The Organizing Committee and the Problem Selection Committee thank the following 31 countries for sending in proposals of problems

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Mihai Bălună	Nicuşor Dan	Dragoş Popescu
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Dan Brânzei	Mihai Piticari	

The 27 problems submitted to the Jury are classified under Number Theory (N1-N6), Geometry (G1-G8), Algebra (A1-A6) and Combinatorics (C1-C7).

Within each subject they are arranged in ascending order of estimated difficulty. A few of them are slightly modified and new solutions are presented.

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Problem N1

n

Find all the pairs of positive integers (x,p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

divides

Solution. Clearly we have the solutions $(1,p)$ and $(2,2)$, and for every other solution $p \geq 3$.

It remains to find the solutions (x,p) with $x \geq 2$ and $p \geq 3$. We claim that in this case x is divisible by p and $x < 2p$, whence $x = p$. This will lead to

$$p^{p-1} | (p-1)^p + 1 = p^2 \left(p^{p-2} - \binom{p}{1} p^{p-3} + \ldots + \binom{p}{p-3} p - \binom{p}{p-2} + 1 \right)$$

therefore, because all the terms in the brackets excepting the last one is divisible by p, $p - 1 \leq 2$. This leaves only $p = 3$ and $x = 3$.

Let us prove now the claim. Since $(p-1)^x + 1$ is odd, so is x (therefore $x < 2p$). Denote by q the smallest prime divisor of x. From $q | (p-1)^x + 1$ we get $(p-1)^x \equiv -1 (\bmod q)$ and $(q, p-1) = 1$. But $(x, q-1) = 1$ (from the choice of q) leads to the existence of integers u, v such that $ux + v(q-1) = 1$, whence $p - 1 \equiv (p-1)^{ux} \cdot (p-1)^{v(q-1)} \equiv (-1)^u \cdot 1^v \equiv -1 (\bmod q)$, because u must be odd. This shows that $q | p$, therefore $q = p$.

In conclusion the required solutions are $(2,2)$, $(3,3)$ and $(1, p)$, where p is an arbitrary prime.

(n, p) p prime

$n \leq 2p$

$n^{p-1} \mid (p-1)^n + 1$

e.g. $2^1 \mid 1+1$

$3^2 \mid 2^3 + 1$

Problem N2

Prove that every positive rational number can be represented in the form

$$\frac{a^3+b^3}{c^3+d^3},$$

where a, b, c, d are positive integers.

Solution. We firstly claim that every rational number from the interval (1,2) can be represented in the form

$$\frac{a^3+b^3}{a^3+d^3}.$$

Indeed, let $m/n \in (1,2)$, where m and n are natural numbers. We will choose a, b, d such that $b \neq d$ and $a^2 - ab + b^2 = a^2 - ad + d^2$, i.e. $b + d = a$. In that case

$$\frac{a^3+b^3}{a^3+d^3} = \frac{a+b}{a+d} = \frac{a+b}{2a-b}.$$

Taking $a + b = 3m$, $2a - b = 3n$, that is $a = m + n$, $b = 2m - n$, the claim is proven.

We can prove now the required conclusion. If $r > 0$ is a rational number, take positive integers p, q such that $1 < \frac{p^3}{q^3}r < 2$. There exists positive integers a, b, d such that

$$\frac{p^3}{q^3}r = \frac{a^3+b^3}{a^3+d^3}.$$

Hence

$$r = \frac{(aq)^3+(bq)^3}{(ap)^3+(dp)^3}.$$

Problem N3

Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n+1)$ divides b_n^2+1 for every natural n.

Solution. A way of constructing such sequences is to use the following

Lemma. *If $a, c \in \mathbb{N}$ and $a^2|c^2+1$ then there exists $b \in \mathbb{N}$ such that $a^2(a^2+1)|b^2+1$.*

Proof. Indeed $a^2|(c+a^2c-a^3)^2+1$ and $a^2+1|(c+a^2c-a^3)^2+1$, so we can take $b = c+a^2c-a^3$. □

Using the lemma we see that it is enough to find strictly increasing sequences (a_n) and (c_n) such that $a_n^2|c_n^2+1$ for every $n \in \mathbb{N}$. This can be realised by taking for instance $a_n = 2^{2n}+1$, $c_n = 2^{na_n}$. In this case

$$c_n^2+1 = (2^{2n})^{a_n}+1 = (a_n-1)^{a_n}+1$$

is divisible by a_n^2.

Another solution. We will use factorizations over the ring $\mathbb{Z}[i]$. Take (p, q, r) a Pythago-
rean triple ($p^2 = q^2+r^2$) and $a = p^2$. Then

$$a(a+1) = (q^2+r^2)(p^2+1) = (q+ri)(p+i)(q-ri)(p-i)$$

must divide $(b+i)(b-i)$. Therefore we have to find a Gaussian integer $x+yi$ such that

$$(q+ri)(p+i)(x+yi) = b+i$$

that is we have to find integers x, y such that

$$(pr+q)x+(pq-r)y = 1.$$

It is easy to see that if p, q, r are coprime then $pr+q$ and $pq-r$ are also coprime: if d is a prime divisor of $pr+q$ and $pq-r$, then

$$d|q(pr+q)-r(pq-r) = q^2+r^2 = p^2,$$

hence $d|p$ and also $d|q, r$, which implies $d = 1$. Thus the existence of x, y is proven.

The construction of the sequences (a_n) and (b_n) is now obvious.

Problem N4

Denote by S the set of all primes p such that the decimal representation of $1/p$ has the fundamental period divisible by 3.

For every $p \in S$ such that $1/p$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0, a_1 a_2 \dots a_{3r} a_1 a_2 \dots a_{3r} \dots,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k,p)$ by

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)}.$$

a) *Prove that S is infinite.*

b) *Find the highest value of $f(k,p)$ for $k \geq 1$ and $p \in S$.*

Solution. a) The fundamental period of $1/p$ is the smallest integer $d \geq 1$ such that $10^d - 1$ is divisible by p.

Let s be a prime and $N_s = 10^{2s} + 10^s + 1$. Clearly $N_s \equiv 3 (\operatorname{mod} 9)$. Let p_s be a prime divisor of $N_s/3$; p_s cannot be 3. Since N_s is a divisor of $10^{3s} - 1$, the decimal representation of $1/p_s$ has a period of length $3s$, so its fundamental period is a divisor of $3s$. The fundamental period cannot be s, because this would imply $10^s \equiv 1 (\operatorname{mod} p_s)$, leading to $N_s \equiv 3 \not\equiv 0 (\operatorname{mod} p_s)$. Also, the fundamental period can be 3 only in the case when p_s is a divisor of $10^3 - 1 = 3^3 \cdot 37$, that is $p_s = 37$. We claim that p_s can be chosen $\neq 37$: otherwise $N_s = 3 \cdot 37^k \equiv 3 (\operatorname{mod} 4)$ and $N_s = 10^{2s} + 10^s + 1 \equiv 1 (\operatorname{mod} 4)$. Hence, for every prime s we can find a prime p_s such that the decimal representation of $1/p_s$ has the fundamental period of length $3s$.

b) Let $3r(p)$ be the fundamental period for a prime $p \in S$. Then p is a divisor of $10^{3r(p)} - 1$ but not a divisor of $10^{r(p)} - 1$, so p is a divisor of $N_p = 10^{2r(p)} + 10^{r(p)} + 1$.

Let $1/p = 0, a_1 a_2 a_3 \dots, x_j = 10^{j-1}/p$ and $y_j = \{x_j\} = 0, a_j a_{j+1} a_{j+2} \dots$.

Clearly $a_j < 10 y_j$, therefore

$$f(k,p) = a_k + a_{k+r(p)} + a_{k+2r(p)} < 10(y_k + y_{k+r(p)} + y_{k+2r(p)}).$$

We notice that

$$x_k + x_{k+r(p)} + x_{k+2r(p)} = \frac{10^{k-1} N_p}{p}$$

is an integer, whence $y_k + y_{k+r(p)} + y_{k+2r(p)}$ is also an integer. Since $y_k + y_{k+r(p)} + y_{k+2r(p)} < 3$, it follows that $y_k + y_{k+r(p)} + y_{k+2r(p)} \leq 2$, therefore $f(k,p) < 20$.

Hence, the highest value for $f(k,p)$ can be at most 19. From $f(2,7) = 4+8+7 = 19$ we conclude that this is the required maximum.

Problem N5

Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

Solution. Let $n = 2^a 5^b p$, where a, b are non-negative integers and $(p, 10) = 1$. We notice that it is enough to find M such that $p | M$ and the sum of M's digits is k (we can take then $m = M \cdot 10^c$, where $c = \max\{a, b\}$).

Since $(p, 10) = 1$, there exists $k \geq 2$ such that $10^k \equiv 1 (\bmod p)$. It follows that $10^{ik} \equiv 1 (\bmod p)$ and $10^{jk+1} \equiv 10 (\bmod p)$ for every non-negative integers i, j. We will look for integers $u, v \geq 0$ so that $M = \sum_{i=1}^{u} 10^{ki} + \sum_{j=1}^{v} 10^{kj+1}$ (if u or v is 0 then the corresponding sum is 0).

Notice that $M \equiv u + 10v (\bmod p)$. Hence M is acceptable if

$$\begin{cases} u + v = k \\ p | u + 10v \end{cases} \Leftrightarrow \begin{cases} u + v = k & (1) \\ p | k + 9v & (2) \end{cases}$$

Because $(p, 3) = 1$, one of the residues $(\bmod p)$ of the numbers $k, k+9, k+18, \dots, k+9(p-1)$ must be nil, so relation (2) holds for some v_0 with $0 \leq v_0 < p$. Taking this v_0 and $u_0 = k - v_0$ we get the wanted M.

Problem N6

Prove that for every real number M there exists an infinite arithmetic progression such that:

– each term is a positive integer and the common difference is not divisible by 10;
– the sum of the digits of each term (in decimal representation) exceeds M.

Solution. We will prove that it is possible to take a common difference of the form $10^m + 1$, where m is a positive integer.

