IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. **31** (2014) 195014 (20pp) doi:10.1088/0264-9381/31/19/195014

A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor

Romain Gicquaud and Quôc Anh Ngô

Laboratoire de Mathématiques et de Physique Théorique, Université François Rabelais de Tours, Parc de Grandmont, F-37200 Tours, France

E-mail: romain.gicquaud@lmpt.univ-tours.fr, quoc-anh.ngo@lmpt.univ-tours.fr and bookworm_vn@yahoo.com

Received 25 March 2014
Accepted for publication 7 July 2014
Published 22 September 2014

Abstract

In this paper, we give a construction of the solutions to the Einstein constraint equations using the well-known conformal method. Our method gives a result similar to the one in [14, 15, 21], namely existence when the so-called TT-tensor σ is small and the Yamabe invariant of the manifold is positive. The method we describe is, however, much simpler than the original method and allows easy extensions to several other problems. Some non-existence results are also considered.

Keywords: Einstein constraint equations, non-CMC, conformal method, implicit function theorem, positive Yamabe invariant, small TT-tensor
PACS numbers: 02.30.Jr, 02.40.Ky, 04.20.Ex

1. Introduction

1.1. The Einstein constraint equations

The initial data for the Cauchy problem in general relativity are usually given in terms of the geometry of the Cauchy surface $(M, \widehat{g})$ in the spacetime $(\mathcal{M}, \mathbf{g})$ of dimension $n+1$ with $n \geqslant 3$. Assuming that the spacetime $\mathcal{M}$ is globally hyperbolic and M is a spacelike Cauchy surface, one can define the metric $\widehat{g}$ induced on M by the spacetime metric $\mathbf{g}$ and the second fundamental form $\widehat{K}$ of M in $\mathcal{M}$. It follows from the Einstein equations together with the

Gauss and Codazzi equations that $\widehat{g}$ and $\widehat{K}$ are related by the following equations

$$\begin{cases} \mathrm{Scal}_{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\,\widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} = 2\rho, & (1.1a) \\ \mathrm{div}_{\widehat{g}}\,\widehat{K} - d(\mathrm{tr}_{\widehat{g}}\,\widehat{K}) = j, & (1.1b) \end{cases}$$

where ρ and j are related to other fields, such as matter fields and the electromagnetic field, that one wants to include in the universe under consideration. Also, in (1.1), $\mathrm{Scal}_{\widehat{g}}$ is the scalar curvature of $\widehat{g}$. To keep things simple, we will consider only the gravitational field, hence forcing $\rho \equiv 0$ and $j \equiv 0$.

A simple dimension-counting argument shows that the system (1.1) is under-determined, and thus it is generally hard to solve (1.1) in this form. To overcome this difficulty, we need to decompose both $\widehat{g}$ and $\widehat{K}$ into given data and unknowns that will have to be adjusted so that equations (1.1*a*) and (1.1*b*) are fulfilled. Several such splittings exist and we refer the reader to [1] for a detailed review of some known results on the constraint equations. In the literature, the most commonly used method is the conformal method, which we briefly describe now. We invite the reader to have a look at the excellent recent work of Maxwell [20] for a deep understanding of this method and its connection to other widely used methods.

The given data in the conformal method consist of

- a Riemannian manifold (M, g),
- a function $\tau\colon M \to \mathbb{R}$,
- and a symmetric 2-tensor σ on M, which is traceless and transverse in the following sense

$$\mathrm{tr}_g \sigma \equiv 0, \qquad \mathrm{div}_g \sigma \equiv 0.$$

As shorthand, we will call σ a TT-tensor. The unknowns in the conformal method are

- a positive function $\varphi\colon M \to \mathbb{R}_+^*$,
- and a 1-form W.

Combining all these elements, one can form $(\widehat{g}, \widehat{K})$ as follows:

$$\begin{aligned} \widehat{g} &= \varphi^{N-2} g, \\ \widehat{K} &= \frac{\tau}{n}\widehat{g} + \varphi^{-2}\left(\sigma + \mathbb{L}_g W\right), \end{aligned} \tag{1.2}$$

where $N := 2n/(n-2)$ and $\mathbb{L}_g$ is the conformal Killing operator given by

$$\mathbb{L}_g W_{ij} := \nabla_i W_j + \nabla_j W_i - \frac{2}{n}\nabla^k W_k g_{ij},$$

with ∇ the Levi–Civita connection associated with the background metric g. Here, τ is the mean curvature of M as a hypersurface of $(\mathcal{M}, \mathbf{g})$ given by

$$\tau = \widehat{g}^{ij}\widehat{K}_{ij}.$$

The choice for σ and W in (1.2) is related to the York splitting; see the remark at the end of section 4.1.

Using the decomposition (1.2), the constraint (1.1) become a system of PDEs for (φ, W) as follows:

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g\varphi + \mathrm{Scal}_g\varphi = -\dfrac{n-1}{n}\tau^2\varphi^{N-1} + \left|\sigma + \mathbb{L}W\right|_g^2\varphi^{-N-1}, & (1.3a)\\ \qquad\qquad \Delta_{\mathbb{L},g}W = \dfrac{n-1}{n}\varphi^N d\tau, & (1.3b)\end{cases}$$

where we denote $\Delta_g\varphi = \mathrm{div}_g(\nabla_g\varphi)$ and $\Delta_{\mathbb{L},g}W = \mathrm{div}_g(\mathbb{L}_gW)$. In the literature, equation (1.3*a*) is commonly known as the Lichnerowicz equation, while equation (1.3*b*) is usually called the vector equation.

The system (1.3) is notoriously hard to solve except in the case when τ is a constant function, which is now well understood; see for instance [16]. Indeed, when τ is constant, equation (1.3*b*) only involves W and generically implies that $W \equiv 0$. Therefore, one is left with solving the Lichnerowicz equation (1.3*a*) without any W. However, everything dramatically changes when τ is no longer constant. Perturbation arguments can be used to address the case when $\mathrm{d}\tau$ is small in some sense. But, until recently, very few results were known for arbitrary τ. Two major breakthroughs were first obtained by Holst *et al* in [15, 16], by Maxwell in [23], and then by Dahl *et al* in [4].

Usually, standard methods to solve elliptic PDEs require an *a priori* knowledge of the solutions, i.e. nice domains in which one can try to apply fixed point theorems, fixed point arguments, etc. However, via a simple scaling argument, changing φ to $\lambda\varphi$ where $\lambda \gg 1$ shows that the two dominant terms in the Lichnerowicz equation are $(n-1)/n\,\tau^2\varphi^{N-1}$ and $|\mathbb{L}_gW|_g^2\varphi^{-N-1}$. These two terms have the same scaling behavior but come up with opposite signs in the Lichnerowicz equation (1.3*a*). Although the first term has the right sign and in fact helps us in applying the maximum principle, the second one has the wrong sign and eventually destroys any attempt to get an *a priori* upper bound for φ when $\mathrm{d}\tau$ is not small.

1.2. The Holst–Nagy–Tsogtgerel–Maxwell method

Losing such an *a priori* estimate, a very nice idea was proposed in [15, 16]. This was pushed further in [23], and consists of looking for solutions of the system (1.3) with φ and W very close to zero to make the two dominant terms irrelevant. To do this, the manifold (M, g) needs to be closed with a positive Yamabe invariant, $\mathcal{Y}(g) > 0$ (see equation (1.4)). Consequently, the scalar curvature Scal_g becomes in some sense the dominant term. In addition, σ needs to be small to control the right-hand side of equation (1.3*a*). The theorem they obtained is the following.

Theorem 1.1. (see [23]). *Let M be a compact Riemannian manifold without boundary. Given $p > n$, let $g \in W^{2,p}$, $\tau \in W^{1,p}$, and $\sigma \in W^{1,p}$, $\sigma \not\equiv 0$ be given data. Assume that the Yamabe invariant $\mathcal{Y}(g)$ is strictly positive and that g has no conformal Killing vector fields. Then, if $\|\sigma\|_{L^\infty}$ is small enough, there exists at least one solution $(\varphi, W) \in W^{2,p}(M, \mathbb{R}) \times W^{2,p}(M, T^*M)$ to the system (1.3).*

Assume that M is a compact manifold without boundary, we recall that the Yamabe invariant $\mathcal{Y}(g)$ of (M, g) is defined as

$$\mathcal{Y}(g) = \inf_{0\not\equiv\varphi\in W^{1,2}(M,\mathbb{R})} \frac{\int_M \left((4(n-1)/(n-2))|\mathrm{d}\varphi|_g^2 + \mathrm{Scal}_g\varphi^2\right)\mathrm{dvol}_g}{\left(\int_M \varphi^N\,\mathrm{dvol}_g\right)^{N/2}} \qquad (1.4)$$

The method of [15, 16] was recently adapted to other situation such as asymptotically Euclidean manifolds in [7], asymptotically cylindrical manifolds in [19], compact manifolds with boundary in [5, 14], and to asymptotically Euclidean manifolds with boundary in [13]. As can be seen from the statement of theorem 1.1, and as we mentioned earlier, the smallness of $\| \sigma \|_{L^\infty}$ was used. However, it is worth mentioning that such an L^∞-smallness assumption can be weakened to small enough $\| \sigma \|_{L^2}$; see [24].

