate (1.13) allows us to freely use derivatives of any order of f . In addition, the set of points $\{a, (a+b)/2, b\}$ which appears in the original estimate (1.8) is now replaced by our knots $\{x_k\}_k$.

Later on, also in the year 2010, by keeping the sequence $\{x_k\}_k$ which satisfies (1.14) above, we obtained the following generalization for (1.9)

$$|I(f) - Q(f, n, x_1, \dots, x_n)| \leq \left(\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m-1}} \right) \frac{(b-a)^{m+1/2}}{m!} \sqrt{\sigma(f^{(m)})} \tag{1.14}$$

for any m -times differentiable function $f : [a, b] \rightarrow \mathbb{R}$ such that $f^{(m)} \in L^2(a, b)$, see [10, Theorem 3]. Again, as in (1.13), the estimate (1.14) allows us to use derivatives of any order of f and the set of point $\{a, (a+b)/2, b\}$ is now the set $\{x_k\}_k$.

Finally, in 2010, we announced a generalization for (1.7). Our generalization has two folds. First we replace the term $(f(a) + f(b))/2$ by the term Q as in the previous works. Second, we replaced f'' by f''' to get a new estimate. Precisely, we proved in [11, Theorems 3 and 4] the following

$$A_{p,q}(b-a)^3 \leq I(f) - Q(f, n, x_1, \dots, x_n) + (b-a)^2 p(f'(b) - f'(a)) \leq B_{p,q}(b-a)^3 \tag{1.15}$$

and

$$\left| I(f) - Q(f, n, x_1, \dots, x_n) + (b-a)^2 \left(\frac{q}{2} - \frac{1}{6} \right) (f'(b) - f'(a)) \right| \leq K_{r,q} (b-a)^{4-1/r} \|f'''\|_r \tag{1.16}$$

where the constant $K_{r,q}$ depends only on q and r while the constants $A_{p,q}$ and $B_{p,q}$ depend on $p, q, \inf_{a \leq x \leq b} f'(x)$, and $\sup_{a \leq x \leq b} f'(x)$. Besides, the sequence $\{x_k\}_{k=1}^n \subset [0, 1]$ is now chosen in such a way that

$$\begin{cases} x_1 + x_2 + \dots + x_n = \frac{n}{2}, \\ x_1^2 + x_2^2 + \dots + x_n^2 = nq, \end{cases}$$

for some $q \in [0, 1/2]$.

While the optimal constants for (1.12), and (1.14)–(1.16) remain unknown, the optimal constant for (1.13) has been recently found. For a detail of the progress of finding the optimal constants, we refer the reader to [27,31,28], especially the work [30, Theorem 2.3]. It is worth noticing that in [30], a beautiful connection between the optimal constant for (1.13) and the well-known Bernoulli polynomials has been established. From our point of view, this could be led to optimal constants for the others inequalities such as (1.12), and (1.14)–(1.16). We hope that we shall soon see some responses on this issue.

1.5. Our main result

In the last paper of the series, our purpose is to make some improvements of Ostrowski type inequalities such as (1.5) and (1.6). In order to see the idea underlying our generalization, let us take a look at the inequalities (1.5)–(1.11). The main difference between the inequalities (1.5) and (1.6) and the others is the presence of $f(x)$. A prior to this work, what we have already done is to keep the integral $\int_a^b f(t)dt$ fixed but freely prescribed the value of f at certain points using our knots. In this work, we make a further step by replacing $f(x)$ in (1.5) and (1.6) by something which is new and depends on more than one parameter. A simple choice that one could think about is to replace $f(x)$ by a set of new knots.