Let a_0 be a positive integer and $a_n = a_0 + n(10^m + 1) = \overline{b_s b_{s-1} \ldots b_0}$, where s and the digits $b_0, b_1, \ldots, b_s$ depend on n. It is easy to see that if $l \equiv k (\bmod 2m)$ then $10^l \equiv 10^k (\bmod (10^m + 1))$. Therefore $a_0 \equiv a_n = \overline{b_s b_{s-1} \ldots b_0} \equiv \sum_{i=0}^{2m-1} c_i 10^i (\bmod 10^m + 1)$, where $c_i = b_i + b_{2m+i} + b_{4m+i} + \ldots$ for $i = 0, 1, \ldots, 2m-1$.

Let $N > M$ be a positive integer. The number of $2m$-uples $(c_0, c_1, \ldots, c_{2m-1})$ of non-negative integers with $c_0 + c_1 + \ldots + c_{2m-1} \le N$ is equal to the number of strictly increasing sequences

$$0 \le c_0 < c_0+c_1+1 < c_0+c_1+c_2+2 < \ldots < c_0+c_1+\ldots+c_{2m-1}+2m-1 \le N+2m-1.$$

This is equal to the number of subsets with $2m$ elements of the set $\{0, 1, \ldots, 2m-1\}$ and is

$$K_{N,2m} = \binom{2m+N}{N} = \frac{(2m+N)(2m+N-1)\cdot\ldots\cdot(2m+1)}{N!}.$$

For sufficiently large m we have $K_{N,2m} < 10^m$. Taking $a_0 \in \{1, 2, \ldots, 10^m\}$ such that a_0 is not congruent $(\bmod 10^m + 1)$ with any of the numbers belonging to the set

$$\{\overline{c_{2m-1} c_{2m-2} \cdots c_0} \mid c_0 + c_1 + \ldots + c_{2m-1} \le N\},$$

we get the required sequence.

Remark. *For large M a "small" common difference cannot do the job, because in such a case the sequence would have at least one of its terms in an interval of the form $[10^k, 10^k + d]$ and all the integers from such an interval have the sum of their digits at most $1 + 9 \cdot \log_{10} d$.*

Problem G1

Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

Lemma. *If M is an interior point of the convex quadrilateral $ABCD$ then $MA + MB < AC + CD + DB$.*

Proof of the lemma (Figure 1). The ray $(AM$ intersects the quadrilateral in N; suppose, for instance, that $N \in [CD]$. Then $MA + MB < MA + MN + NB \leq AN + NC + CB \leq AD + DN + NC + CB = AD + DC + CB$.

Solution of the problem (Figure 2). The median triangle DEF divides triangle ABC into four regions. Each region is covered by at least two of the convex quadrilaterals $ABDE$, $BCEF$, $CAFD$. If, for instance, M belongs to $[ABDE]$ and $[BCEF]$ then $MA + MB < BD + DE + EA$ and $MB + MC < CE + EF + FB$. By adding these two inequalities we get $MB + (MA + MB + MC) \leq AB + BC + CA$, which implies the required conclusion.

Another solution. Let O be the circumcenter of ΔABC.
Case 1: $O \in [ABC]$ (Figure 3). Suppose $M \in [ADOE]$. Then $MA = \min\{MA, MB, MC\}$ and $MA + MB \leq NA + NB$, $MA + MC \leq NA + NC$. Since $NA + NB \leq OA + OB$ and $NA + NC \leq AD + DC$ (for $O([ANC]$; otherwise $NA + NC < BD + DC$) it is enough to prove that $m_c + (c/2) + 2R < a + b + c$ (with the usual notations). Since $m_c < (a + b)/2$, it is enough to prove that $4R < a + b + c$ (in a nonobtuse-angled triangle).
This reduces to $\sin A + \sin B + \sin C > 2$, or

$$\sin\frac{A}{2}\cos\frac{A}{2} + \cos\frac{A}{2}\cos\frac{B-C}{2} > 1.$$

For fixed A the minimum value of the left member is obtained when $B - C$ has maximum value, that is $B = \pi/2$, $C + A = \pi/2$ and the minimum is $(1/2)(\sin A + 1 + \cos A)$ therefore it is greater than $(1/2)(1 + 1) = 1$.

Case 2: ABC is obtuse-angled (Figure 4). Suppose for instance that $m(\widehat{B}) > 90^\circ$. Let P and Q be the intersections of the perpendicular bisectors of the sides $[AB]$ and $[BC]$ with AC.
For $M \in [ADP]$, $\min\{MA, MB, MC\} = MA$ and, as above, $MA + MB \leq PA + PB = 2 \cdot AP$, $MA + MC \leq m_c + (c/2) < (a + b + c)/2$. It is enough to prove that $4 \cdot AP < a + b + c$, which is implied by $4AP < 2b < a + b + c$. A similar argument works in the case $M \in [CEQ]$.
For $M \in [BEQPD]$, let $BM \cap AC = N$.
Then $\min\{MA, MB, MC\} = MB$ and $MB + MC \leq NB + NC$, $MB + MA \leq NB + NA$, $NB \leq \max\{PB, QB\}$, therefore it is enough to prove that $2 \cdot PB < AB + BC$. Since $PB < BD + DP < BD + PP'$, the needed relation is obvious.

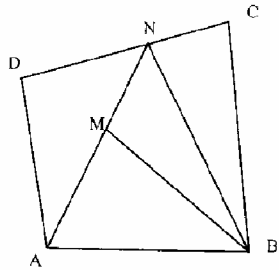


Figure 1

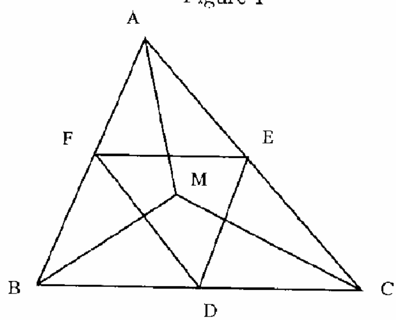


Figure 2

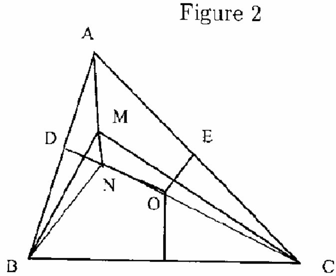


Figure 3

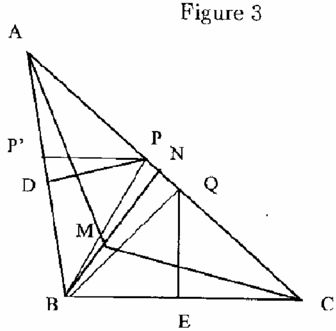


Figure 4

Problem G2

A circle is called a **separator** *for a set of five points in a plane if it passes through three of these points, it contains a fourth point inside and the fifth point is outside the circle.*

Prove that every set of five points such that no three are collinear and no four are concyclic has exactly four separators.

Solution. Let $\{A, B, C, D\}$ be the inverses of four of the points of the set through an inversion having as pole the fifth point. Notice that a separator which passes through the pole is transformed into a straight line which passes through two of the points A, B, C, D and separates the other two. Notice also that a separator which does not pass through the pole is transformed into a circle which passes through three of the points A, B, C, D and contains the fourth point inside.

Considering K-the convex hull of the set $\{A, B, C, D\}$, two cases may appear.

Case 1: K =quadrilateral (for instance $ABCD$). In this case we have as separators the two diagonals of the quadrilateral and one circle from each pair $\{(ABC), (ABD)\}$, $\{(CDA), (CDB)\}$ (the one corresponding to the smaller angle from $\{\widehat{ACB}, \widehat{ADB}\}$ and $\{\widehat{CAD}, \widehat{CBD}\}$ respectively).

Case 2: K=triangle (for instance ΔABC). In this case we have as separators, with obvious arguments, the straight lines DA, DB, DC and the circle (ABC).

Another solution. Consider coordinates and the points $A_i(x_i, y_i)$, $i = 1, 2, 3, 4, 5$. Let d_{ijkl} be the value of the determinant $|x_p^2 + y_p^2 \; x_p \; y_p 1 \; 1|_{p=i,j,k,l}$. The circle $(A_1A_2A_3)$ is a separator iff d_{1234} and d_{1235} have different signs. Let us look at the ten pairs (d_{1234}, d_{1235}), (d_{1243}, d_{1245}), etc. corresponding to the ten circles which pass through three of the five given points. Denoting by a_n the number d_{ijkl} ($i < j < k < l$, $\{i, j, k, l, n\} = \{1, 2, 3, 4, 5\}$), the ten pairs are (a_5, a_4), $(-a_5, a_3)$, $(-a_4, -a_3)$, (a_5, a_2), $(a_4, -a_2)$, (a_3, a_2), $(-a_5, a_1)$, $(-a_4, -a_1)$, $(-a_3, a_1)$, $(-a_2, -a_1)$.

We notice that the number of separators is equal to the number of pairs of terms with the same sign from the sequence $S = (a_1, -a_2, a_3, -a_4, a_5)$. We also remark (using the determinant $|x_p^2 + y_p^2 \;\; x_p \;\; y_p \;\; 1 \;\; 1|_{p=1,2,3,4,5}$) that $a_1 - a_2 + a_3 - a_4 + a_5 = 0$. This shows that S cannot have all its terms of the same sign.