1.3. The Dahl–Gicquaud–Humbert method

The idea of [4] goes in the opposite direction to the method used in section 1.2. Intuitively, the idea of [4] is to study what happens if φ and W become very large, i.e. what prevents the existence of an *a priori* estimate. The answer to this is heuristically that if φ can become very large, by setting $\gamma = \| \varphi \|_{L^\infty}$ and by renormalizing φ, W, and σ as follows:

$$\widetilde{\varphi} = \gamma^{-1}\varphi, \qquad \widetilde{W} = \gamma^{-N} W, \qquad \widetilde{\sigma} = \gamma^{-N}\sigma,$$

it turns out that $\widetilde{\varphi}$ and $\widetilde{W}$ satisfy the following system:

$$\begin{cases} \dfrac{1}{\gamma^{N-2}}\left(-\dfrac{4(n-1)}{n-2}\Delta_g \widetilde{\varphi} + \mathrm{Scal}_g \widetilde{\varphi}\right) & = & -\dfrac{n-1}{n}\tau^2 \widetilde{\varphi}^{N-1} + \left|\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}\right|_g^2 \widetilde{\varphi}^{-N-1}, \\ \Delta_{\mathbb{L},g} \widetilde{W} & = & \dfrac{n-1}{n}\widetilde{\varphi}^N \mathrm{d}\tau. \end{cases}$$

In the limit as $\gamma \to +\infty$, one is left with

$$\frac{n-1}{n}\tau^2 \widetilde{\varphi}^{N-1} = \left|\mathbb{L}_g \widetilde{W}\right|_g^2 \widetilde{\varphi}^{-N-1}.$$

Therefore, $\widetilde{W}$ becomes a non-trivial solution to the so-called limit equation

$$\Delta_{\mathbb{L},g} \widetilde{W} = \sqrt{\frac{n-1}{n}} \left|\mathbb{L}_g \widetilde{W}\right|_g \frac{\mathrm{d}\tau}{\tau}. \tag{1.5}$$

The rigorous argument leads to a similar limit equation with a parameter $\alpha \in (0, 1]$ given as follows:

$$\Delta_{\mathbb{L},g} \widetilde{W} = \alpha\sqrt{\frac{n-1}{n}} \left|\mathbb{L}_g \widetilde{W}\right|_g \frac{\mathrm{d}\tau}{\tau}. \tag{1.6}$$

The main theorem of [4] can be stated as follows.

Theorem 1.2. *Let M be a compact Riemannian manifold without boundary. Given $p > n$, let $g \in W^{2,p}$, $\tau \in W^{1,p}$, and $\sigma \in W^{1,p}$ be given data. Assume that g has no conformal Killing vector fields, $\tau > 0$ and that $\sigma \not\equiv 0$ if $\mathcal{Y}(g) \geqslant 0$. If the limit equation (1.6) admits no non-zero solution $\widetilde{W}$ for all values of the parameter $\alpha \in (0, 1]$, then there exists at least one solution $(\varphi, W) \in W^{2,p}(M, \mathbb{R}) \times W^{2,p}(M, T^*M)$ to the system (1.3).*

It is worth noticing that the result in [4] requires that τ is bounded away from zero, however, it involves no assumption on the Yamabe invariant $\mathcal{Y}(g)$. A simplified proof of theorem 1.2 appears in [24].

This method was adapted to several other contexts such as asymptotically hyperbolic manifolds in [12] and asymptotically cylindrical manifolds in [8]. In particular, strong results are obtained for negatively curved manifolds; see [12, proposition 6.2 and remark 6.3]. The case of asymptotically Euclidean manifolds and compact manifolds with boundary is currently work in progress [6, 11]. New difficulties show up in these cases.

1.4. Objective and outline of the paper

As we have already seen from sections 1.2 and 1.3, both of the approaches we presented are dual in a certain sense. The first one constructs solutions that are very close to zero, while the second is a means to ensure control on the size of the solutions. In this note, we emphasize the duality between both methods, showing that the Holst *et al* method can be rephrased as a scaling argument. This duality can potentially deepen further, recasting both methods in a single framework; see remark 2.3. This also sheds light on the role of the assumptions of the main theorem of [23].

Nevertheless, our new method leads to a result that is not as good as the one of [23] but much simpler than the original one, and also appears quite versatile.

In section 2, we present in detail the simplest case of our method, namely when the manifold is closed. Also in this section, a non-existence result is presented. Then, we have a quick look at the asymptotically Euclidean case in section 3 and at compact manifolds with boundary in section 4.

2. The closed case

In this section, we are interested in studying solutions of (1.3) when the underlying manifold M is compact without boundary. In the first part of this section, we prove a result that basically says that (1.3) is solvable when $\mathcal{Y}(g) > 0$ and $\sigma \not\equiv 0$ is small enough; see theorem 2.1. Then, we improve [4, theorem 1.7] by showing that (1.3) admits no solution provided $\mathcal{Y}(g) > 0$, $\sigma \equiv 0$, and $d\tau/\tau$ is small in the L^n-norm.

2.1. Existence results for small but non-vanishing TT-tensors

The main result of this subsection is the following.

Theorem 2.1. *Let M be a compact manifold without boundary. Given $p > n$, let $g \in W^{2,p}(M, S^2(M))$, $\tau \in W^{1,p}(M, \mathbb{R})$ and $\widetilde{\sigma} \in W^{1,p}(M, S^2(M))$, $\widetilde{\sigma} \not\equiv 0$ be given data. Assume that the Yamabe invariant $\mathcal{Y}(g)$ is strictly positive and that g has no conformal Killing vector fields. There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ there exists at least one solution $(\varphi, W) \in W^{2,p}(M, \mathbb{R}) \times W^{2,p}(M, T^*M)$ to the system (1.3) with $\sigma = \eta\widetilde{\sigma}$.*

Note that this theorem is not as good as theorem 1.1. Indeed, η_0 depends *a priori* on $\widetilde{\sigma}$ in an unknown way, while theorem 1.1 asserts that the system (1.3) with $\sigma = \eta\widetilde{\sigma}$ has a solution provided that $\| \sigma \|_{L^\infty} = |\eta| \| \widetilde{\sigma} \|_{L^\infty}$ is small enough (less than some $\varepsilon > 0$). So the corresponding η_0 would be $\varepsilon/\| \widetilde{\sigma} \|_{L^\infty}$. Nevertheless, the proof appears to be constructive since it relies on the sub- and super-solutions method and on the implicit function theorem. For the sake of clarity, we divide the proof into several claims.

Claim 1. Let $\widetilde{\sigma} \not\equiv 0$ be a TT-tensor belonging to $W^{1,p}(M, S^2(M))$. Then there exists a unique solution $\widetilde{\varphi}_0 \in W^{2,p}(M, \mathbb{R})$ to the following equation

$$-\frac{4(n-1)}{n-2}\Delta_g \widetilde{\varphi} + \mathrm{Scal}_g \widetilde{\varphi} = \left|\widetilde{\sigma}\right|_g^2 \widetilde{\varphi}^{-N-1}, \tag{2.1}$$

Proof. The proof is standard; see [21]. Note that this equation is nothing but the Lichnerowicz equation with $\tau \equiv 0$ and $W \equiv 0$. To prove existence, we rely on the classical sub- and super-solutions method described, for example, in [17, proposition 2]. Since $\mathcal{Y}(g) > 0$, there exists a positive $W^{2,p}(M)$ function ψ so that the metric $\overline{g} = \psi^{N-2} g$ has positive constant scalar curvature. Setting $\overline{\varphi} = \psi^{-1}\widetilde{\varphi}$, equation (2.1) transforms into

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\overline{\varphi} + \mathrm{Scal}_{\overline{g}}\overline{\varphi} = \left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}} \overline{\varphi}^{-N-1}. \tag{2.2}$$

To solve (2.2) for $\overline{\varphi}$, we follow the method of sub- and super-solutions by constructing a sub-solution $\overline{\varphi}_-$ and a super-solution $\overline{\varphi}^+$ as follows. Let $\overline{u} \in W^{2,p}(M)$ denote the solution to the following linear equation:

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\overline{u} + \mathrm{Scal}_{\overline{g}}\overline{u} = \left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}}.$$

It follows from the strong maximum principle that $\overline{u} > 0$ in M. By setting

$$\overline{\varphi}_- = (\max \overline{u})^{-\frac{N+1}{N+2}}\overline{u}$$

and

$$\overline{\varphi}_+ = \left(\min \overline{u}\right)^{-\frac{N+1}{N+2}}\overline{u},$$

one readily checks that $\overline{\varphi}_+$ and $\overline{\varphi}_-$ are super- and sub-solutions for (2.2), respectively, meaning that

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\overline{\varphi}_- + \mathrm{Scal}_{\overline{g}}\overline{\varphi}_- \leqslant \left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}} \left(\overline{\varphi}_-\right)^{-N-1}$$

and that

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\overline{\varphi}_+ + \mathrm{Scal}_{\overline{g}}\overline{\varphi}_+ \geqslant \left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}} \left(\overline{\varphi}_+\right)^{-N-1}.$$

Hence, there exists (at least) one solution $\overline{\varphi}$ to equation (2.2) and it leads to a solution $\widetilde{\varphi}_0 = \psi\overline{\varphi}$ to equation (2.1) as well.