Our present work has three folds. First, we generalize (1.5). Before doing so, let us further introduce some notation. Let $\alpha_i \geq 0$ be satisfied

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = n. \quad (1.17)$$

For each $i = \overline{1, n}$, we assume $0 \leq y_i \leq 1$. Instead of using f mentioned above, we then use the following quantity

$$Q(f, y_1, \dots, y_n) = \frac{b-a}{n} \sum_{k=1}^n \alpha_k f(a + y_k(b-a)). \quad (1.18)$$

We note that this new Q given in (1.18) is different from the previous one by the weights α_k . Besides, $Q(f, y_1, \dots, y_n)/(b-a)$ goes back to $f(x)$ if one sets $n = 1$, $\alpha_1 = 1$, and $y_1 = (x-a)/(b-a)$. We are now in a position to state our main result for this generalization.

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \rightarrow \mathbb{R}$ be an differentiable function. We also let $\Gamma = \sup_{x \in [a, b]} f'(x)$ and $\gamma = \inf_{x \in [a, b]} f'(x)$. Then the following estimate holds*

$$\left| \frac{1}{b-a} (Q(f, y_1, y_2, \dots, y_n) - I(f)) - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| \leq \frac{9}{4} (b-a) (\Gamma - \gamma) \quad (1.19)$$

for arbitrary sequences $\{x_k\}_{k=1}^n \subset [0, 1]$ and $\{y_k\}_{k=1}^n \subset [0, 1]$ with $n \geq 1$ and

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \frac{n}{2}.$$

Clearly, the estimate in (1.19) still makes use of f' on the interval $[a, b]$. However, the term $f(x)$ which appears in (1.5) had been changed to $Q(f, y_1, y_2, \dots, y_n)/(b-a)$. In order to see the difference, let us now consider a very special case of (1.19). By choosing $n = 1$ and $\alpha_1 = 1$ we see that we have no choice for x_1 but $x_1 = 1/2$. If we choose $y_1 = (x-a)/(b-a)$ where $x \in [a, b]$ then, by changing variables, (1.19) tells us that

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{n} \underbrace{\left(\frac{x-a}{b-a} - \frac{1}{2} \right)}_{\frac{f(b)-f(a)}{b-a} \left(x - \frac{a+b}{2} \right)} \right| \leq \frac{9}{4} (b-a) (\Gamma - \gamma)$$

which is nothing but an Ostrowski–Grüss type inequality of the form (1.5).

Second, we generalize (1.6). Unlike the previous approach, for simplicity, we shall use $\|f''\|_p$ instead of $\|f''\|_2$. We prove the following result.

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \rightarrow \mathbb{R}$ be an twice-times differentiable function such that $f'' \in L^p(a, b)$, $1 \leq p \leq \infty$. Then we have*

$$\left| \frac{1}{b-a} (Q(f, y_1, y_2, \dots, y_n) - I(f)) - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| \leq \frac{9}{4} (b-a)^{2-1/p} \|f''\|_p \quad (1.20)$$

for arbitrary sequences $\{x_k\}_{k=1}^n \subset [0, 1]$ and $\{y_k\}_{k=1}^n \subset [0, 1]$ with $n \geq 1$ and

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \frac{n}{2}.$$

As an immediate application of [Theorem 1.2](#), we also obtain

$$\left| f(a + (b - a)x) - \frac{1}{b - a} \int_a^b f(t) dt - (f(b) - f(a)) \left(x - \frac{1}{2} \right) \right| \leq \frac{9}{4} (b - a)^{2-1/p} \|f''\|_p$$

for any $x \in [a, b]$ and any $1 \leq p \leq \infty$.

In the last part of the present paper, we slightly improve [\(1.12\)](#) and [\(1.13\)](#) with weights α_k . Concerning [\(1.13\)](#), we prove the following result theorem.