We claim also that S cannot have four terms of the same sign. Indeed, if four terms have the same sign then all the six circles passing through one point of the given five are separators. Taking this point as origin, the Ox axis through an other of the given points and passing to polar coordinates we would get, for instance,

$$\begin{vmatrix} r_1 & \cos a & \sin a \\ r_2 & \cos b & \sin b \\ r_3 & \cos c & \sin c \end{vmatrix} > 0, \quad \begin{vmatrix} r_1 & \cos a & \sin a \\ r_2 & \cos b & \sin b \\ r_4 & 1 & 0 \end{vmatrix} < 0,$$

$$\begin{vmatrix} r_1 & \cos a & \sin a \\ r_3 & \cos c & \sin c \\ r_4 & 1 & 0 \end{vmatrix} > 0, \quad \text{and} \quad \begin{vmatrix} r_2 & \cos b & \sin b \\ r_3 & \cos c & \sin c \\ r_4 & 1 & 0 \end{vmatrix} < 0.$$

This would lead to

$$r_1 \sin(c-b) + r_2 \sin(a-c) + r_3 \sin(b-a) > 0,$$
$$-r_1 \sin b + r_2 \sin a + r_4 \sin(b-a) < 0,$$
$$-r_1 \sin c + r_3 \sin a + r_4 \sin(c-a) > 0,$$
$$-r_2 \sin c + r_3 \sin b + r_4 \sin(c-b) < 0.$$

Multiplying the last three relations by r_3, $-r_2$ and r_1 respectively and adding the results we would get $r_4[r_1 \sin(c-b) + r_2 \sin(a-c) + r_3 \sin(b-a)] < 0$, impossible.

Hence three terms of S have the same sign and the other two have the other sign, therefore there are exactly four separators.

Problem G3

A set S of points from the space will be called **completely symmetric** *if it has at least three elements and fulfils the condition:*

for every two distinct points A,B from S the perpendicular bisector plane of the segment AB is a plane of symmetry for S.

Prove that if a completely symmetric set is finite then it consists of the vertices of either of a regular polygon, or a regular tetrahedron or a regular octahedron.

Solution. Denote by r_{PQ} the reflection in the perpendicular bisector plane of a segment PQ and let G be the barycenter of S. From $r_{AB}(S) = S$ we get $r_{AB}(G) = G$ for every $A, B \in S$, therefore all the points of S are at the same distance from G. This shows that S is included in a sphere Σ.

Case 1: S is included in a plane π. In this case S is included in the circle $\Sigma \cap \pi$ and its points form a convex polygon $A_1A_2...A_n$. The reflection in the perpendicular bisector of A_1A_3 transforms each half-plane bordered by A_1A_3 into itself, therefore the point $r_{A_1A_3}(A_2)$ can be only A_2. Hence $A_1A_2 = A_2A_3$. In the same way $A_2A_3 = A_3A_4 = ... = A_nA_1$. Since S is included in a circle, this proves that $A_1A_2...A_n$ is a regular polygon.

Case 2: the points of S are not coplanar. In this case the points of S are the vertices of a convex polyhedron P (since they are on the sphere Σ). Each face $A_1A_2...A_k$ of P is invariant under every reflection $r_{A_iA_j}$, $1 \le i < j \le k$, therefore it is a regular polygon (from case 1).

Notice also that every reflection r_{AB}, $A, B \in S$ transforms a face of P into a face of P.

Take now a vertex V of P and denote P's edges issuing from V by $VV_1, VV_2, \ldots, VV_s$, such that $(VV_1, VV_2), (VV_2, VV_3), \ldots, (VV_s, VV_1)$ are on the same face (Figure 1). Notice that the intersection of the half-planes $[V_1V_3, V_2$ and $[V_1V_3, V$ with P are triangles $V_1V_2V_3$ and V_1V_3V respectively. The reflection $r_{V_1V_3}$ transforms each of these half-planes into itself, therefore it can transform V_2 and V only into themselves. This shows that $r_{V_1V_3}$ must transform the face containing (VV_2, VV_3) into the face containing (VV_2, VV_1). Hence these faces are congruent.

In the same way every two faces of P having a common side are congruent. This shows that all P's faces are congruent, because every two faces can be 'linked' by a chain of faces so that every two consecutive faces of the chain have a common edge. It follows that P is a regular polyhedron (a similar argument shows that from each vertex emerges the same number of edges).

It remains to rule out the cube, the regular dodecahedron and the regular icosahedron. The cube (Figure 2) is ruled out because of the reflection $r_{AC'}$ (the rectangle $ACC'A'$ should be invariant, but it isn't). The dodecahedron (Figure 3) is excluded because of the reflection $r_{A_3B_1}$ (same argument for the rectangle $A_1A_3B_3B_1$). Finally, the icosahedron (Figure 4) is eliminated because of the reflection r_{AB} (use rectangle AA_1BB_1).

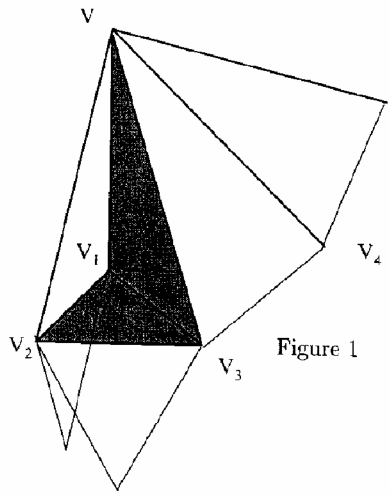


Figure 1

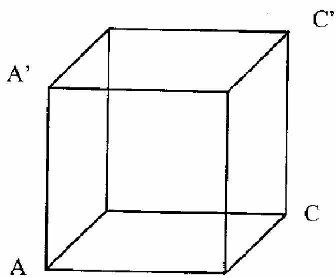


Figure 2

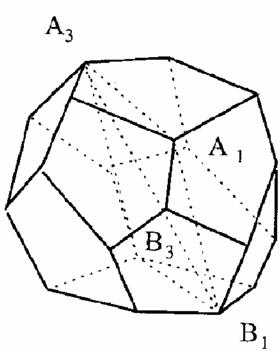


Figure 3

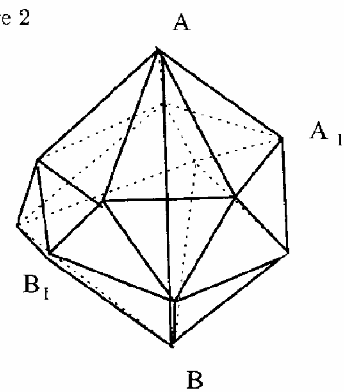


Figure 4

Problem G4

For a triangle $T = ABC$ we take the point X on the side (AB) such that $AX/XB = 4/5$, the point Y on the segment (CX) such that $CY = 2YX$ and, if possible, the point Z on the ray $(CA$ such that $\widehat{CXZ} = 180^\circ - \widehat{ABC}$. We denote by Σ the set of all triangles T for which $\widehat{XYZ} = 45^\circ$.

Prove that all the triangles from Σ are similar and find the measure of their smallest angle.

Solution. A convenient way to describe the position of the points using trigonometry is to employ the cotangent.We firstly prove the following

Lemma (Figure 1). In a triangle ABC, $X \in (AB)$, $XA : XB = m : n$, $\widehat{CXB} = \alpha$ and $\widehat{ACX} = \beta$. Then

$$(m+n)\cot\alpha = n\cot A - m\cot B \quad \text{and} \quad m\cot\beta = (m+n)\cot C + n\cot A.$$

Proof of the lemma. Let $CF = h$ be the altitude from C. Then, using oriented segments, $AX = |\overline{AF} + \overline{FX}| = h \cdot \cot A - h \cdot \cot\alpha$ and $BX = |\overline{BF} + \overline{FX}| = h\cot B + h\cot\alpha$. The first part of the conclusion follows now from $n \cdot XA = m \cdot XB$.

For the second part take $XT \| BC$, $T \in (AB)$. Then $\widehat{XTA} = C$ and $CT : TA = n : m$. The required result follows from the first part applied in $\triangle AXC$. □

We have (Figure 2) by the lemma and the hypothesis $4\cot\widehat{ACX} = 9\cot C + 5\cot A$ and

$$\cot\widehat{ACX} - 2\cot\widehat{CXZ} = 3\cot\widehat{XYZ} = 3.$$

We also have $\widehat{CXZ} = 180^\circ - B$, therefore $(9\cot C + 5\cot A)/4 + 2 \cdot \cot B = 3$, that is

$$45\cot A + 8\cot B + 9\cot C = 12.$$

We will prove that this equation specifies the angles of the triangle ABC. Denoting $\cot A = x$, $\cot B = y$, $\cot C = z$ we have $5x+8y+9z = 12$ and the well-known relation $xy + yz + zx = 1$.

Eliminating z we get $(x+y)(12-5x-8y)+9xy = 9$, that is $(4y+x-3)^2+9(x-1)^2 = 0$. This shows that $x = 1$, $y = \dfrac{1}{2}$ and $z = \dfrac{1}{3}$, therefore all the triangles from Σ are similar and their smallest angle is $A = 45^\circ$.

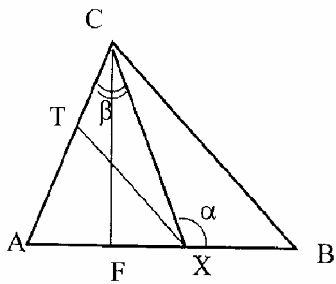


Figure 1

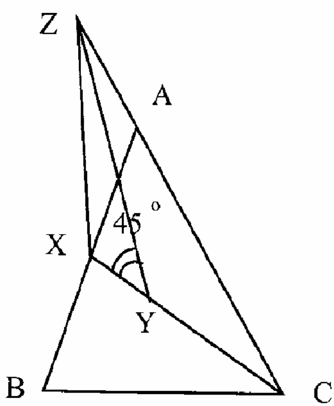


Figure 2

Problem G5

Let ABC be a triangle, Ω its incircle and $\Omega_a, \Omega_b, \Omega_c$ three circles orthogonal to Ω passing through (B,C), (A,C) and (A,B) respectively. The circles Ω_a and Ω_b meet again in C'; in the same way we obtain the points B' and A'. Prove that the radius of the circumcicle of $A'B'C'$ is half the radius of Ω.

Solution. Denote by I the incenter, by r the inradius, by D, E, F the contacts of the incircle with BC, CA, AB respectively and by P, Q, R the midpoints of the segments $[EF], [FD], [DE]$.