Uniqueness is also easy to prove. Indeed, let $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_0'$ be two solutions to equations (2.1) and denote $\overline{\varphi}_0 = \psi^{-1}\widetilde{\varphi}_0$ and $\overline{\varphi}_0' = \psi^{-1}\widetilde{\varphi}_0'$. A simple calculation leads us to the following equality:

$$\begin{aligned}
-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\left(\overline{\varphi}_0 - \overline{\varphi}_0'\right) + {} & \mathrm{Scal}_{\overline{g}}\left(\overline{\varphi}_0 - \overline{\varphi}_0'\right) \\
&= \left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}} \left(\frac{1}{\left(\overline{\varphi}_0\right)^{N+1}} - \frac{1}{\left(\overline{\varphi}_0'\right)^{N+1}}\right) \\
&= \underbrace{-(N+1)\left|\psi^{-2}\widetilde{\sigma}\right|^2_{\overline{g}} \int_0^1 \frac{\mathrm{d}x}{\left(x\overline{\varphi}_0 + (1-x)\overline{\varphi}_0'\right)^{N+2}}}_{:=f}\left(\overline{\varphi}_0 - \overline{\varphi}_0'\right),
\end{aligned}$$

where the term f is obviously non-negative. This then implies

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\left(\overline{\varphi}_0 - \overline{\varphi}_0'\right) + \left(\mathrm{Scal}_{\overline{g}} + f\right)\left(\overline{\varphi}_0 - \overline{\varphi}_0'\right) = 0.$$

Since $\mathrm{Scal}_{\overline{g}} + f > 0$, we immediately conclude that $\overline{\varphi}_0 - \overline{\varphi}_0' \equiv 0$. This proves the uniqueness of the solution $\widetilde{\varphi}_0$ as claimed. □

Remark 2.2. As can be seen, the existence of such a metric $\overline{g}$ in the proof of claim 1 does not need the full strength of the Yamabe theorem, we could only require that $\overline{g}$ has positive scalar curvature. However, this claim strongly relies on the positivity of the Yamabe invariant $\mathscr{Y}(g)$. Indeed, assume that there exists a positive solution $\widetilde{\varphi}$ to equation (2.1), the scalar curvature $\mathrm{Scal}_{\widehat{g}}$ of the metric $\widehat{g} = \varphi^{N-1}g$ satisfies

$$\mathrm{Scal}_{\widehat{g}} = \varphi^{1-N}\left(-\frac{4(n-1)}{n-2}\Delta_g\varphi + \mathrm{Scal}_g\varphi\right) = \varphi^{-2N}\left|\widetilde{\sigma}\right|_g^2.$$

Hence, the scalar curvature of $\widehat{g}$ is non-negative and not identically zero.

Thus, $\mathcal{Y}(g) = \mathcal{Y}(\widehat{g}) > 0$. This partially explains why this method cannot be adapted to asymptotically hyperbolic manifolds.

Now, we introduce the following μ–deformed system of (1.3):

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g\widetilde{\varphi} + \mathrm{Scal}_g\widetilde{\varphi} = -\dfrac{n-1}{n}\tau^2\mu^2\widetilde{\varphi}^{N-1} + \left|\widetilde{\sigma} + \mathbb{L}\widetilde{W}\right|_g^2\widetilde{\varphi}^{-N-1}, & (2.3a) \\ \Delta_{\mathbb{L},g}\widetilde{W} = \dfrac{n-1}{n}\widetilde{\varphi}^N\mu d\tau. & (2.3b) \end{cases}$$

Note that this system is obtained from (1.3) by changing the mean curvature τ simply by $\mu\tau$.

Claim 2. There exists $\varepsilon > 0$ such that the system (2.3) admits a solution $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu) \in W^{2,p}(M, \mathbb{R}) \times W^{2,p}(M, T^*M)$ for all $\mu \in [0, \varepsilon)$.

Proof. The proof is based on the implicit function theorem. First, we define the operator

$$F: \mathbb{R} \times W_+^{2,p}(M, \mathbb{R}) \times W^{2,p}\left(M, T^*M\right) \to L^p(M, \mathbb{R}) \times L^p\left(M, T^*M\right)$$

as follows:

$$F\left(\mu, \widetilde{\varphi}, \widetilde{W}\right) = \begin{pmatrix} -\dfrac{4(n-1)}{n-2}\Delta_g\widetilde{\varphi} + \mathrm{Scal}_g\widetilde{\varphi} + \dfrac{n-1}{n}\tau^2\mu^2\widetilde{\varphi}^{N-1} - \left|\widetilde{\sigma} + \mathbb{L}_g\widetilde{W}\right|_g^2\widetilde{\varphi}^{-N-1} \\ \Delta_{\mathbb{L},g}\widetilde{W} - \dfrac{n-1}{n}\widetilde{\varphi}^N\mu\mathrm{d}\tau \end{pmatrix}.$$

It is readily checked that F is a C^1-mapping. Notice that

$$F\left(0, \widetilde{\varphi}_0, 0\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\widetilde{\varphi}_0$ is the solution found in claim 1. All we need to do is prove that the partial derivative of F with respect to $(\widetilde{\varphi}, \widetilde{W})$ is an isomorphism at $(0, \widetilde{\varphi}_0, 0)$. To this end, we first observe that the differential $\mathcal{D}F_{(0,\widetilde{\varphi}_0,0)}$ is given by

$$\mathcal{D}F_{(0,\widetilde{\varphi}_0,0)}(0,\widetilde{\theta},\widetilde{Z}) = \left(\begin{array}{c|c} -\frac{4(n-1)}{n-2}\Delta_g + \mathrm{Scal}_g + (N+1)\left|\widetilde{\sigma}\right|_g^2 \widetilde{\varphi}_0^{-N-2} & -2\widetilde{\varphi}_0^{-N-1}\langle\widetilde{\sigma}_g, \mathbb{L}_g\cdot\rangle \\ \hline 0 & \Delta_{\mathbb{L},g} \end{array}\right)\begin{pmatrix}\widetilde{\theta}\\ \widetilde{Z}\end{pmatrix}.$$

Note that $\mathcal{D}F_{(0,\widetilde{\varphi}_0,0)}(0,\widetilde{\theta},\widetilde{Z})$ is triangular, meaning that the second line of the two-by-two block matrix above does not depend on $\widetilde{\theta}$. Thus, the invertibility of $\mathcal{D}F_{(0,\widetilde{\varphi}_0,0)}$ follows from the fact that the diagonal terms

$$\begin{aligned} H: W^{2,p}(M,\mathbb{R}) \to &\ L^p(M,\mathbb{R}) \\ \widetilde{\theta} \mapsto &\ -\frac{4(n-1)}{n-2}\Delta_g\widetilde{\theta} + \mathrm{Scal}_g\widetilde{\theta} + (N+1)\left|\widetilde{\sigma}\right|_g^2 \widetilde{\varphi}_0^{-N-2}\widetilde{\theta} \end{aligned}$$

and

$$\begin{aligned} V: W^{2,p}\left(M, T^*M\right) \to &\ L^p\left(M, T^*M\right) \\ \widetilde{Z} \mapsto &\ \Delta_{\mathbb{L},g}\widetilde{Z} \end{aligned}$$

are invertible. The invertibility of V follows from [23, proposition 5], while H is a Fredholm map of index 0. Since $\mathcal{Y}(g) > 0$, the conformal Laplacian is positive definite. Hence, for any given $u \in W^{2,p}(M)$ with $u \not\equiv 0$, we calculate to obtain

$$\begin{aligned} \int_M uH(u)\,\mathrm{dvol}_g = &\ \underbrace{\int_M \left(\frac{4(n-1)}{n-2}\left|du\right|_g^2 + \mathrm{Scal}_g u^2\right)\mathrm{dvol}_g}_{>0} \\ &+ \underbrace{\int_M (N+1)\left|\widetilde{\sigma}\right|_g^2 \widetilde{\varphi}^{-N-2}u^2\,\mathrm{dvol}_g}_{\geqslant 0} > 0. \end{aligned}$$

Hence, H has a trivial kernel. Thus, we have shown that $\mathcal{D}F_{(0,\widetilde{\varphi}_0,0)}$ is an isomorphism as claimed. □

The last claim is just a straightforward calculation, therefore we omit its proof.