Theorem 1.3. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $m \geq 2$ be arbitrary. We also let $f : I \rightarrow \mathbb{R}$ be a m th differentiable function and denote $S = \sup_{x \in [a, b]} f^{(m)}(x)$ and $s = \inf_{x \in [a, b]} f^{(m)}(x)$. Then we have*

$$|f - Q(f, x_1, \dots, x_n)| \leq \frac{2m + 5}{4} \frac{(b - a)^{m+1}}{(m + 1)!} (S - s), \tag{1.21}$$

for arbitrary sequences $\{x_k\}_{k=1}^n \subset [0, 1]$ with $n \geq 1$ and

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \frac{n}{2}, \\ \dots \\ \alpha_1 x_1^j + \alpha_2 x_2^j + \dots + \alpha_n x_n^j = \frac{n}{j+1}, \\ \dots \\ \alpha_1 x_1^m + \alpha_2 x_2^m + \dots + \alpha_n x_n^m = \frac{n}{m+1}, \end{cases} \tag{1.22}$$

Regarding to [\(1.12\)](#), we prove the following result.

Theorem 1.4. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \rightarrow \mathbb{R}$ be a m th differentiable function with $m \geq 2$ such that $f^{(m)} \in L^p(a, b)$, $1 \leq p \leq \infty$. Then the following estimate holds*

$$|f - Q(f, x_1, \dots, x_n)| \leq \frac{1}{m!} \left(\left(\frac{1}{mq + 1} \right)^{1/q} + \left(\frac{1}{(m - 1)q + 1} \right)^{1/q} \right) (b - a)^{m+1/q} \|f^{(m)}\|_p, \tag{1.23}$$

for arbitrary sequences $\{x_k\}_{k=1}^n \subset [0, 1]$ satisfying

$$\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \frac{n}{2}, \\ \dots \\ \alpha_1 x_1^j + \alpha_2 x_2^j + \dots + \alpha_n x_n^j = \frac{n}{j+1}, \\ \dots \\ \alpha_1 x_1^{m-1} + \alpha_2 x_2^{m-1} + \dots + \alpha_n x_n^{m-1} = \frac{n}{m}, \end{cases} \tag{1.24}$$

and $1/p + 1/q = 1$.

Before closing this section, we would like to mention that due to the restriction of the technique that we use, inequalities [\(1.19\)](#), [\(1.20\)](#), [\(1.23\)](#), and [\(1.21\)](#) are not sharp. However, the presence of the paper [\[30\]](#) strongly proves that there could be some possibility to get optimal constants for all these inequalities. Besides, it turns out that the right hand sides of [\(1.19\)](#), [\(1.20\)](#), [\(1.23\)](#), and [\(1.21\)](#) do not depend on n but the regularity of the function f . This is because we want to unify all the number of (interpolation) points appearing in all known inequalities mentioned at the beginning of the present paper by n , see [\(1.8\)](#)–[\(1.11\)](#).

Finally, it is worth noting that rather than the classical inequalities mentioned above, other classical inequalities such as the Fejér and Hermite–Hadamard inequalities have also been studied, for example, see [\[13\]](#).

2. Proofs

We spend this section to prove Theorems [\(1.1\)](#)–[\(1.4\)](#). First, we prove [Theorem 1.1](#).

Proof of Theorem 1.1. By using the Taylor formula with the integral remainder, it is not hard to check that

$$f(a + (b - a)y_k) = f(a) + \int_0^{(b-a)y_k} f'(a + t) dt = f(a) + \int_0^{(b-a)} y_k f'(a + y_k t) dt = f(a) + \int_a^b y_k f'(a(1 - y_k) + y_k t) dt.$$

Therefore, by taking the sum for k from 1 to n , we get

$$\sum_{k=1}^n \alpha_k f(a + (b - a)y_k) = nf(a) + \sum_{k=1}^n \left(\alpha_k \int_a^b y_k f'(a(1 - y_k) + y_k t) dt \right),$$

which can be rewritten using our notation as

$$\frac{1}{b-a}Q(f, y_1, y_2, \dots, y_n) = f(a) + \frac{1}{n} \sum_{k=1}^n \left(\alpha_k \int_a^b y_k f'(a(1-y_k) + y_k t) dt \right).$$