We will prove that Ω_a is the circle (B,C,Q,R). We firstly notice that from the right-angled triangles IBD and IDQ we get $IQ \cdot IB = ID^2 = r^2$ and in the same way $IR \cdot IC = r^2$, therefore the points B, C, R, Q are on a circle Γ_a. The points Q and R belong to the segments (IB) and (IC), so I is exterior to the circle (B,Q,R,C) and I's power with respect to the circle (B,Q,R,C) is $IB \cdot IQ = r^2$, which is the condition of Ω beeing orthogonal to the circle (B,Q,R,C).

In the same way Ω_b is the circle (C,R,P,A) and Ω_c is the circle (A,P,Q,B). It follows that A', B', C' coincide with P, Q, R and the required conclusion is now obvious.

Another solution. Let O_a be the center of Ω_a and M be the midpoint of (BC). Denote, using oriented segments, $\overline{MO_a} = x$ (the positive sense on the perpendicular bisector of (BC) beeing $\overrightarrow{ID}$).

The radius of Ω_a is $x^2 + \dfrac{a^2}{4}$ and

$$IO_a^2 = DM^2 + (\overline{ID} + \overline{MO_a})^2 = \frac{a}{2} - \left(\frac{p-b}{2}\right)^2 + (r+x)^2.$$

The condition of Ω and Ω_a beeing orthogonal is $O_aI^2 = O_aB^2 + r^2$, that is

$$x^2 + \frac{a^2}{4} + r^2 = \left(\frac{b-c}{2}\right)^2 + (r+x)^2 \Leftrightarrow x = \frac{(p-b)(p-c)}{2r} \Leftrightarrow X = 2R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

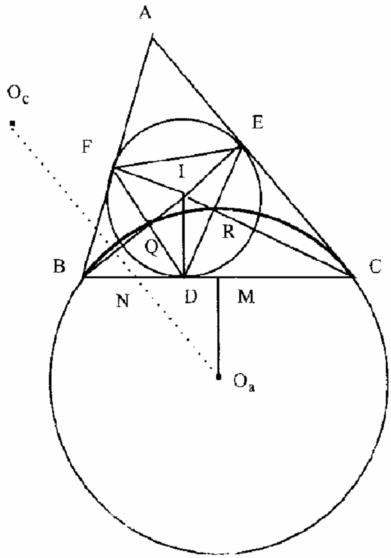
It follows that

$$OO_a = \overline{OM} + \overline{MO_a} = 2R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = R + \frac{r}{2}.$$

Therefore O_a, O_b, O_c are on the circle of center O and radius $R + \dfrac{r}{2}$. Let $\{N\} = O_aO_c \cap BC$. The angle $\widehat{O_aOO_c}$ has measure $\pi - B$ (regardless of the position of B), so $\widehat{OO_aO_c} = \dfrac{B}{2}$ and

$$MN = x\tan\frac{B}{2} = 2R\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{p-c}{2}, \quad BN = \frac{a}{2} - \frac{p-c}{2} = \frac{p-b}{2}.$$

Hence $BN = \dfrac{BD}{2}$ and, because $MN < BM$, $N \in (BM)$. The same holds for the common point of AB and O_aO_c, therefore the reflection of B in O_aO_c is on DF. This proves that B' — the second common point of Ω_a and Ω_c — is the midpoint of (DF). The conclusion follows now easily.


A
O_c
E
F
I
R
Q
B
C
N
D
M
O_a

Problem G6

Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersect Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

Solution.

Lemma. *The circle k_1 touches internally circle k at A and touches one of k's chords MN at B. Let C be the mid-point of k's arc $\widehat{MN}$ which does not contain A. Then the points A, B, C are collinear and $CA \cdot CB = CM^2$.*

Proof of the lemma (fig.1). The homothety with center A which transforms k_1 into k transforms MN into a tangent at k parallel to MN, i.e. into the tangent at C to k, so A, B, C are collinear.

For the second part notice that $\widehat{NMC} \equiv \widehat{CAM}$, therefore $\Delta ACM \sim \Delta MCB$, whence $CA \cdot CB = CM^2$.

Solution of the problem. Let O_1 and O_2 be the centers of Ω_1 and Ω_2 respectively and t_1, t_2 their common tangents (fig.2). Let α, β be the arcs cut from Ω by t_1 and t_2, positioned like in the lemma.

Their midpoints have, according to the lemma, equal powers with respect to Ω_1 and Ω_2, therefore they are on the radical axis of the two circles. Thus A and B are the midpoints of α and β. From the lemma we also conclude that C and D are the points in which the tangents t_1 and t_2 touch Ω_1. If H is the homothety with center M transforming Ω_1 into Ω_2 then $H : CD \mapsto AB$ whence $AB \| CD$. Therefore $CD \perp O_1O_2$ and O_2 is midpoint of one of the arcs $\widehat{CD}$ from Ω_2.

Let X be the point in which t_1 touches Ω_2. We get $\widehat{XCO_2} \equiv (1/2)\widehat{CO_1O_2} \equiv \widehat{DCO_2}$, so O_2 lies on the bisector of the angle $\widehat{XCD}$, therefore CD is tangent to the circle Ω_2.

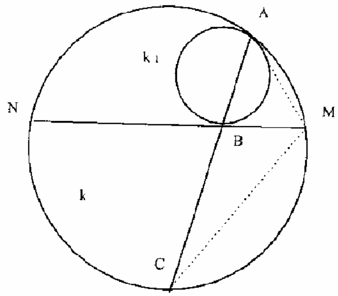


Figure 1

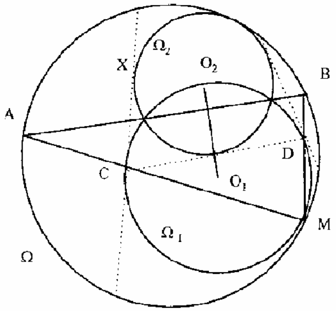


Figure 2

Another solution (Figure 3). An inversion of pole N transforms the figure as follows: Ω and Ω_2 in parallel straight lines, Ω_1 in a circle which passes through the reflection of N in Ω_2 and is tangent to Ω, AB in the circle (congruent with Ω_1) passing through N and through $\Omega_1 \cap \Omega_2$, AM and BM into the circles (congruent) passing through N, M and A, B respectively, therefore CD becomes circle (NCD). We have to prove that (NCD) is tangent to Ω_2.

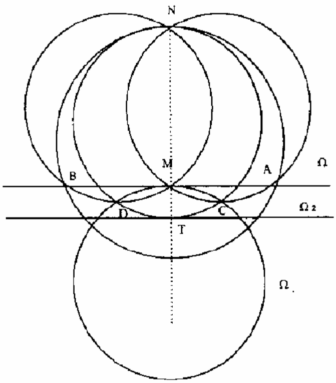


Figure 3

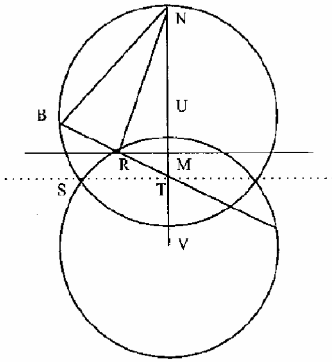


Figure 4

Denote by d the distance between the centers U and V of the circles Ω_1 and (NAB), by r their radiuses and by T the midpoint of the common chord Ω_2 (Figure 4). It follows that

$$TU = \frac{d}{2}, \quad TS^2 = r^2 - \frac{d^2}{4}, \quad TM = r - \frac{d}{2}, \quad TN = r + \frac{d}{2}.$$

The circle (CDN) has its center on MN and $ND \perp BD$, therefore it is enough to prove that B, D, T are collinear, that is the common point R of BT and Ω_2 is the projection of N on BT. This results from $BR \cdot 2BT = BM^2 = BN^2 - NM^2 = BN^2 + BT^2 - NT^2$, the last equality being justified by $NM^2 = d^2$, $BT^2 - NT^2 = BU^2 + UM^2 - 2 \cdot UM \cdot UT - NT^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2 \cdot \frac{d}{2}(d-r) - (r + \frac{d}{2})^2 = -d^2$.

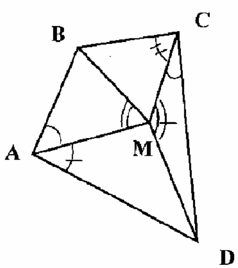
Problem G7

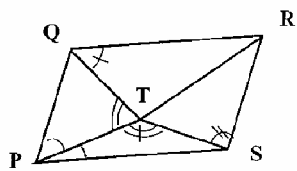
The point M is inside the convex quadrilateral $ABCD$, such that

$$MA = MC, \widehat{AMB} = \widehat{MAD} + \widehat{MCD} \text{ and } \widehat{CMD} = \widehat{MCB} + \widehat{MAB}.$$

Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

Solution. Construct the convex quadrilateral $PQRS$ and the interior point T such that $\Delta PTQ \equiv \Delta AMB$, $\Delta QTR \sim \Delta AMD$ and $\Delta PTS \sim \Delta CMD$.





It follows that

$$TS = \frac{MD \cdot PT}{MC} = MD, \quad \frac{TR}{TS} = \frac{MD \cdot MB}{MA \cdot MD} = \frac{MB}{MC}$$

and $\widehat{STR} \equiv \widehat{BMC}$, therefore $\Delta RTS \sim \Delta BMC$. The assumption on angles leads to

$$\widehat{QPS} + \widehat{RSP} = \widehat{QPT} + \widehat{TPS} + \widehat{TSP} + \widehat{TSR} = \widehat{PTS} + \widehat{TPS} + \widehat{TSP} = 180^\circ$$

and

$$\widehat{RQP} + \widehat{SPQ} = \widehat{RQT} + \widehat{TQP} + \widehat{TPQ} + \widehat{TRS} = \widehat{QTP} + \widehat{TQP} + \widehat{TPQ} = 180^\circ,$$

so $PQRS$ is a parallelogram.

Hence $PQ = RS$ and $QR = PS$, that is

$$AB = \frac{BC \cdot TS}{MC} = \frac{BC \cdot MD}{MC} \text{ and } \frac{AD \cdot QT}{AM} = \frac{CD \cdot TS}{MD} = CD.$$

The conclusion is now obvious.