Claim 3. Set

$$\left\{\begin{array}{rcl} \varphi_\mu &=& \mu^{\frac{2}{N-2}}\widetilde{\varphi}_\mu, \\ W_\mu &=& \mu^{\frac{N+2}{N-2}}\widetilde{W}_\mu, \\ \sigma_\mu &=& \mu^{\frac{N+2}{N-2}}\widetilde{\sigma}. \end{array}\right.$$

If $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu)$ solves (2.3), the (φ_μ, W_μ) solves (1.3) with $\sigma = \sigma_\mu$.

Finally, the proof of theorem 2.1 follows by setting $\eta_0 = \varepsilon^{(N+2)/(N-2)}$, where ε is the constant appearing in claim 2.

Remark 2.3. It is quite appealing to use the deformation (2.3) of the conformal constraint equations to get a new proof of the limit equation criterion as in [4]. Indeed, the system (2.3) could be studied using the Leray–Schauder fixed point theorem, which would allow μ to go up to 1 (hence $\widetilde{\sigma}$ would be set equal to the desired σ). Assuming that the set of $(\widetilde{\varphi}, \widetilde{W}, \mu)$ solutions to (2.3) with $0 \leqslant \mu \leqslant 1$ is bounded, the Leray–Schauder theorem would guarantee that the system (1.3) has (at least) one solution. If this set is unbounded, the argument presented in section 1.3 would lead to the existence of a non-trivial solution to equation (1.5). Hence, the main result of [4] could be strengthened, getting rid of parameter α (which appears because we introduce a different deformed system there). Such a result would show that the methods of [15, 16] and [4] are two facets of a deeper method. However, one serious difficulty appears in attempting this proof: one has to ensure that if $\widetilde{\varphi}$ (or $\widetilde{W}$) diverges, then μ stays away from 0.

2.2. A non-existence result

The assumption on σ, namely that it has to be small but cannot be zero, looks weird at first sight and one can wonder if the hypothesis $\sigma \not\equiv 0$ is purely technical. As can be seen from [4, 18, 23], σ is used to show that the function φ solving the Lichnerowicz equation (1.3*a*) is bounded away from zero. We give a slight improvement of [18] and [3, theorem 1.7] to the class of metrics with positive Yamabe invariant, showing that the assumption $\sigma \not\equiv 0$ is needed.

As in [4], the manifold M is still assumed to admit no conformal Killing vector fields. Recall that the proof presented in [4] depends on a Sobolev quotient for the operator $\mathbb{L}_g$, i.e. whenever M admits no non-zero conformal Killing vector fields, the following holds:

$$C_g = \inf_{0 \not\equiv V \in W^{1,2}\left(M, T^*M\right)} \frac{\left(\int_M \left|\mathbb{L}_g V\right|_g^2 \mathrm{dvol}_g\right)^{1/2}}{\left(\int_M |V|_g^N \mathrm{dvol}_g\right)^{1/N}} > 0 \tag{2.4}$$

The main result in this subsection is the following.

Theorem 2.4. *Assume that $g \in W^{2,p}(M, S^2(M))$ has non-negative Yamabe invariant $\mathcal{Y}(g)$ and (M,g) has no conformal Killing vector fields. If $\sigma \equiv 0$ and $\tau \in W^{1,p}(M, \mathbb{R})$, there exists a positive constant $C(g)$ independent of $\tau \in W^{1,p}(M, \mathbb{R})$ such that if*

$$\left\| \frac{\mathrm{d}\tau}{\tau} \right\|_n < C,$$

there is no solution (φ, W) to the system (1.3) with $\varphi > 0$.

Note that this allows (*a priori*) τ to have isolated non degenerate zeros. But, if τ changes sign, it can be proven that $d\tau/\tau$ does not belong to any L^p space for any $p \geqslant 1$. Hence, such a case is out of reach from this theorem.

Proof. Let us first assume that the system (1.3a) admits a solution (φ, W) with $\varphi > 0$ and $\sigma \equiv 0$. To prove the result, we denote by $\overline{g}$ the conformal metric $\psi^{N-2}g$ where a positive function $\psi \in W^{2,p}(M, \mathbb{R})$ is chosen in such a way that $\mathrm{Scal}_{\overline{g}} \geqslant 0$. Such a function ψ exists since $\mathcal{Y}(g) \geqslant 0$. In terms of the metric $\overline{g}$, equation (1.3a) becomes

$$-\frac{4(n-1)}{n-2}\Delta_{\overline{g}}\left(\psi^{-1}\varphi\right)+\mathrm{Scal}_{\overline{g}}\left(\psi^{-1}\varphi\right)= -\frac{n-1}{n}\tau^2\left(\psi^{-1}\varphi\right)^{N-1} + \left|\psi^{-2}\mathbb{L}_g W\right|^2_{\overline{g}}\left(\psi^{-1}\varphi\right)^{-N-1}. \tag{2.5}$$

Consequently, if we denote $\overline{\varphi}:=\psi^{-1}\varphi$, multiply both sides of (2.5) by $\overline{\varphi}^{N+1}$ and integrate both sides of the resulting equation with respect to the conformal metric $\overline{g}$, we get

$$\frac{3n-2}{n-2}\int_M\left|\mathrm{d}\overline{\varphi}^{N/2+1}\right|^2_{\overline{g}}\,\mathrm{dvol}_{\overline{g}}+\int_M \mathrm{Scal}_{\overline{g}}\overline{\varphi}^{N+2}\,\mathrm{dvol}_{\overline{g}} + \frac{n-1}{n}\int_M\tau^2\overline{\varphi}^{2N}\,\mathrm{dvol}_{\overline{g}}\leqslant\int_M\left|\psi^{-2}\mathbb{L}_gW\right|^2_{\overline{g}}\,\mathrm{dvol}_{\overline{g}}. \tag{2.6}$$

Under our conformal change $\overline{g}=\psi^{N-2}g$, there holds

$$\begin{aligned}\mathrm{dvol}_{\overline{g}}&=\psi^N\,\mathrm{dvol}_g,\\ \left|\psi^{-2}\mathbb{L}_gW\right|^2_{\overline{g}}&=\psi^{-2N}\left|\mathbb{L}_gW\right|^2_g.\end{aligned}\tag{2.7}$$

Therefore, in terms of the background metric g, (2.6) implies

$$\frac{n-1}{n}\int_M\tau^2\psi^{-N}\varphi^{2N}\,\mathrm{dvol}_g\leqslant\int_M\psi^{-N}\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g. \tag{2.8}$$

Since $\psi\in W^{2p}(M)$ is strictly positive, (2.8) immediately implies

$$\int_M\tau^2\varphi^{2N}\,\mathrm{dvol}_g\leqslant\frac{n}{n-1}\left(\frac{\max\psi}{\min\psi}\right)^N\int_M\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g. \tag{2.9}$$

We take the scalar product of the vector equation (1.3*b*) with W and integrate over M with respect to the background metric g to get

$$-\frac{1}{2}\int_M\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g=\frac{n-1}{n}\int_M\varphi^N\langle\mathrm{d}\tau,W\rangle\,\mathrm{dvol}_g. \tag{2.10}$$

Using the Hölder inequality, we can estimate (2.10) as follows:

$$\begin{aligned}&\frac{1}{2}\int_M\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g\\
&\leqslant\frac{n-1}{n}\left(\int_M\tau^2\varphi^{2N}\,\mathrm{dvol}_g\right)^{1/2}\left(\int_M\left|\frac{\mathrm{d}\tau}{\tau}\right|^n_g\,\mathrm{dvol}_g\right)^{1/n}\left(\int_M|W|^N_g\,\mathrm{dvol}_g\right)^{1/N}\\
&\leqslant\frac{n-1}{n}\left(\frac{n}{n-1}\left(\frac{\max\psi}{\min\psi}\right)^N\int_M\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g\right)^{1/2}\\
&\quad\times\left(\int_M\left|\frac{d\tau}{\tau}\right|^n_g\,\mathrm{dvol}_g\right)^{1/n}C_g^{-1}\left(\int_M\left|\mathbb{L}_gW\right|^2_g\mathrm{dvol}_g\right)^{1/2}\\
&\leqslant\sqrt{\frac{n-1}{n}}C_g^{-1}\left(\frac{\max\psi}{\min\psi}\right)^{N/2}\left(\int_M\left|\frac{d\tau}{\tau}\right|^n_g\,\mathrm{dvol}_g\right)^{1/n}\int_M\left|\mathbb{L}_gW\right|^2\,\mathrm{dvol}_g.\end{aligned}$$

By setting

$$C = \frac{1}{2}\sqrt{\frac{n}{n-1}}\,C_g\left(\frac{\min\psi}{\max\psi}\right)^{N/2},$$

one gets that

$$\int_M \left|\frac{d\tau}{\tau}\right|_g^n \mathrm{dvol}_g \geqslant C,$$

unless

$$\int_M \left|\mathbb{L}_g W\right|_g^2 \mathrm{dvol}_g = 0.$$

However, in the second case, we conclude from inequality (2.8) that

$$\int_M \tau^2 \psi^{-N} \overline{\varphi}^{2N}\,\mathrm{dvol}_g = 0.$$

Hence, $\overline{\varphi} \equiv 0$, which contradicts the fact that $\varphi > 0$. Thus, we have proved that if $\mathrm{d}\tau/\tau$ is small in the L^n sense, then the constraint equations (1.3) with vanishing σ admit no solution. $\square$

Since our assumptions are weaker than those in [4, theorem 1.7], the constant C appearing in theorem 2.4 is smaller than the constant appearing in [4, theorem 1.7].