Similarly, we obtain

$$\frac{1}{b-a}Q(f, x_1, x_2, \dots, x_n) = f(a) + \frac{1}{n} \sum_{k=1}^n \left(\alpha_k \int_a^b x_k f'(a(1-x_k) + x_k t) dt \right). \tag{2.1}$$

Hence, by subtracting, we arrive at

$$\begin{aligned} & \left| \frac{Q(f, y_1, y_2, \dots, y_n)}{b-a} - \frac{Q(f, x_1, x_2, \dots, x_n)}{b-a} - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| \\ &= \left| \frac{1}{n} \sum_{k=1}^n \alpha_k y_k \int_a^b \left(f'(a(1-y_k) + y_k t) - \frac{f(b) - f(a)}{b-a} \right) dt - \frac{1}{n} \sum_{k=1}^n \alpha_k x_k \int_a^b \left(f'(a(1-x_k) + x_k t) - \frac{f(b) - f(a)}{b-a} \right) dt \right| \\ &\leq \frac{1}{n} \underbrace{\sum_{k=1}^n \alpha_k y_k}_{\leq n} \int_a^b (\Gamma - \gamma) dt + \frac{1}{n} \underbrace{\sum_{k=1}^n \alpha_k x_k}_{=n/2} \int_a^b (\Gamma - \gamma) dt \leq \frac{3}{2} (b-a)(\Gamma - \gamma), \end{aligned} \tag{2.2}$$

where we have used the fact that f' and $(f(b) - f(a))/(b - a)$ belong to $[\gamma, \Gamma]$. From the estimate (2.2), it is necessary to control $Q(f, x_1, x_2, \dots, x_n)$. This can be done if we use $\int_a^b f(t) dt$. This is the content of the next part of the proof. Indeed, thanks to

$$\int_a^b f(t) dt = \int_a^b (b-t)f'(t) dt + (b-a)f(a)$$

and (2.1), some easy calculation first shows that

$$\begin{aligned} \left| \int_a^b f(t) dt - Q(f, x_1, x_2, \dots, x_n) \right| &= \left| \int_a^b (b-t)f'(t) dt - \frac{1}{b-a} \left(\int_a^b (b-t) dt \right) \left(\int_a^b f'(t) dt \right) \right. \\ &\quad + \underbrace{\frac{1}{b-a} \left(\int_a^b (b-t) dt \right) \left(\int_a^b f'(t) dt \right)}_M \\ &\quad - \frac{b-a}{n} \left(\sum_{k=1}^n \left(\alpha_k \int_a^b x_k f'((1-x_k)a + x_k t) dt \right) \right. \\ &\quad \left. - \frac{1}{b-a} \left(\int_a^b \alpha_k x_k dt \right) \left(\int_a^b f'((1-x_k)a + x_k t) dt \right) \right) \Bigg| \\ &= \underbrace{\left| \frac{b-a}{n} \sum_{k=1}^n \frac{1}{b-a} \left(\int_a^b \alpha_k x_k dt \right) \left(\int_a^b f'((1-x_k)a + x_k t) dt \right) \right|}_N. \end{aligned}$$

Clearly, $M = (b-a)(f(b) - f(a))/2$ which implies

$$\frac{1}{2}(b-a)^2\gamma \leq M \leq \frac{1}{2}(b-a)^2\Gamma.$$

For the term N , it is clear that

$$N = \frac{1}{n} \sum_{k=1}^n \left(\int_a^b \alpha_k x_k dt \right) \left(\int_a^b f'((1-x_k)a + x_k t) dt \right) = \frac{1}{n} \sum_{k=1}^n \left((b-a)\alpha_k x_k \int_a^b f'((1-x_k)a + x_k t) dt \right)$$

which yields

$$\underbrace{\frac{1}{n} \sum_{k=1}^n \left((b-a)\alpha_k x_k \int_a^b \gamma dt \right)}_{\gamma(b-a)^2/2} \leq N \leq \underbrace{\frac{1}{n} \sum_{k=1}^n \left((b-a)\alpha_k x_k \int_a^b \Gamma dt \right)}_{\Gamma(b-a)^2/2}.$$