Problem A1

Let $n \geq 2$ be a fixed integer. Find the least constant C such that the inequality

$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C\Big(\sum_i x_i\Big)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands).
For this constant C, characterize the instances of equality.

Solution. The inequality is symmetric and homogeneous, so we can suppose that $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ and $\sum_i x_i = 1$. In this case we have to maximize the sum

$$F(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j (x_i^2 + x_j^2).$$

We try to increase the value of F by replacing the vector $x = (x_1, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$ with $x' = (x_1, \ldots, x_{k-1}, x_k + x_{k+1}, 0, \ldots, 0)$ (x_{k+1} is the last nonzero coordinate and we suppose that $k \geq 2$):

$$\begin{aligned} F(x') \quad F(x) &= x_k x_{k+1}\Big[3(x_k + x_{k+1}) \sum_{i=1}^{k-1} x_i - x_k^2 - x_{k+1}^2\Big] = \\ &= x_k x_{k+1}[3(x_k + x_{k+1})(1 - x_k - x_{k+1}) - x_k^2 - x_{k+1}^2] = \\ &= x_k x_{k+1}[(x_k + x_{k+1})(3 - 4(x_k + x_{k+1})) + 2x_k x_{k+1}]. \end{aligned}$$

From

$$1 \geq x_1 + x_k + x_{k+1} \geq \frac{1}{2}(x_k + x_{k+1}) + x_k + x_{k+1}$$

it follows that $2/3 \geq x_k + x_{k-1}$, and therefore

$$F(x') - F(x) > 0.$$

Applying the above replacements several times we obtain

$$\begin{aligned} F(x) &\leq F(a, b, 0, \ldots, 0) = ab(a^2 + b^2) = \frac{1}{2}(2ab)(1 - 2ab) \leq \\ &\leq \frac{1}{8} = F\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right). \end{aligned}$$

Thus the constant C is equal to $1/8$. The equality occurs if and only if two of the x_i's are equal (possibly zero) and the remaining ones are zero.

Another solution. We shall use the symmetric polynomials $p_k = \sum_{i=1}^{n} x_i^k$ and $s_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$ ($s_k = 0$ if $k > n$) and Newton's relation

$$p_4 - s_1 p_3 + s_2 p_2 - s_3 p_1 + 4s_4 = 0.$$

We still take $p_1 = s_1 = 1$ and we obtain:

$$\begin{aligned} F(x_1, \ldots, x_n) &= \sum_{i<j} x_i x_j (x_i^2 + x_j^2) = \sum_{i=1}^{n} \left(x_i^3 \sum_{j \neq i} x_j \right) = \\ &= \sum_{i=1}^{n} x_i^3 (1 - x_i) = p_3 - p_4 = s_2 p_2 - s_3 + 4 s_4 = \\ &= s_2 (1 - 2 s_2) + 4 s_4 - s_3. \end{aligned}$$

By adding the inequalities :

$$s_2(1 - 2s_2) \leq \frac{1}{8} \tag{1}$$

and

$$\begin{aligned} 4s_4 - s_3 &= \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} \left(\sum_{i \notin \{i_1, i_2, i_3\}} x_i \right) - \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} = \\ &= \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} \left(\sum_{i \notin \{i_1, i_2, i_3\}} x_i - 1 \right) = \\ &= - \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} (x_{i_1} + x_{i_2} + x_{i_3}) \leq 0 \end{aligned} \tag{2}$$

we obtain $F(x) \leq 1/8$.

The inequalities (1) and (2) turn into equalities if and only if $s_4 = 1/4$ and at least $n - 2$ of x_i's are zero. This gives the instances of equality obtained in the first solution.

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From

$$1 \geq x_1 + x_k + x_{k+1} \geq \frac{1}{2}(x_k + x_{k+1}) + x_k + x_{k+1}$$

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Applying the above replacements several times we obtain

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Thus the constant C is equal to $1/8$. The equality occurs if and only if two of the x_i's are equal (possibly zero) and the remaining ones are zero.

Another solution. We shall use the symmetric polynomials $p_k = \sum\limits_{i=1}^{n} x_i^k$ and $s_k = \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$ ($s_k = 0$ if $k > n$) and Newton's relation

$$p_4 - s_1 p_3 + s_2 p_2 - s_3 p_1 + 4 s_4 = 0.$$

We still take $p_1 = s_1 = 1$ and we obtain:

$$\begin{aligned} F(x_1, \ldots, x_n) &= \sum_{i<j} x_i x_j (x_i^2 + x_j^2) = \sum_{i=1}^{n} \left(x_i^3 \sum_{j \neq i} x_j \right) = \\ &= \sum_{i=1}^{n} x_i^3 (1 - x_i) = p_3 - p_4 = s_2 p_2 - s_3 + 4 s_4 = \\ &= s_2(1 - 2s_2) + 4s_4 - s_3. \end{aligned}$$

By adding the inequalities :

$$(1) \qquad s_2(1 - 2s_2) \leq \frac{1}{8}$$

and

$$(2) \qquad \begin{aligned} 4s_4 - s_3 &= \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} \left(\sum_{i \notin \{i_1, i_2, i_3\}} x_i \right) - \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} = \\ &= \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} \left(\sum_{i \notin \{i_1, i_2, i_3\}} x_i - 1 \right) = \\ &= - \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} (x_{i_1} + x_{i_2} + x_{i_3}) \leq 0 \end{aligned}$$

we obtain $F(x) \leq 1/8$.

The inequalities (1) and (2) turn into equalities if and only if $s_4 = 1/4$ and at least $n - 2$ of x_i's are zero. This gives the instances of equality obtained in the first solution.

Problem A2

The numbers from 1 to n^2 are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any pair of numbers situated on the same row or on the same column the ratio of the greater number to the smaller one is calculated.

Let us call the **characteristic** *of the arrangement the smallest of these $n^2(n-1)$ fractions. What is the highest possible value of the characteristic?*

Solution. Let us firstly prove that for any arrangement A its characteristic $C(A)$ is less or equal than $(n+1)/n$.

If two of the greatest numbers $\{n^2-n+1, n^2-n+2, \ldots, n^2-1, n^2\}$ lie on the same row or column, we have

$$C(A) \leq \frac{a}{b} \leq \frac{n^2}{n^2-n+1} < \frac{n+1}{n}.$$

If all these numbers are in different rows and columns, then two of them are on the same row or column with the number n^2-n, so we have

$$C(A) \leq \frac{a}{n^2-n} \leq \frac{n^2-1}{n^2-n} = \frac{n+1}{n}.$$

Now the arrangement $a_{ij} = \begin{cases} i+n(j-i-1) & \text{if } i<j \\ i+n(n-i+j-1) & \text{if } i \geq j \end{cases}$

$1+(n-1)n$	1	$1+n$	$\cdots$	$1+(n-3)n$	$1+(n-2)n$
$2+(n-2)n$	$2+(n-1)n$	2	$\cdots$	$2+(n-4)n$	$2+(n-3)n$
$3+(n-3)n$	$3+(n-2)n$	$3+(n-1)n$	$\cdots$	$3+(n-5)n$	$3+(n-4)n$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$(n-2)+2n$	$(n-2)+3n$	$(n-2)+4n$	$\cdots$	$n-2$	$(n-2)+n$
$(n-1)+n$	$(n-1)+2n$	$(n-1)+3n$	$\cdots$	$n-1+(n-1)n$	$n-1$
n	$n+n$	$n+2n$	$\cdots$	$n+(n-2)n$	$n+(n-1)n$

has the characteristic $C(A) = (n+1)/n$:

- the difference of any two numbers lying on the same row is a multiple of n, therefore

$$\frac{a_{ik}}{a_{ij}} = \frac{a_{ij}+hn}{a_{ij}} \geq \frac{a_{ij}+n}{a_{ij}} \geq \frac{n^2}{n^2-n} > \frac{n+1}{n};$$

- on the first column we have the arithmetic progression

$$n \leq (n-1)+n \leq (n-2)+2n \leq \ldots \leq 2+(n-2)n \leq 1+(n-1)n.$$

Thus

$$\frac{a_{i1}}{a_{k1}} \geq \frac{n^2-n+1}{n^2-2n+2} \geq \frac{n+1}{n} \quad \text{(with equality if } n=2\text{)}.$$

- the column $j = 2, \ldots, n-1$ contains the elements

$$j - 1, j - 2 + n, j - 3 + 2n, \ldots, 1 + (j-2)n, n + j(n-1), \ldots, j + (n-1)n.$$

Therefore

$$\frac{a_{ij}}{a_{kj}} \geq \frac{j + (n-1)n}{j + 1 + (n-2)n} \geq \frac{n+1}{n} \quad \text{(with equality for } j = n-1\text{)}.$$

Problem A3

A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, in an arbitrary order, exchange the balls they have at that moment. The game is called **nice** *if at the end nobody has her own ball and it is called* **tiresome** *if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.*

Solution. A game with n players is determinated by ordering the $N = \binom{n}{2}$ transpositions (i,j) of the set $\{1,\ldots,n\} : t_1,\ldots,t_N$. The game is nice if the permutation $P = t_N \circ \ldots \circ t_2 \circ t_1$ has no fixed point and it is tiresome if P is the identity.

Claim 1. There exists a nice game with n players if and only if $n \neq 3$.

If $n = 3$ and the players are denoted such that $t_1 = (a,b)$, $t_2 = (a,c)$, $t_3 = (b,c)$, then

$$P = t_3 t_2 t_1 = (a,c)$$

has a fixed point.

For every n and the changes made in the order $(1,2),(1,3),\ldots,(1,n),(2,3)$, $(2,4),\ldots,(2,n),\ldots,(n-1,n)$ we get, using induction

$$\begin{aligned} P &= (n-1,n)(n-2,n)(n-2,n-1)..(2,3)(1,n)(1,n-1)..(1,3)(1,2) = \\ &= \begin{pmatrix} 2 & 3 & .. & i & .. & n \\ n & n-1 & .. & n-i+2 & .. & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & .. & n \\ 2 & 3 & .. & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 2 & .. & i & .. & n \\ n & n-1 & .. & n-i+1 & .. & 1 \end{pmatrix} \end{aligned}$$

and, for $n = 2k$, we have no fixed point: $i \neq 2k - i + 1$.