3. The asymptotically Euclidean case

We now study the situation in the asymptotically Euclidean case. For relevant results on Sobolev spaces on asymptotically Euclidean manifolds, we refer the reader to [1] or [21]. See also the forthcoming article [5].

Let (M^n, g) be a complete non-compact Riemannian manifold. We say that (M, g) is $W^{k,p}_\delta$-asymptotically Euclidean if there exist a compact set $K \subset M$, a real number $R > 0$, and a diffeomorphism $\Psi: M \backslash K \to \mathbb{R}^n \backslash B_R(0)$ such that, denoting b the flat (background) metric on $\mathbb{R}^n$ and setting $e := \Psi_* g - b$, we have

$$\sum_{0\leqslant i\leqslant k} \int_{\mathbb{R}^n\backslash B_R} \left|\partial^{(i)} e\right|_b^p \left(1 + |x|^2\right)^{-(\delta + n/p - |i|)p/2} \mathrm{dvol}_b(x) < \infty$$

for some $k \geqslant 2$, $p > n$ and $\delta > 0$. Here, we denoted by $\partial^{(i)}e$ the ith order derivative (in the sense of distributions) of e and $\left|\partial^{(i)}e\right|_b$ its (pointwise) norm with respect to the Euclidean metric.

Given an asymptotically Euclidean manifold (M, g) we denote by r the pullback of the distance function from the origin in $\mathbb{R}^n$: $r = |\cdot|\circ\Psi$ and extend it to a positive continuous function on K. For any natural tensor bundle $E \to M$ and any section $\xi \in \Gamma(E)$, we define the following weighted Sobolev norm:

$$\|\xi\|_{W^{s,q}_\gamma(M,E)} := \left(\sum_{0\leqslant i\leqslant s} \int_M \left|\nabla^{(i)}\xi\right|_g^q \left(1 + r^2\right)^{-(\gamma + n/q - i)q/2} \mathrm{dvol}_g\right)^{1/q},$$

and the associated Sobolev space

$$W^{s,q}_\delta(M, E) := \left\{\xi \in W^{s,q}_{\mathrm{loc}},\ \|\xi\|_{W^{s,q}_\delta(M,E)} < \infty\right\}.$$

We also recall that the Yamabe invariant for an asymptotically Euclidean manifold (M, g) is given by (1.4) even if the solution to the Yamabe problem in this case does not belong to $W^{1,2}$ since it tends to some positive constant at infinity.

We prove the following theorem.

Theorem 3.1. *Let (M, g) be a $W^{2,p}_\delta$–asymptotically Euclidean manifold for some $p > n$ and some $\delta \in (2–n, 0)$. Assume that the Yamabe invariant $\mathcal{Y}(g)$ of the manifold (M, g) is positive. Then given any $\tau \in W^{1,p}_\delta(M, \mathbb{R})$, $\widetilde{\sigma} \in W^{1,p}_\delta(M, S^2(M))$, $\widetilde{\sigma} \not\equiv 0$, and $\widetilde{\varphi}_\infty \in \mathbb{R}^*_+$, there exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ there exists at least one solution to the system (1.3a)–(1.3b) with $\sigma = \eta\sigma_0$ and $(\varphi - \eta^{2/(N-2)}\widetilde{\varphi}_\infty, W) \in W^{2,p}_\delta(M, \mathbb{R}) \times W^{2,p}_\delta(M, T^*M)$.*

Note that the condition $\varphi - \eta\widetilde{\varphi}_\infty \in W^{2,p}_\delta(M, \mathbb{R})$ immediately implies that $\varphi \to \eta\widetilde{\varphi}_\infty$ at infinity. The proof of this theorem mimics that of theorem 2.1, replacing the $W^{k,p}$ spaces by the $W^{k,p}_\delta$ ones. We only give the analogs of each of the four claims and a proof of the significantly different steps.

Claim 1′. There exists a unique solution $\widetilde{\varphi}_0$ to the equation (2.1) such that $\widetilde{\varphi}_0 - \widetilde{\varphi}_\infty \in W^{2,p}_\delta(M, \mathbb{R})$.

Proof. To simplify the proof, we assume that the manifold (M, g) has zero scalar curvature. This assumption is harmless since it is known that any asymptotically Euclidean metric g with positive Yamabe invariant $\mathcal{Y}(g)$ is conformally related to a metric $\overline{g} = \psi^{N-2}g$ with zero scalar curvature with $\psi - 1 \in W^{2,p}_\delta(M, \mathbb{R})$ (for instance, see [21, proposition 3]). Hence, one can proceed as in the proof of claim 1, working with metric $\overline{g}$ and replacing $\left|\widetilde{\sigma}\right|^2_g$ by $\left|\psi^{-2}\widetilde{\sigma}\right|^2_g$.

To prove the existence part, we first decompose $\widetilde{\varphi} = \widetilde{\varphi}_\infty + \widetilde{v}$ and wish to look for $\widetilde{v}$ in $W^{2,p}_\delta(M, \mathbb{R})$ solving the following PDE:

$$-\frac{4(n-1)}{n-2}\Delta_g \widetilde{v} = \frac{\left|\widetilde{\sigma}\right|^2_g}{\left(\widetilde{\varphi}_\infty + \widetilde{v}\right)^{N+1}}. \tag{3.1}$$

Note that $\widetilde{v}_- \equiv 0$ is always a sub-solution to (3.1). To construct a super-solution to (3.1), let $\widetilde{v}_+ \in W^{2,p}_\delta(M, \mathbb{R})$ denote the solution to the following Poisson equation:

$$-\frac{4(n-1)}{n-2}\Delta_g \widetilde{v}_+ = \frac{\left|\widetilde{\sigma}\right|^2_g}{\left(\widetilde{\varphi}_\infty\right)^{N+1}}.$$

From the strong maximum principle it follows that $\widetilde{v}_+ > 0$. As a consequence, there holds

$$-\frac{4(n-1)}{n-2}\Delta_g \widetilde{v}_+ \geqslant \frac{\left|\widetilde{\sigma}\right|^2_g}{\left(\widetilde{\varphi}_\infty + \widetilde{v}_+\right)^{N+1}},$$

this is to say that $\widetilde{v}_+$ is a super-solution to (3.1). The standard sub- and super-solutions method applies, giving rise to the existence of a solution $\widetilde{\varphi}_0$ solving (2.1) and satisfying $\widetilde{\varphi}_0 - \widetilde{\varphi}_\infty \in W^{2,p}_\delta(M, \mathbb{R})$. The proof of the uniqueness property is then entirely similar to the compact case, therefore we omit it. □

Claim 2′. There exists $\varepsilon > 0$ such that the system (2.3) admits a solution $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu)$ such that $\widetilde{\varphi}_\mu - \widetilde{\varphi}_\infty \in W^{2,p}_\delta(M, \mathbb{R})$ and $\widetilde{W}_\mu \in W^{2,p}_\delta(M, T^*M)$ for all $\mu \in [0, \varepsilon)$.

Proof. The proof of claim 2 translates *mutatis mutandis*, the only difference being that we need to work on the affine space $(\widetilde{\varphi}_\infty, 0) + W^{2,p}_\delta(M, \mathbb{R}) \times W^{2,p}_\delta(M, T^*M)$. The relevant properties for the operator $\Delta_{\mathbb{L},g}$ on asymptotically Euclidean manifolds can be found in [24, theorem 5.4]. $\square$

Claim 3′. Set

$$\begin{cases} \varphi_\mu &= \mu^{\frac{2}{N-2}} \widetilde{\varphi}_\mu, \\ W_\mu &= \mu^{\frac{N+2}{N-2}} \widetilde{W}_\mu, \\ \sigma_\mu &= \mu^{\frac{N+2}{N-2}} \widetilde{\sigma}. \end{cases}$$

If $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu)$ solves (2.3) with $\widetilde{\varphi}_\mu \to \widetilde{\varphi}_\infty$ at infinity, then (φ_μ, W_μ) solves (1.3) with $\sigma = \sigma_\mu$ and $\varphi_\mu \to \mu^{2/(N-2)} \widetilde{\varphi}_\infty$ at infinity.