Therefore, the difference $M - N$ is now easy to handle as follows

$$|M - N| \leq \frac{1}{2}(b-a)^2(\Gamma - \gamma).$$

For the remaining terms in the expansion of $\int_a^b f(t)dt - Q(f, x_1, x_2, \dots, x_n)$ above, one may consult the Grüss inequality. Indeed, we can estimate further as follows

$$\left| \int_a^b (b-t)f'(t)dt - \frac{1}{b-a} \left(\int_a^b (b-t)dt \right) \left(\int_a^b f'(t)dt \right) \right| \leq \frac{1}{4} (b-a)^2 (\Gamma - \gamma).$$

We note that

$$\sum_{k=1}^n \alpha_k \int_a^b x_k f'((1-x_k)a + x_k t)dt - \frac{1}{b-a} \sum_{k=1}^n \left(\int_a^b \alpha_k x_k dt \right) \left(\int_a^b f'((1-x_k)a + x_k t)dt \right) = 0.$$

Hence, all in one, we arrive at

$$\begin{aligned} & \frac{1}{b-a} \left| \int_a^b f(t)dt - Q(f, x_1, x_2, \dots, x_n) \right| \\ & \leq \frac{1}{b-a} \left| \int_a^b (b-t)f'(t)dt - \frac{1}{b-a} \left(\int_a^b (b-t)dt \right) \left(\int_a^b f'(t)dt \right) \right| \\ & \quad + \frac{1}{b-a} |M - N| \\ & \leq \frac{1}{b-a} \left(\frac{(b-a)^2}{4} (\Gamma - \gamma) + \frac{1}{2} (b-a)^2 (\Gamma - \gamma) \right) \\ & = \frac{3}{4} (b-a) (\Gamma - \gamma). \end{aligned} \tag{2.3}$$

Having (2.2) and (2.3) yields

$$\begin{aligned} & \left| \frac{Q(f, y_1, y_2, \dots, y_n)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| \\ & \leq \left| \frac{Q(f, y_1, y_2, \dots, y_n)}{b-a} - \frac{Q(f, x_1, x_2, \dots, x_n)}{b-a} \right. \\ & \quad \left. - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| + \frac{1}{b-a} \left| \int_a^b f(t)dt - Q(f, x_1, x_2, \dots, x_n) \right| \\ & \leq \frac{9}{4} (b-a) (\Gamma - \gamma). \end{aligned}$$

The proof is now complete. \square

We now prove Theorem 1.2 whose proof is basically based on Theorem 1.1. The idea is to control $\Gamma - \gamma$ from the above in terms of f'' .

Proof of Theorem 1.2. To prove the theorem, we observe from the Hölder inequality that, for all $u, v \in [a, b]$ satisfying $u \leq v$, there holds

$$|f'(u) - f'(v)| = \left| \int_u^v f''(t)dt \right| \leq \left(\int_u^v |f''(t)|^p dt \right)^{1/p} (v-u)^{1/q} \leq \|f''\|_p (b-a)^{1/q},$$

where $1/p + 1/q = 1$. Thanks to $\Gamma = \sup_{x \in [a,b]} f'(x)$, $\gamma = \inf_{x \in [a,b]} f'(x)$, we immediately have

$$\Gamma - \gamma \leq \|f''\|_p (b-a)^{1/q}.$$

Making use of this and Theorem 1.1, we obtain

$$\left| \frac{1}{b-a} (Q(f, y_1, y_2, \dots, y_n) - I(f)) - \frac{f(b) - f(a)}{n} \sum_{k=1}^n \alpha_k (y_k - x_k) \right| \leq \frac{9}{4} (b-a)^{1+1/q} \|f''\|_p,$$

which now completes the proof because $1 + 1/q = 2 - 1/p$. \square

To prove Theorem 1.3, we follow the same idea and method used in [9] and refer the reader to [9] for details.