For $n = 2k+1$, $k \geq 2$, prolonge the previous ordering by

$$(1,2k+1),(2,2k+1),..,(k,2k+1),(2k,2k+1),(2k-1,2k+1),..,(k+1,2k+1):$$

$$\begin{aligned} P &= (k+1,2k+1)..(2k,2k+1)(k,2k+1)..(1,2k+1)(2k-1,2k)..(1,2) = \\ &= \begin{pmatrix} 1 & 2 & .. & k-1 & k & k+1 & k+2 & .. & 2k & 2k+1 \\ 2 & 3 & .. & k & 2k & 2k+1 & k+1 & .. & 2k-1 & 1 \end{pmatrix} \circ \\ &\quad \circ \begin{pmatrix} 1 & 2 & .. & k-1 & k & k+1 & .. & 2k \\ 2k & 2k-1 & .. & k+2 & k+1 & k & .. & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 2 & .. & k-1 & k & k+1 & k+2 & .. & 2k & 2k+1 \\ 2k-1 & 2k-2 & .. & k+1 & 2k+1 & 2k & k & .. & 2 & 1 \end{pmatrix} \end{aligned}$$

has no fixed points ($k+1 \neq 2k$ because $k \neq 1$).

Claim 2. There exists a tiresome game with n players if and only $n = 4k$ or $n = 4k+1$.

Indeed, the signature of the permutation P is equal to $(-1)^{\binom{n}{2}}$ and thus n must be $4k$ or $4k+1$. If $n = 4$ we have the solution:

(1) $$(3,4)(1,3)(2,4)(2,3)(1,4)(1,2) = \text{id}.$$

Between two groups of four players we choose the partial game:

$$(4,7)(3,7)(4,6)(1,6)(2,8)(3,8)(2,7)(2,6)(4,5)\\ (4,8)(1,7)(1,8)(3,5)(3,6)(2,5)(1,5) = \text{id}. \tag{2}$$

With (1) and (2) we can build a tiresome $4k$-game: divide the $4k$ players into k groups, each of them containing 4 players; then play the game (1) inside each group and the game (2) between any two different groups, in an arbitrary order.

An identity $4k+1$-game can be obtained from a $4k$-game by inserting in each block of moves of type (1) the transpositions with $4k+1$ such that the block is not disturbed:

$$[(3,5)(3,4)(4,5)](1,3)(2,4)(2,3)(1,4)[(1,5)(1,2)(2,5)] =\\ = (3,4)(1,3)(2,4)(2,3)(1,4)(1,2) = \text{id}. \tag{3}$$

Another solution. We can prove the second claim in the case $n = 4k$ by induction on k. If there exists a tiresome game G_k for $4k$ then, noticing that

$$\begin{aligned} H_{k+1} &= [(1,n+2)(2,n+2)..(n,n+2)] \circ \\ &\quad \circ (n+2,n+3)[(n,n+3)(n-1,n+3)..(1,n+3)] = \\ &= (n+2,n+3) \end{aligned}$$

and

$$\begin{aligned} I_{k+1} &= [(1,n+4)(2,n+4)..(n,n+4)] \circ \\ &\quad \circ (n+1,n+4)[(n,n+1)(n-1,n+1)..((1,n+1)] = \\ &= (n+1,n+4), \end{aligned}$$

we can take for $n = 4k+4$ the game

$$G_{k+1} = G_k(n+3,n+4)(n+1,n+3)(n+2,n+4)H_{k+1}I_{k+1}(n+1,n+2)$$

and, according to (1), this is equal to id.

The case $n = 4k+1$ can be solved as above.

Problem A4

Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

Solution. Denote $f(x,y) = x^2 - xy + y^2$ and suppose that one can find such a partition: $\mathbb{N}^* = A \coprod B \coprod C$.

Suppose also that $1 \in A$, $b \in B$, $c \in C$ $(b < c)$ are the first elements of the three classes (this implies that $1, 2, \ldots, b-1 \in A$).

Lemma 1. *x, y and $x+y$ cannot belong to the three different subsets.*

Proof. If $x \in A$, $y \in B$, $x + y \in C$ then $z = f(x+y, x) = f(x+y, y)$ should be both in B and A. □

Lemma 2. *The subset C contains a multiple of b. If kb is the first such multiple, then $(k-1)b \in B$.*

Proof. Let r be the residue of $c (\bmod b)$. If $r = 0$, then $c = nb \in C$. If $r > 0$, $c - r$ is not in C because c is minimal; also $c - r$ is not in B because $r \leq b - 1$ is in A and $r + (c - r) = c$. Hence $c - r \in A$, $b \in B$ and $f(c - r, b) = nb$ belongs to C.

The last statement of the lemma follows immediately from Lemma 1.□

Lemma 3. *For every positive integer n we have $(nk-1)b+1 \in A$ and $nkb+1 \in A$.*

Proof. Use induction on n. Take $n = 1$. Then $(k-1)b + 1 \notin C$ because $1 \in A$, $(k-1)b \in B$. Also $(k-1)b \notin B$ because $b - 1 \in A$, $kb \in C$. Hence $(k-1)b + 1 \in A$.

Similarly, $kb + 1 \notin C$ because $(k-1)b + 1 \in A$, $b \in B$ and $kb + 1 \notin B$ because $1 \in A$, $kb \in C$. Therefore $kb + 1 \in A$.

Suppose now that $((n-1)k - 1)b + 1$ and $(n-1)kb + 1$ belong to A.

We have $(n-1)kb + 1 \in A$, $(k-1)b \in B$, hence $(nk-1)b + 1 \notin C$ and also $((n-1)k-1)b+1 \in A$, $kb \in C$, hence $(nk-1)b+1 \notin B$. Thus we get $(nk-1)b+1 \in A$. Further, $(nk-1)b+1 \in A$, $b \in B$ implies $nkb+1 \notin C$ and $(n-1)kb+1 \in A$, $kb \in C$ implies $nkb + 1 \notin B$. Therefore $nkb + 1 \in A$ and the lemma is proven. □

A contradiction follows now easily: $kb+1 \in A$ and $kb \in C$, therefore $f(kb+1, kb) =$ $= (kb+1)kb + 1$ should be in A (Lemma 3) and also in B.

Problem A5

Find all the functions $f : \mathbb{R} \to \mathbb{R}$ ~~*which satisfy*~~ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution. Let $A = \operatorname{Im} f$ and $c = f(0)$. By putting $x = y = 0$ we get $f(-c) = f(c) + c - 1$, so $c \neq 0$.

It is easy to find the restriction $f|A$: take $x = f(y)$ and obtain

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2} \tag{1}$$

for all x in A.

The main step is to show that $A - A = \mathbb{R}$. Indeed, for $y = 0$ we get:

$$\{f(x - c) - f(x) \mid x \in \mathbb{R}\} = \{cx + f(c) - 1 \mid x \in \mathbb{R}\} = \mathbb{R}$$

because c is not zero.

Now we can obtain the value of $f(x)$ for an arbitrary x: if we choose $y_1, y_2 \in A$ such that $x = y_1 - y_2$ and use (1) we find that

$$\begin{aligned} f(x) &= f(y_1 - y_2) = f(y_2) + y_1 y_2 + f(y_1) - 1 \\ &= \frac{c+1}{2} - \frac{y_2^2}{2} + y_1 y_2 + \frac{c+1}{2} - \frac{y_1^2}{2} - 1 \qquad (2) \\ &= c - \frac{(y_1 - y_2)^2}{2} = c - \frac{x^2}{2}. \end{aligned}$$

Comparing (1) and (2) we obtain $c = 1$ and therefore

$$f(x) = 1 - \frac{x^2}{2}$$

for all $x \in \mathbb{R}$.

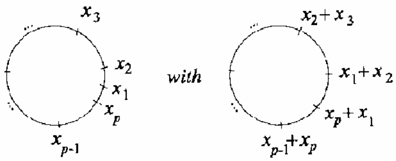
Problem A6

For $n \geq 3$ and $a_1 \leq a_2 \leq \ldots \leq a_n$ given real numbers we have the following instructions:

(1) *place out the numbers in some order in a ring;*

(2) *delete one of the numbers from the ring;*

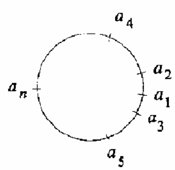
(3) *if just two numbers are remaining in the ring: let S be the sum of these two numbers. Otherwise, if there are more then two numbers in the ring, replace*



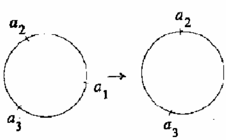
Afterwards start again with the step (2).
Show that the largest sum S which can result in this way is given by the formula

$$S_{\max} = \sum_{k=2}^{n} \binom{n-2}{[\frac{k}{2}]-1} a_k.$$

Solution. First we shall show that $S_{\max}$ can be reached by the ordering

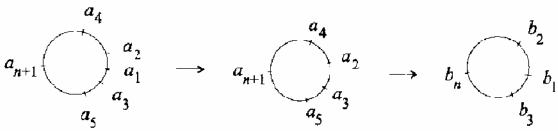


and repeatedly deleting the smallest term.
Start the induction with $n = 3$:



$$\rightarrow S_3 = a_2 + a_3 = \binom{1}{0} a_2 + \binom{1}{0} a_3.$$

Now start with the steps (1) and (2) for $n+1$ numbers



where

$$
\begin{aligned}
&b_1 = a_2 + a_3 \le b_2 = a_2 + a_4 \le b_3 = a_3 + a_5 \le \ldots \le \\
&\le b_k = a_k + a_{k+2} \le b_{k+1} = a_{k+1} + a_{k+3} \le \ldots \le \\
&\le b_{n-1} = a_{n-1} + a_{n+1} \le b_n = a_n + a_{n+1}.
\end{aligned}
$$