4. The compact with boundary case

4.1. Boundary conditions

A natural issue in the study of the Einstein constraint equations is the construction of initial data modeling black holes. While the definition of a black hole requires knowledge of the whole solution $(\mathcal{M}, \mathbf{g})$ of the Einstein equations, it is natural to construct initial data containing apparent horizons. For an overview, we refer the reader to [3]. A natural way to construct such solutions is to excise the inside of the apparent horizon and thus construct solutions to the constraint equations on the outside. As a consequence, we fix a manifold M with boundary ∂M, and solve the constraint equations on M in such a way that ∂M becomes an apparent horizon.

The first articles where such solutions to the constraint equations were constructed dealt with the constant mean curvature case; see e.g. [10, 21]. Very recently, people have turned their attention to compact manifolds with boundary with a varying τ; see for example [5, 14].

To go further, let us roughly reformulate this problem. For a detailed explanation and calculations, we refer the reader to [5, 11, 14]. Let $\hat{\nu}$ be the (spacelike) unit normal vector field to ∂M in M pointing towards the outside of M (hence to the 'inside' of the apparent horizon) and let $\hat{n}$ be the future directed unit normal spacetime vector field to M. Then, by means of apparent horizon boundaries, in addition to the constraint (1.1), we further require

$$\begin{cases} \widehat{\Theta}_- &\leqslant 0, \\ \widehat{\Theta}_+ &= 0, \end{cases} \tag{4.1}$$

where $\widehat{\Theta}_\pm$, known as the null expansion with respect to the null normal $\ell_\pm := \hat{n} \mp \hat{\nu}$, are given as follows:

$$\widehat{\Theta}_\pm = \mathrm{tr}_{\hat{g}} \widehat{K} - \widehat{K}\left(\hat{\nu}, \hat{\nu}\right) \mp H_{\hat{g}},$$

where $H_{\hat{g}}$ is the (unnormalized) mean curvature of ∂M in M evaluated with respect to $\hat{\nu}$, that is to say

$$H_{\widehat{g}} = \widehat{g}^{ij}\widehat{\nabla}_i\widehat{\nu}_j,$$

where we denote by $\widehat{\nabla}$ the Levi–Civita connection for the metric $\widehat{g}$. Since we require $\widehat{\Theta}_+ \equiv 0$ on ∂M, the conditions can be rewritten as

$$\begin{cases} \mathrm{tr}_{\widehat{g}}\widehat{K} - \widehat{K}\left(\widehat{\nu}, \widehat{\nu}\right) &= \dfrac{\widehat{\Theta}_+ + \widehat{\Theta}_-}{2} = \dfrac{\widehat{\Theta}_-}{2}, \\ H_{\widehat{g}} &= \dfrac{\widehat{\Theta}_- - \widehat{\Theta}_+}{2} = \dfrac{\widehat{\Theta}_-}{2}. \end{cases}$$

On the other hand, recalling that $\widehat{g} = \varphi^{N-2}g$, one has the following formula relating: $H_{\widehat{g}}$ and H_g:

$$\frac{2(n-1)}{n-2}\partial_\nu\varphi + H_g\varphi = H_{\widehat{g}}\varphi^{N/2},$$

where $\nu = \varphi^{N/2-1}\widehat{\nu}$ is the unit vector field normal to Σ calculated with respect to the metric g. Hence, we get the following condition for φ:

$$\frac{2(n-1)}{n-2}\partial_\nu\varphi + H_g\varphi = \frac{\widehat{\Theta}_-}{2}\varphi^{N/2}. \tag{4.2}$$

Next, thanks to $\mathrm{tr}_{\widehat{g}}\widehat{K} = \tau$ and the fact that

$$\widehat{K}\left(\widehat{\nu}, \widehat{\nu}\right) = \frac{\tau}{n} + \left(\sigma + \mathbb{L}_g W\right)(\nu, \nu)\varphi^{-N},$$

we obtain the following identity:

$$\frac{\widehat{\Theta}_-}{2} = \frac{n-1}{n}\tau - \left(\sigma + \mathbb{L}_g W\right)(\nu, \nu)\varphi^{-N}. \tag{4.3}$$

Contrary to (4.2), this does not give a boundary condition that complements equation (1.3*b*). In this context, it is natural to prescribe $(\sigma + \mathbb{L}_g W)(\nu, \cdot\,)$ as follows:

$$\left(\sigma + \mathbb{L}_g W\right)(\nu, \cdot\,) = \left(\frac{n-1}{n}\tau - \frac{\widehat{\Theta}_-}{2}\right)\varphi^N\nu^\flat + \xi \tag{4.4}$$

where ξ is a 1-form on ∂M that we extend to the restriction of TM to ∂M by setting $\xi(\nu) = 0$ so that condition (4.3) is satisfied. Also, in (4.4), we use $\nu^\flat$ to denote the 1-form dual to the normal vector field ν, which is given by $\nu^\flat(X) = g(\nu, X)$ for any vector field X on ∂M. Having had the discussion above, we are now in a position to write the following system of PDEs:

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g\varphi + \mathrm{Scal}_g\varphi &= -\dfrac{n-1}{n}\tau^2\varphi^{N-1} + \left|\sigma + \mathbb{L}_g W\right|_g^2 \varphi^{-N-1}, \\ \Delta_{\mathbb{L},g} W &= \dfrac{n-1}{n}\varphi^N \mathrm{d}\tau, \\ \dfrac{2(n-1)}{n-2}\partial_\nu\varphi + H_g\varphi &= \dfrac{\widehat{\Theta}_-}{2}\varphi^{N/2}, \\ \left(\sigma + \mathbb{L}_g W\right)(\nu, \cdot\,) &= \left(\dfrac{n-1}{n}\tau - \dfrac{\widehat{\Theta}_-}{2}\right)\varphi^N\nu^\flat + \xi, \end{cases} \tag{4.5}$$

where the given data are now (M, g) a compact Riemannian manifold with boundary ∂M, τ a function on M, σ a TT-tensor, $\widehat{\Theta}_-$ a nonpositive function on $\Sigma = \partial M$ and $\xi \in \Gamma(\partial M, T^*M)$ a 1-form.

In the presence of the boundary ∂M, instead of using the sign of $\mathcal{Y}(g)$, we use the sign of the Yamabe invariant $\mathcal{Y}(g, \partial M)$ introduced by Escobar [9]:

$$\mathcal{Y}(g, \partial M) := \inf_{0 \not\equiv \varphi \in W^{1,2}(M, \mathbb{R})} \frac{\int_M \left(\frac{4(n-1)}{n-2} |d\varphi|_g^2 + \mathrm{Scal}_g \varphi^2\right) \mathrm{dvol}_g + \int_{\partial M} H_g \varphi^2 \, \mathrm{ds}_g}{\left(\int_M \varphi^N \, \mathrm{dvol}_g\right)^{N/2}}.$$

We also comment on the York splitting on compact manifolds with boundary. While on closed manifolds we have that the set of (say) $W^{1,2}$-TT-tensors is L^2-orthogonal to the set $\{\mathbb{L}_g W, W \in W^{2,2}(M, T^*M)\}$, this is no longer true for compact manifolds with boundary. Indeed, let σ be a TT-tensor and W be an arbitrary 1-form, then if we denote by $W^\sharp$ the vector field dual to the 1-form W, then by a direct calculation together with the Stokes theorem, we have

$$\begin{aligned}
\int_M \left\langle \sigma, \mathbb{L}_g W \right\rangle \mathrm{dvol}_g &= 2 \int_M \langle \sigma, \nabla W \rangle \, \mathrm{dvol}_g \\
&= 2 \int_M \mathrm{div}\left(\sigma\left(W^\sharp, \cdot\right)\right) \mathrm{dvol}_g - 2 \int_M (\mathrm{div}\, \sigma)\left(W^\sharp\right) \mathrm{dvol}_g \\
&= 2 \int_{\partial M} \sigma\left(W^\sharp, \nu\right) \mathrm{ds}_g,
\end{aligned}$$

where $\mathrm{tr}_g \sigma = 0$ and $\mathrm{div}_g\, \sigma = 0$ were also used to obtain the first and last lines, respectively. Since the restriction of W to ∂M can be arbitrary, σ belongs to the orthogonal of the set of $\mathbb{L}_g W$'s if and only if we also impose that $\sigma(\nu, \cdot) \equiv 0$ on ∂M. We will make this assumption from now on.

4.2. Existence result

The main result of this subsection is the following.