Proof of Theorem 1.3. By applying the Taylor formula with the integral remainder to the function $\int_a^x f(t)dt$, we arrive at

$$I(f) = \sum_{k=0}^{m-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) + \int_0^{b-a} \frac{(b-a-t)^m}{m!} f^{(m)}(a+t)dt. \tag{2.4}$$

For each $1 \leq i \leq n$, applying the Taylor formula with the integral remainder again to the function $f(x)$, we now get

$$\begin{aligned} f(a + x_i(b - a)) &= \sum_{k=0}^{m-1} \frac{x_i^k (b - a)^k}{k!} f^{(k)}(a) + \int_0^{x_i(b-a)} \frac{(x_i(b - a) - t)^{m-1}}{(m - 1)!} f^{(m)}(a + t) dt \\ &= \sum_{k=0}^{m-1} \frac{x_i^k (b - a)^k}{k!} f^{(k)}(a) + \int_0^{b-a} \frac{x_i^m (b - a - t)^{m-1}}{(m - 1)!} f^{(m)}(a + x_i t) dt. \end{aligned}$$

Then by summing up and thanks to the first $m - 1$ equations in (1.22), we deduce that

$$\sum_{i=1}^n \alpha_i f(a + x_i(b - a)) = n \sum_{k=0}^{m-1} \frac{(b - a)^k}{(k + 1)!} f^{(k)}(a) + \sum_{i=1}^n \int_0^{b-a} \frac{\alpha_i x_i^m (b - a - t)^{m-1}}{(m - 1)!} f^{(m)}(a + x_i t) dt.$$

In other words, we have proved that

$$Q(f, x_1, \dots, x_n) = \sum_{k=0}^{m-1} \frac{(b - a)^{k+1}}{(k + 1)!} f^{(k)}(a) + \frac{b - a}{n} \sum_{i=1}^n \int_0^{b-a} \frac{\alpha_i x_i^m (b - a - t)^{m-1}}{(m - 1)!} f^{(m)}(a + x_i t) dt. \tag{2.5}$$

Combining (2.4) and (2.5) gives

$$I(f) - Q(f, x_1, \dots, x_n) = I\left(\frac{(b - \cdot)^m}{m!} f^{(m)}\right) - \frac{b - a}{n} \sum_{i=1}^n I\left(\frac{\alpha_i x_i^m (b - \cdot)^{m-1}}{(m - 1)!} f^{(m)}((1 - x_i)a + x_i \cdot)\right).$$

Observe that

$$\frac{(b - a)^m}{(m + 1)!} (f^{(m-1)}(b) - f^{(m-1)}(a)) = \frac{1}{b - a} I\left(\frac{(b - \cdot)^m}{m!}\right) \cdot I(f^{(m)}).$$

Therefore, we can write

$$|I(f) - Q(f, x_1, \dots, x_n)| = \left| \frac{(b - a)^m}{(m + 1)!} (f^{(m-1)}(b) - f^{(m-1)}(a)) + M - \frac{b - a}{n} N - \frac{P}{n} \right|,$$

with

$$M = I\left(\frac{(b - \cdot)^m}{m!} f^{(m)}\right) - \frac{1}{b - a} I\left(\frac{(b - \cdot)^m}{m!}\right) I(f^{(m)}),$$

$$N = \sum_{i=1}^n \left(I\left(\frac{\alpha_i x_i^m (b - \cdot)^{m-1}}{(m - 1)!} f^{(m)}((1 - x_i)a + x_i \cdot)\right) \right) - \frac{1}{b - a} I\left(\frac{\alpha_i x_i^m (b - \cdot)^{m-1}}{(m - 1)!}\right) I(f^{(m)}((1 - x_i)a + x_i \cdot)),$$

$$P = I\left(\frac{\alpha_i x_i^m (b - \cdot)^{m-1}}{(m - 1)!}\right) I(f^{(m)}((1 - x_i)a + x_i \cdot)).$$