According to the induction assumption

$$
\begin{aligned}
&S_{n+1}(a_1,\ldots,a_{n+1}) = S_n(b_1,\ldots,b_n) = \sum_{k=2}^{n} \binom{n-2}{[\frac{k}{2}]-1} b_k = \\
&= \sum_{k=2}^{n-1} \binom{n-2}{[\frac{k}{2}]-1} (a_k + a_{k+2}) + \binom{n-2}{[\frac{n}{2}]-1}(a_n + a_{n-1}) = \\
&= a_2 + a_3 + \sum_{k=4}^{n-1} \left(\binom{n-2}{[\frac{k}{2}]-1} + \binom{n-2}{[\frac{k}{2}]-2} \right) a_k + \left(\binom{n-2}{[\frac{n}{2}]-2} + \binom{n-2}{[\frac{n}{2}]-1} \right) a_n + \\
&\qquad + \left(\binom{n-2}{[\frac{n-1}{2}]-1} + \binom{n-2}{[\frac{n}{2}]-1} \right) a_{n+1} = \\
&= \binom{n-1}{0} a_2 + \binom{n-1}{0} a_3 + \sum_{k=4}^{n} \binom{n-1}{[\frac{k}{2}]-1} a_k + \binom{n-1}{[\frac{n+1}{2}]-1} a_{n+1}.
\end{aligned}
$$

For the last coefficient we used $[\frac{n}{2}] + [\frac{n-1}{2}] = n$ and

$$
\begin{aligned}
\binom{n-2}{[\frac{n-1}{2}]-1} + \binom{n-2}{[\frac{n}{2}]-1} &= \binom{n-2}{[\frac{n-1}{2}]-1} + \binom{n-2}{n-1-[\frac{n+1}{2}]} = \\
&= \binom{n-2}{[\frac{n+1}{2}]-2} + \binom{n-2}{[\frac{n+1}{2}]-1} = \binom{n-1}{[\frac{n+1}{2}]-1}.
\end{aligned}
$$

Now we have to show that for any numbers $a_1 \le \ldots \le a_n$ and for any choice of steps (1) and (2) we get a final sum $S(a_1,\ldots,a_n) \le S_{\max}(a_1,\ldots,a_n)$.

For every k-tuple $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ denote by $(x'_1, x'_2, \ldots, x'_k)$ the k-tuple having the same elements rearranged in increasing order.

Define now the following partial order:

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_k) \le (y_1, y_2, \ldots, y_k) \quad \text{if and only if} \\
&x'_k \le y'_k \\
&x'_k + x'_{k-1} \le y'_k + y'_{k-1} \\
&\ldots \\
&x'_k + x'_{k-1} + \ldots + x'_1 \le y'_k + y'_{k-1} + \ldots + y'_1.
\end{aligned}
$$

Lemma. *Let $x_1, x_2, \ldots, x_k$ and $y_1 \le y_2 \le \ldots \le y_k$ $(k \ge 3)$ be real numbers such that $(x_1, x_2, \ldots, x_k) \le (y_1, y_2, \ldots, y_k)$. Let $(z_1, z_2, \ldots, z_{k-1})$ be the $(k-1)$-tuple resulting from $(x_1, x_2, \ldots, x_k)$ by an application of steps* (1), (2), (3). *Then*

$$(z_1, z_2, \ldots, z_{k-1}) \le (y_2 + y_3, y_2 + y_4, y_3 + y_5, \ldots, y_{k-2} + y_k, y_{k-1} + y_k)$$

Proof of the lemma. Notice that the sum $z'_{k-1} + z'_{k-2} + \ldots + z'_i$ $(i \ge 2)$ contains $2(k-i)$ numbers of the form x_p, each x_p appears at most twice and two of the x_p's appear only once. Therefore

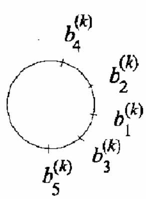
$$\begin{aligned} & z'_{k-1} + z'_{k-2} + \ldots + z'_i \le 2(x'_k + x'_{k-1} + \ldots + x'_{i+2}) + x'_{i+1} + x'_i \le \\ & \le 2(y_k + y_{k-1} + \ldots + y_{i+2}) + y_{i+1} + y_i = \\ & = (y_{k-1} + y_k) + (y_{k-2} + y_k) + (y_{k-3} + y_{k-1}) + \ldots + (y_{i+1} + y_{i+3}) + (y_i + y_{i+2}). \end{aligned}$$

Also

$$\begin{aligned} & z'_{k-1} + z'_{k-2} + \ldots + z'_1 \le 2(x'_k + x'_{k-1} + \ldots + x'_2) \le 2(y_k + y_{k-1} + \ldots + y_2) = \\ & = (y_{k-1} + y_k) + (y_{k-2} + y_k) + (y_{k-3} + y_{k-1}) + \ldots + (y_2 + y_4) + (y_2 + y_3). \qquad \square \end{aligned}$$

Denote now by $(a_1^{(k)}, a_2^{(k)}, \ldots, a_{n-k}^{(k)})$ the $(n-k)$-tuple obtained from $(a_1, a_2, \ldots, a_n)$ after k random iterations of the given algorithm.

Denote by $(b_1^{(k)}, b_2^{(k)}, \ldots, b_{n-k}^{(k)})$ the $(n-k)$-tuple obtained from $(a_1, \ldots, a_n)$ after k-iterations of the $S_{\max}$-algorithm, where $b_1^{(k)}, b_2^{(k)}, \ldots, b_{n-k}^{(k)}$ are placed



We notice that

$$b_1^{(0)} \le b_2^{(0)} \le \ldots \le b_n^{(0)}$$

and that

$$b_1^{(k)} \le b_2^{(k)} \le \ldots \le b_{n-k}^{(k)} \Rightarrow b_1^{(k+1)} \le b_2^{(k+1)} \le \ldots \le b_{n-k-1}^{(k+1)}.$$

Starting with the obvious relation $(a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)}) \le (b_1^{(0)}, b_2^{(0)}, \ldots, b_n^{(0)})$ and applying the lemma we get by induction

$$(a_1^{(k)}, a_2^{(k)}, \ldots, a_{n-k}^{(k)}) \le (b_1^{(k)}, b_2^{(k)}, \ldots, b_{n-k}^{(k)}) \quad \text{for } k = 1, \ldots, n-2,$$

which implies

$$S(a_1, \ldots, a_n) = a_1^{(n-2)} + a_2^{(n-2)} \le b_1^{(n-2)} + b_2^{(n-2)} = S_{\max}(a_1, \ldots, a_n).$$

Problem C1

Let $n \geq 1$ be an integer. A **path** *from $(0,0)$ to (n,n) in the xy plane is a chain of consecutive unit moves either to the right (move denoted by E) or upwards (move denoted by N), all the moves being made inside the halfplane $x \geq y$. A* **step** *in a path is the occurence of two consecutive moves of the form EN.*

Show that the number of paths from $(0,0)$ to (n,n) that contain exactly s steps $(n \geq s \geq 1)$ is

$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

Solution. A path with s steps from $(0,0)$ to (n,n) will be called a path of type (n,s). Let $f(n,s)$ denote the number of paths of type (n,s) and let

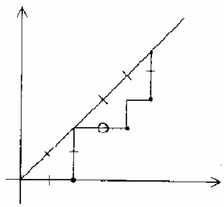
$$g(n,s) = \frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

We will prove that $f(n,s) = g(n,s)$ for $s = 1, 2, \ldots, n$ by induction on n.
It is easy to see that

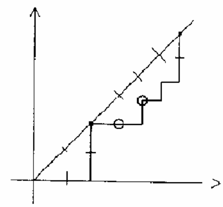
$$f(1,1) = 1 = g(1,1),$$
$$f(2,1) = 1 = g(2,1),$$
$$f(2,2) = 1 = g(2,2).$$

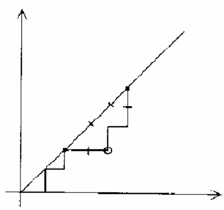
Let $n \geq 2$ and assume that $f(m,s) = g(m,s)$ for $1 \leq s \leq m \leq n$. It is clear that $f(n+1,1) = 1 = g(n+1,1)$. We will show that $f(n+1,s+1) = g(n+1,s+1)$ for $1 \leq s \leq n$.

We say that a $(n,s)-$ path and a $(n+1,s+1)-$ path are related if the latter is obtained from the former either by inserting in the first path a pair EN between two succesive moves of the form (E,E), (N,N) or (N,E) or by adding a pair EN at the end of it. We say also that a $(n,s+1)-$ path and a $(n+1,s+1)-$ path are related if the longer path is obtained from the shorter one by inserting a pair EN between (E,N).

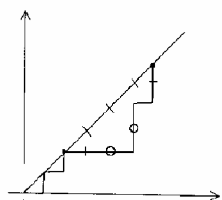


(5,3) <-> (6,4)





(5,4) <-> (6,4)



Each (n,s)−path is related to $2n+1-s$ different $(n+1,s+1)$− paths; each $(n,s+1)$− path is related to $s+1$ different $(n+1,s+1)$− paths; each $(n+1,s+1)$− path is related to exactly $s+1$ paths of type (n,s) or $(n,s+1)$. Therefore the number of related pairs is

$$(s+1)f(n+1,s+1) = (2n+1-s)f(n,s) + (s+1)f(n,s+1)\,.$$

It is easy to verify that

$$(s+1)g(n+1,s+1) = (2n+1-s)g(n,s) + (s+1)g(n,s+1)$$

and thus

$$f(n+1,s+1) = g(n+1,s+1).$$

Remark. If $m \geq n \geq s \geq 1$, the number of paths from $(0,0)$ to (m,n) with s steps is given by

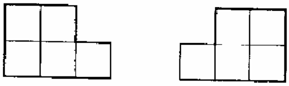
$$f(m,n,s) = \binom{m}{s}\binom{n-1}{s-1} - \binom{m}{s-1}\binom{n-1}{s}.$$

The proof is by induction on m:

$$(s+1)f(m+1,n+1,s+1) = (m+n+1-s)f(m,n,s) + (s+1)f(m,n,s+1)\,.$$

Problem C2

a) If a $5 \times n$ rectangle can be tiled using n pieces like those shown in the diagram, prove that n is even.