Theorem 4.1. *Let M be a compact manifold with boundary. Given $p > n$, let $g \in W^{2,p}(M, S^2(M))$, $\tau \in W^{1,p}(M, \mathbb{R})$, and $\widetilde{\sigma} \in W^{1,p}(M, S^2(M))$, $\widehat{\Theta}_- \in W^{1-1/p,p}(\partial M, \mathbb{R})$, $\widetilde{\xi} \in W^{1-1/p,p}(\partial M, T^*M)$ be given data, where $\widetilde{\sigma}$ is a TT-tensor such that $\widetilde{\sigma}(\nu, \cdot) \equiv 0$ on ∂M. Assume that the Escobar invariant $\mathcal{Y}(g, \partial M)$ is strictly positive, that g has no conformal Killing vector fields and either $\widetilde{\sigma} \not\equiv 0$ or $\widetilde{\xi} \not\equiv 0$. There exists $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$ there exists at least one solution $(\varphi, W) \in W^{2,p}(M, \mathbb{R}) \times W^{2,p}(M, T^*M)$ to the system (4.5) with $\sigma = \eta \widetilde{\sigma}$ and $\xi = \eta \widetilde{\xi}$.*

We initiate the proof of theorem 4.1 by proving that the right-hand side of the analog of equation (2.1) (see equation (4.6)) is actually non-zero.

Claim 0″. Let $\widetilde{W}_0 \in W^{2,p}(M, T^*M)$ be the unique solution of

$$\begin{cases} \Delta_{\mathbb{L},g} \widetilde{W}_0 &= 0, \\ \mathbb{L}_g(\nu, \cdot) &= \widetilde{\xi}. \end{cases}$$

Then under the assumptions of theorem 4.1, we have

$$\left|\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}_0\right|_g^2 \not\equiv 0.$$

Proof. The existence, uniqueness, and regularity of $\widetilde{W}_0$ are proved in [13, theorem 4.5]. See also [21, proposition 5.1] and [10, theorem 8.6] for earlier references. From the remark at the end of section 4.1, we have

$$\int_M \left|\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}_0\right|_g^2 \mathrm{dvol}_g = \int_M \left|\widetilde{\sigma}\right|_g^2 \mathrm{dvol}_g + \int_M \left|\mathbb{L}_g \widetilde{W}_0\right|_g^2 \mathrm{dvol}_g.$$

Hence if $\widetilde{\sigma} \not\equiv 0$, the claim follows. Otherwise if $\widetilde{\xi} \not\equiv 0$, $\widetilde{W}_0$ is a non-trivial element of $W^{2,p}(M, T^*M)$. Since (M, g) has no non-zero conformal Killing vector field, it follows that

$$\int_M \left|\mathbb{L}_g \widetilde{W}_0\right|_g^2 \mathrm{dvol}_g > 0,$$

which proves the claim. □

Claim 1″. Under the assumptions of theorem 4.1, there exists a unique solution $\widetilde{\varphi}_0 \in W^{2,p}(M, \mathbb{R})$ to the following system:

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g \widetilde{\varphi}_0 + \mathrm{Scal}_g \widetilde{\varphi}_0 &= \left|\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}_0\right|_g^2 \widetilde{\varphi}_0^{-N-1}, \\ \dfrac{2(n-1)}{n-2}\partial_\nu \widetilde{\varphi}_0 + H_g \widetilde{\varphi}_0 &= 0. \end{cases} \tag{4.6}$$

Proof. The proof of this claim is similar to the proof of claim 1. From the work of Escobar [8, lemma 1.1], there exists a conformal factor $\psi \in W^{2,p}(M, \mathbb{R})$ such that the metric $\overline{g} = \psi^{N-2} g$ has $\mathrm{Scal}_{\overline{g}} > 0$ and the mean curvature of the boundary ∂M vanishes identically: $H_{\overline{g}} \equiv 0$[1]. The equation for $\overline{\varphi}_0 := \psi^{-1}\widetilde{\varphi}_0$ reads

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g \overline{\varphi}_0 + \mathrm{Scal}_g \overline{\varphi}_0 &= \left|\psi^{-2}\left(\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}_0\right)\right|_{\overline{g}}^2 \overline{\varphi}_0^{-N-1}, \\ \partial_{\overline{\nu}} \overline{\varphi}_0 &= 0, \end{cases} \tag{4.7}$$

where $\overline{\nu} = \psi^{1-N/2}\nu$ is the unit normal to ∂M for the metric $\overline{g}$. There exists a unique function $\overline{u} \in W^{2,p}(M, \mathbb{R})$ solving

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g \overline{u} + \mathrm{Scal}_g \overline{u} &= \left|\psi^{-2}\left(\widetilde{\sigma} + \mathbb{L}_g \widetilde{W}_0\right)\right|_{\overline{g}}^2, \\ \partial_{\overline{\nu}} \overline{u}_0 &= 0. \end{cases} \tag{4.8}$$

Further, the function u is positive. By setting

$$\underline{\overline{\varphi}} = (\max \overline{u})^{(N+1)/(N+2)} \overline{u}$$

[1] As pointed out by the referee, [8, lemma 1.1] is only stated for smooth metrics. However, the proof works for $W^{2,p}$ metrics without any change.

and

$$\overline{\varphi}_+ = \left(\min \overline{u}\right)^{(N+1)/(N+2)} \overline{u},$$

one readily checks that $\overline{\varphi}_+$ and $\underline{\varphi}_-$ are super- and sub-solutions for (4.7). Hence, by the sub- and super-solution method, we conclude that there exists a solution $\overline{\varphi}_0$ to (4.7). The function $\widetilde{\varphi}_0 := \psi\overline{\varphi}_0$ is then a solution to (4.6). The proof of uniqueness is a rephrasing of that in claim 1 with a Neumann boundary condition. □

Similar to (2.3) for the closed case, in view of (4.5) we now introduce the following μ-deformed system for the compact with boundary case:

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_g\widetilde{\varphi} + \mathrm{Scal}_g\widetilde{\varphi} &= -\dfrac{n-1}{n}\tau^2\mu^2\widetilde{\varphi}^{N-1} + \left|\widetilde{\sigma} + \mathbb{L}_g\widetilde{W}\right|_g^2\widetilde{\varphi}^{-N-1}, \\ \Delta_{\mathbb{L},g}\widetilde{W} &= \dfrac{n-1}{n}\widetilde{\varphi}^N\mu\mathrm{d}\tau, \\ \dfrac{2(n-1)}{n-2}\partial_\nu\widetilde{\varphi} + H_g\widetilde{\varphi} &= \dfrac{\widehat{\Theta}_-}{2}\mu\widetilde{\varphi}^{N/2}, \\ \mathbb{L}_g\widetilde{W}(\nu,\cdot) &= \mu\left(\dfrac{n-1}{n}\tau - \dfrac{\widehat{\Theta}_-}{2}\right)\widetilde{\varphi}^N\nu^\flat + \widetilde{\xi}. \end{cases} \tag{4.9}$$

This system is obtained from (4.5) by replacing τ by $\mu\tau$ and $\widehat{\Theta}_-$ by $\mu\widehat{\Theta}_-$.

Claim 2″. There exists $\varepsilon > 0$ such that (4.9) admits a solution $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu)$ for all $\mu \in [0, \varepsilon)$.

Proof. We define the following operator:

$$F: \quad \mathbb{R} \times W^{2,p}_+(M,\mathbb{R}) \times W^{2,p}\left(M, T^*M\right)$$
$$\downarrow$$
$$L^p(M,\mathbb{R}) \times W^{(1-(1/p)),p}(\partial M,\mathbb{R}) \times L^p\left(M, T^*M\right) \times W^{(1-(1/p)),p}\left(\partial M, T^*M\right)$$

given by

$$F\left(\mu, \widetilde{\varphi}, \widetilde{W}\right) = \begin{pmatrix} -\dfrac{4(n-1)}{n-2}\Delta_g\widetilde{\varphi} + \mathrm{Scal}_g\widetilde{\varphi} + \dfrac{n-1}{n}\tau^2\mu^2\widetilde{\varphi}^{N-1} - \left|\sigma + \mathbb{L}_g\widetilde{W}\right|_g^2\widetilde{\varphi}^{-N-1} \\ \dfrac{2(n-1)}{n-2}\partial_\nu\widetilde{\varphi} + H_g\widetilde{\varphi} - \dfrac{\widehat{\Theta}_-}{2}\mu\widetilde{\varphi}^{N/2} \\ \Delta_{\mathbb{L},g}\widetilde{W} - \dfrac{n-1}{n}\widetilde{\varphi}^N\mu\mathrm{d}\tau \\ \mathbb{L}_g\widetilde{W}(\nu,\cdot) - \mu\left(\dfrac{n-1}{n}\tau - \dfrac{\widehat{\Theta}_-}{2}\right)\widetilde{\varphi}^N\nu^\flat - \widetilde{\xi} \end{pmatrix}.$$