Making use of the Grüss inequality, see [9, Lemma 5], gives that

$$|M| \leq \frac{1}{4} \frac{(b - a)^{m+1}}{m!} (S - s)$$

and that

$$|N| \leq \frac{1}{4} \sum_{i=1}^n \frac{(b - a)^m \alpha_i x_i^m}{(m - 1)!} (S - s) = \frac{n}{4} \frac{(b - a)^m}{(m + 1)(m - 1)!} (S - s).$$

For remaining terms, it is clear that

$$\frac{(b - a)^{m+1}}{(m + 1)!} s \leq \frac{(b - a)^m}{(m + 1)!} (f^{(m-1)}(b) - f^{(m-1)}(a)) \leq \frac{(b - a)^{m+1}}{(m + 1)!} S,$$

while a direct calculation shows

$$P = \sum_{i=1}^n \frac{\alpha_i x_i^m (b - a)^m}{m!} I(f^{(m)}((1 - x_i)a + x_i \cdot)).$$

Consequently, thanks to $\sum_{k=1}^n \alpha_k x_k^m = n/(m + 1)$ and here is the only place we make use of the last equation in (1.22), there holds

$$\frac{n(b-a)^{m+1}}{(m+1)!} s \leq P \leq \frac{n(b-a)^{m+1}}{(m+1)!} S.$$

In other words, we have proved that

$$\left| \frac{(b-a)^m}{(m+1)!} (f^{(m-1)}(b) - f^{(m-1)}(a)) - \frac{P}{n} \right| \leq \frac{(b-a)^{m+1}}{(m+1)!} (S-s).$$

Thus, **Theorem 1.3** follows by using the triangle inequality. \square

We now prove **Theorem 1.4**. To this purpose, we follow the same idea and method used in [8] and we refer the reader to [8] for details.

Proof of Theorem 1.4. From the proof of **Theorem 1.3** and using the triangle inequality, we obtain

$$|f - Q(f, x_1, \dots, x_n)| \leq \left\| \frac{(b-\cdot)^m}{m!} f^{(m)} \right\|_1 + \frac{b-a}{n} \sum_{i=1}^n \left\| \frac{\alpha_i x_i^m (b-\cdot)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i \cdot) \right\|_1. \tag{2.6}$$

Thanks to [8, Eq. (10)], the first term sitting on the right hand side of (2.6) can be estimated as follows

$$\left\| \frac{(b-\cdot)^m}{m!} f^{(m)} \right\|_1 \leq \frac{1}{m!} \left(\frac{(b-a)^{mq+1}}{mq+1} \right)^{1/q} \|f^{(m)}\|_p. \tag{2.7}$$

For the second term, we also note from [8] that

$$x_i \|f^{(m)}((1-x_i)a + x_i \cdot)\|_p \leq \begin{cases} x_i \|f^{(m)}\|_\infty, & \text{if } p = \infty, \\ \|f^{(m)}\|_p, & \text{if } 1 \leq p < \infty. \end{cases}$$

Thanks to $x_i \in [0, 1]$, we can write $x_i \|f^{(m)}((1-x_i)a + x_i \cdot)\|_p \leq \|f^{(m)}\|_p$ in any case. Making use of the Hölder inequality, one can estimate the second term on the right hand side of (2.6) as follows

$$\begin{aligned} & \frac{b-a}{n} \sum_{i=1}^n \left\| \frac{\alpha_i x_i^m (b-\cdot)^{m-1}}{(m-1)!} f^{(m)}((1-x_i)a + x_i \cdot) \right\|_1 \\ & \leq \frac{b-a}{n} \sum_{i=1}^n \frac{\alpha_i x_i^m}{(m-1)!} \|f^{(m)}((1-x_i)a + x_i \cdot)\|_p \| (b-\cdot)^{m-1} \|_q \\ & \leq \frac{b-a}{n} \sum_{i=1}^n \frac{\alpha_i x_i^{m-1}}{(m-1)!} \|f^{(m)}\|_p \| (b-\cdot)^{m-1} \|_q \\ & = \frac{\|f^{(m)}\|_p}{m!} \left(\frac{(b-a)^{mq+1}}{(m-1)q+1} \right)^{1/q}. \end{aligned} \tag{2.8}$$