It is not hard to see that the mapping F is of class C^1 and

$$F\left(0, \widetilde{\varphi}_0, \widetilde{W}_0\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\widetilde{\varphi}_0$ and $\widetilde{W}_0$ are given in claims $0''$ and $1''$. Again, all we need to do is to prove that the derivative of F with respect to $(\widetilde{\varphi}, \widetilde{W})$ is an isomorphism at $(0, \widetilde{\varphi}_0, \widetilde{W}_0)$. To do so, we need to study the following mapping:

$$\begin{aligned} \mathcal{D}F_{\left(0,\widetilde{\varphi}_0,\widetilde{W}_0\right)}: \quad & W^{2,p}(M, \mathbb{R}) \times W^{2,p}\left(M, T^*M\right) \\ & \downarrow \\ L^p(M, \mathbb{R}) \times W^{(1-(1/p)),p}(\partial M, \mathbb{R}) & \times L^p\left(M, T^*M\right) \times W^{(1-(1/p)),p}\left(\partial M, T^*M\right). \end{aligned}$$

A direct computation shows that this derivative is given by

$$\mathcal{D}F_{(0,\widetilde{\varphi}_0,\widetilde{W}_0)}(\widetilde{\theta}, \widetilde{Z}) = \left(\begin{array}{c:c} -\frac{4(n-1)}{n-2}\Delta_g + \mathrm{Scal}_g + (N+1)|\widetilde{\sigma} + \mathbb{L}\widetilde{W}_0|_g^2\widetilde{\varphi}_0^{-N-2} & -2\langle \widetilde{\sigma} + \mathbb{L}\widetilde{W}_0, \mathbb{L}\,\cdot\,\rangle \\ \hdashline \frac{2(n-1)}{n-2}\partial_\nu + H_g & 0 \\ \hdashline 0 & \Delta_{\mathbb{L},g} \\ \hdashline 0 & \mathbb{L}_g\cdot(\nu,\cdot) \end{array}\right)\begin{pmatrix} \widetilde{\theta} \\ \widetilde{Z} \end{pmatrix}.$$

Clearly, $\mathcal{D}F_{(0,\widetilde{\varphi}_0,\widetilde{W}_0)}$ is continuous. To prove that $\mathcal{D}F_{(0,\widetilde{\varphi}_0,\widetilde{W}_0)}$ is invertible, we observe that $\mathcal{D}F_{(0,\widetilde{\varphi}_0,\widetilde{W}_0)}$ is block upper-triangular, where the diagonal blocks are

$$\begin{pmatrix} -\dfrac{4(n-1)}{n-2}\Delta_g + \mathrm{Scal}_g + (N+1)|\sigma|_g^2\widetilde{\varphi}_0^{-N-2} \\ \dfrac{2(n-1)}{n-2}\partial_\nu + H_g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Delta_{\mathbb{L},g} \\ \mathbb{L}_g\cdot(\nu,\cdot) \end{pmatrix}$$

which are invertible. Hence, the derivative $\mathcal{D}F_{(0,\widetilde{\varphi}_0,\widetilde{W}_0)}$ is an isomorphism at $(0, \widetilde{\varphi}_0, \widetilde{W}_0)$ as claimed. □

Claim 3″. Set

$$\begin{cases} \varphi_\mu = \mu^{\frac{2}{N-2}}\widetilde{\varphi}_\mu, \\ W_\mu = \mu^{\frac{N+2}{N-2}}\widetilde{W}_\mu. \end{cases}$$

If $(\widetilde{\varphi}_\mu, \widetilde{W}_\mu)$ solves (4.9), then (φ_μ, W_μ) solves (4.5) with $\sigma = \sigma_\mu := \mu^{(N+2)/(N-2)}\widetilde{\sigma}$ and $\xi = \xi_\mu := \mu^{(N+2)/(N-2)}\widetilde{\xi}$.

Finally, the proof of theorem 4.1 follows by setting $\eta_0 = \varepsilon^{(N+2)/(N-2)}$, where ε is the constant appearing in claim $2''$.

Remark 4.2. It is tempting to prove an analog of the non-existence result for the case of a compact manifold with boundary as in theorem 2.4. The natural assumptions in this theorem would then be $\sigma \equiv 0$, $\xi \equiv 0$ and $\mathcal{Y}(g, \partial M) > 0$. The proof is, however, not just an extension of that of theorem 2.4, it relies on the techniques developed in [11], so we choose to defer this to that article.

Acknowledgments

The authors are grateful to Michael Holst, Emmanuel Humbert, Jim Isenberg, David Maxwell and Daniel Pollack for useful discussions and advice. We would also like to warmly thank the referee and Cang Nguyen for their careful proofreading of this paper, and the editors of *CQG* for taking into account the lengthy list of modifications we had. Part of this work was performed when RG was attending the *Mathematical General Relativity* program at the Mathematical Sciences Research Institute in Berkeley, California, USA, during the Fall of 2013. QAN would like to acknowledge the support of the Région Centre through the convention no. 00078780 during the period 2012–2014.

References

[1] Bartnik R 1986 The mass of an asymptotically flat manifold *Comm. Pure Appl. Math.* **39** 661–93
[2] Bartnik R and Isenberg J 2004 The constraint equations *The Einstein Equations and the Large Scale Behavior of Gravitational Fields* (Basel: Birkhäuser) pp 1–38
[3] Chruściel P T, Galloway G J and Pollack D 2010 Mathematical general relativity: a sampler *Bull. Am. Math. Soc. (N.S.)* **47** 567–638
[4] Dahl M, Gicquaud R and Humbert E 2012 A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method *Duke Math. J.* **161** 2669–97
[5] Dilts J 2014 The Einstein constraint equations on compact manifolds with boundary *Class. Quantum Grav.* **31** 125009
[6] Dilts J, Gicquaud R and Isenberg J A limit equation criterion for applying the conformal method to asymptotically Euclidean initial data sets (to appear on arXiv)
[7] Dilts J, Isenberg J, Mazzeo R and Meier C 2014 Non-CMC solutions of the Einstein constraint equations on asymptotically Euclidean manifolds *Class. Quantum Grav.* **31** 065001
[8] Dilts J and Leach J 2014 A limit equation criterion for applying the conformal method to asymptotically cylindrical initial data sets (arXiv:1401.5369)
[9] Escobar J F 1992 The Yamabe problem on manifolds with boundary *J. Differ. Geom.* **35** 21–84
[10] Gicquaud R 2010 De l'équation de prescription de courbure scalaire aux équations de contrainte en relativité générale sur une variété asymptotiquement hyperbolique *J. Math. Pures Appl.* **94** 200–27
[11] Gicquaud R, Humbert E and Ngô Q A The vacuum Einstein constraint equations with apparent horizon boundaries (in preparation)
[12] Gicquaud R and Sakovich A 2012 A large class of non-constant mean curvature solutions of the Einstein constraint equations on an asymptotically hyperbolic manifold *Commun. Math. Phys.* **310** 705–63
[13] Holst M and Meier C 2014 Non-CMC solutions to the Einstein constraint equations on asymptotically Euclidean manifolds with apparent horizon boundaries (arXiv:1403.4549)
[14] Holst M, Meier C and Tsogtgerel G 2013 Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries (arXiv:1310.2302)
[15] Holst M, Nagy G and Tsogtgerel G 2008 Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics *Phys. Rev. Lett.* **100** 161101
[16] Holst M, Nagy G and Tsogtgerel G 2009 Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions *Commun. Math. Phys.* **288** 547–613
[17] Isenberg J 1995 Constant mean curvature solutions of the Einstein constraint equations on closed manifolds *Class. Quantum Grav.* **12** 2249–74

[18] Isenberg J and Murchadha N Ó 2004 Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations *Class. Quantum Grav.* **21** S233–41 (A spacetime safari: essays in honour of Vincent Moncrief)

[19] Leach J 2014 A far-from-CMC existence result for the constraint on manifolds with ends of cylindrical type *Class. Quantum Grav.* **31** 035003

[20] Maxwell D 2014 The conformal method and the conformal thin-sandwich method are the same *Class. Quantum Grav.* **31** 145006

Maxwell D 2014 Initial data in general relativity described by expansion, conformal deformation and drift (arXiv:1407.1467)

Maxwell D 2011 A model problem for conformal parameterizations of the Einstein constraint equations *Comm. Math. Phys.* **302** 697–736

[21] Maxwell D 2005 Solutions of the Einstein constraint equations with apparent horizon boundaries *Commun. Math. Phys.* **253** 561–83

[22] Maxwell D 2005 Rough solutions of the Einstein constraint equations on compact manifolds *J. Hyperbolic Differ. Equ.* **2** 521–46

[23] Maxwell D 2009 A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature *Math. Res. Lett.* **16** 627–45

[24] Nguyen T C Applications of fixed point theorems to the vacuum Einstein constraint equations with non-constant mean curvature to appear (arXiv:1405.7731)