Combining relations (2.6)–(2.8), we conclude that

$$|f - Q(f, x_1, \dots, x_n)| \leq \frac{1}{m!} \left(\frac{(b-a)^{mq+1}}{mq+1} \right)^{1/q} \|f^{(m)}\|_p + \frac{\|f^{(m)}\|_p}{m!} \left(\frac{(b-a)^{mq+1}}{(m-1)q+1} \right)^{1/q}$$

and **Theorem 1.4** follows. \square

It is interesting to note that a weaker version for the inequality (1.23) can be derived from **Theorem 1.3** so long as $m \geq 3$. Indeed, similar to the proof of **Theorem 1.2**, we can estimate $f^{(m-1)}$ in terms of $\|f^{(m)}\|_p$ to obtain

$$S - s \leq \|f^{(m)}\|_p (b-a)^{1/q}.$$

From this, **Theorem 1.3** with m replaced by $m-1$, and thanks to $m \geq 3$, we obtain

$$|f - Q(f, x_1, \dots, x_n)| \leq \frac{2m+3}{4m!} (b-a)^{m+1/q} \|f^{(m)}\|_p.$$

Clearly, the preceding estimate is weaker than that of **Theorem 1.4** since

$$\frac{1}{m!} \left(\left(\frac{1}{mq+1} \right)^{1/q} + \left(\frac{1}{(m-1)q+1} \right)^{1/q} \right) < \frac{2}{m!} < \frac{2m+3}{4m!}$$

for any $m \geq 3$ and any $q \geq 1$. Note that here we require $m \geq 3$ rather than $m \geq 2$ as in **Theorem 1.4** since we have to make use of **Theorem 1.3** with m replaced by $m-1$. Having this fact, the condition $m-1 \geq 2$ automatically leads to $m \geq 3$ as claimed.

Before closing our paper, we would like to comment on the similarity between the right hand sides of (1.12) and (1.20). Indeed, more or less, the right hand side of (1.20) is a particular case of that of (1.12) when one uses $m = 2$ except the power of $b - a$. Note that, in the case $m = 2$, the right hand side of the estimate (1.12) reads as

$$\frac{1}{2} \left(\left(\frac{1}{2q+1} \right)^{1/q} + \left(\frac{1}{q+1} \right)^{1/q} \right) (b-a)^{3-1/p} \|f''\|_p.$$

The difference between the power of $b - a$ in (1.12) and (1.20) relies on the fact that our expansion (2.1) is only up to f' . One way to equalize these orders is to use some expansion for higher orders together with the Hölder inequality. For interested reader, we refer to the proof of (1.12) in [8, Theorem 4]. Therefore, we expect that there exists a positive constant C which only depends on p and m such that

$$\left| \frac{1}{b-a} (Q(f, y_1, y_2, \dots, y_n) - I(f)) - \frac{f(b) - f(a)}{n} \sum_{k=1}^n (y_k - x_k) \right| \leq C(b-a)^{m+1-1/p} \|f^{(m)}\|_p, \quad (2.9)$$

where the sequence $\{x_k\}_k$ satisfies the same system of equations for (1.12). We shall not prove (2.9) here and leave it for interested reader.

Acknowledgments

The authors would like to thank the two anonymous referees for careful reading of our manuscript and the valuable comment and suggestions, especially for pointing out a mistake in the first paragraph after the proof of Theorem 1.4 in the previous version of the paper. V.N. Huy was partially supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 101.01-2011.32. Q.A. Ngô acknowledges the support of the Région Centre through the convention n° 00078780 during the period 2012-2014.

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