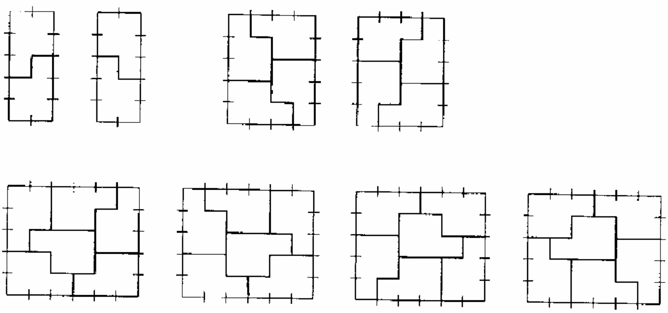


b) Show that there are more than $2 \cdot 3^{k-1}$ ways to tile a fixed $5 \times 2k$ rectangle ($k \geq 3$) with $2k$ pieces. (Symmetric constructions are supposed to be different.)

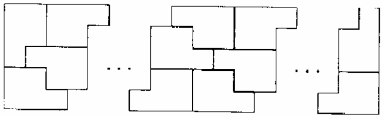
Solution. Colour in red the first, third and fifth row of a tiled rectangle and colour in white the second and fourth row. We get $3n$ red squares and $2n$ white squares. Each copy of the figure can cover at most 3 red squares. It follows that each copy must cover exactly 3 red and 2 white squares. The shape of the figure implies that the 2 white squares are on the same row. Therefore a white line must have an even number of squares, that is n is even.

Denote by a_k the number of possibilities to tile a $5 \times 2k$ rectangle and by p_k the number of such tilings not containing a smaller $5 \times 2i$ rectangle.

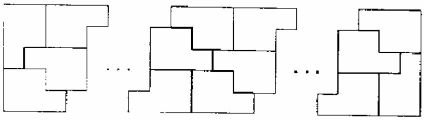
The diagrams



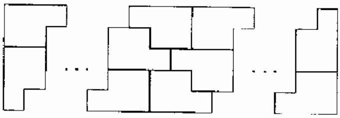
show that $p_1, p_2 \geq 2, p_3 \geq 4$ and the following diagrams show that, in general, $p_k \geq 2$:



$p_{3n} \geqslant 4$



$p_{3n+1} \geqslant 2$



$p_{3n+2} \geqslant 2$

Looking at the leftmost $5 \times 2i$ $(i = 1, 2, \ldots, k-1)$ rectangle of a $5 \times 2k$ rectangle, we find that

$$\begin{aligned} a_k &= p_1 a_{k-1} + p_2 a_{k-2} + \ldots + p_{k-1} a_1 + p_k \geq \\ &\geq 2a_{k-1} + 2a_{k-2} + \ldots + 2a_1 + 2 \,. \end{aligned}$$

The inequality $p_3 \geq 4$ implies that the last inequality is strict for $k \geq 3$.

Consider the sequence $b_1 = 2, b_k = 2b_{k-1} + 2b_{k-2} + \ldots + 2b_1 + 2$. It is easy to see that $b_k - b_{k-1} = 2b_{k-1}$ and also, by induction, that

$$a_k \geq b_k = 2 \cdot 3^{k-1}$$

(with equality only for $k = 1, 2$).

Remark. It is possible to prove that $p_{3n} = 4, p_{3n+1} = p_{3n+2} = 2$. Therefore we have

$$a_k = 2a_{k-1} + 2a_{k-2} + 5a_{k-3} \,.$$

Problem C3

A biologist watches a cameleon. The cameleon catches flies and rests after each catch. The biologist notices that:

- *the first fly is caught after a resting period of one minute;*
- *the resting period before catching the $2m^{th}$ fly is the same as the resting period before catching the m^{th} fly and one minute shorter than the resting period before catching the $(2m+1)^{th}$ fly;*
- *when the cameleon stops resting, he catches a fly instantly.*

a) How many flies were caught by the cameleon before his first resting period of 9 *minutes in a row?*

b) After how many minutes will the cameleon catch his 98^{th} fly?

c) How many flies were caught by the cameleon after 1999 *minutes have passed?*

Solution. Denote by $r(m)$ the length of the resting period before the m^{th} catch. The problem says that $r(1) = 1, r(2m) = r(m), r(2m+1) = $ $= r(m)+1$. This shows that $r(m)$ is equal to the number of 1's in the binary representation of m.

Denote also by $t(m)$ the moment of the m^{th} catch and by $f(n)$ the number of flies caught after n minutes have passed. We notice that

$$t(m) = \sum_{i=1}^{m} r(i) \text{ and } f(t(m)) = m$$

for every m. The following recurrence formulae hold:

$$(*) \qquad \begin{aligned} &t(2m+1) = 2t(m) + m + 1; \\ &t(2m) = 2t(m) + m - r(m); \\ &t(2^p m) = 2^p t(m) + p \cdot m \cdot 2^{p-1} - (2^p - 1)r(m). \end{aligned}$$

The first formula follows from

$$\begin{aligned} &\textstyle\sum_{i=1}^{m} r(2i) = \sum_{i=1}^{m} r(i) = t(m) \text{ and} \\ &\textstyle\sum_{i=0}^{m} r(2i+1) = 1 + \sum_{i=1}^{m}(r(i)+1) = t(m) + m + 1. \end{aligned}$$

The second formula is justified by

$$t(2m) = t(2m+1) - r(2m+1) = 2t(m) + m - r(m).$$

An easy induction on p proves the third formula.

a) We have to find the first m so that $r(m+1) = 9$. The smallest number having 9 binary unit digits is $\overline{11\ldots1}_2 = 2^9 - 1 = 511$, so the required m is 510.

b) Using $(*)$ we get

$$\begin{aligned} &t(98) = 2t(49) + 49 - r(49); \\ &t(49) = 2t(24) + 25; \\ &t(24) = 2^3 t(3) + 3 \cdot 3 \cdot 2^2 - (2^3 - 1)r(3); \\ &r(1) = r(2) = 1,\ r(3) = 2,\ r(49) = r(\overline{110001}_2) = 3 \text{ whence} \\ &t(3) = 4,\ t(24) = 54,\ t(49) = 133 \text{ and } t(98) = 312. \end{aligned}$$

c)We have $f(n) = m$ iff $n \in [t(m), t(m+1))$, so we have to find m_0 such that $t(m_0) \leq 1999 < t(m_0 + 1)$. We start with some more values of the function t:

- $t(2^p - 1) = t(2(2^{p-1} - 1) + 1) = 2t(2^{p-1} - 1) + 2^{p-1}$, therefore $t(2^p - 1) = p \cdot 2^{p-1}$ and $t(2^p) = t(2^p - 1) + r(2^p) = p \cdot 2^{p-1} + 1$;
- $t(\overline{\underbrace{11\ldots1}_{q}\underbrace{00\ldots0}_{p}}_{\,2}) = t(2^p \cdot (2^q - 1)) =$
  $= 2^p t(2^q - 1) + p(2^q - 1)2^{p-1} - (2^p - 1) \cdot r(2^q - 1) =$
  $= (p+q)2^{p+q-1} - p \cdot 2^{p-1} - q \cdot 2^p + q.$

Now we can estimate: from $t(2^8) = 8 \cdot 2^7 + 1 < 1999 < 9 \cdot 2^8 = t(2^9)$ we get $2^8 < m_0 < 2^9$, hence the binary representation of m_0 has 9 digits. Taking $q = 3, p = 6$ and $q = 4, p = 5$ we find $t(\overline{111000000_2}) = 1923$ and $t(\overline{111100000_2}) = 2100$, therefore the first binary digits of m_0 are 1110. Since $t(\overline{111010000_2}) = 2004$, $t(\overline{111001111_2}) = 2000$ and $t(\overline{111001110_2}) = 1993$ it follows that $f(1999) = \overline{111001110}_2 = 462$.

Problem C4

Let A be a set of N residues $(\bmod N^2)$. Prove that there exists a set B of N residues $(\bmod N^2)$ such that the set $A+B=\{a+b \mid a\in A, b\in B\}$ contains at least half of all the residues $(\bmod N^2)$.

Solution. Let $S=\{0,1,\ldots,N^2-1\}$ be a complete residue system (mod N^2) and $A\subset S$ a N-subset of S. We shall prove that there exists a N-subset $B\subset S$ such that the number of residues of $A+B$ is greater than $(1-1/e)N^2$.

We use the notations:

$$\begin{aligned} |X| &= \text{the number of elements of a subset } X\subset S\\ \overline{X} &= S-X \text{ the complement of } X\\ C_i &= \text{the set of residues (mod } N^2) \text{ of the elements}\\ &\quad\ a+i, a\in A \text{ (where } i\in S\text{)}. \end{aligned}$$

Notice that $|C_i|=N$ and $\cup_{i\in S}C_i=S$. Every $x\in S$ appears in exactly $d(x)=N$ sets C_i. Counting in two ways the pairs

$$\{(x,(i_1<\ldots<i_N)) \mid x\in S, x\notin C_{i_1},\ldots,x\notin C_{i_N}\}$$

we get that

$$\sum_{x\in S}|\{(i_1<\ldots<i_N) \mid x\notin C_{i_1},\ldots,x\notin C_{i_N}\}|=$$

$$=\sum_{x\in S}\binom{N^2-d(x)}{N}=\sum_{x\in S}\binom{N^2-N}{N}=\binom{N^2-N}{N}|S|$$

is equal to

$$\sum_{i_1<\ldots<i_N}|\{x\in S \mid x\notin C_{i_1},\ldots,x\notin C_{i_N}\}|=\sum_{i_1<\ldots<i_N}|\overline{C}_{i_1}\cap\ldots\cap\overline{C}_{i_N}|.$$

We obtain

$$\sum_{0\le i_1<\cdots<i_N\le N^2-1}|C_{i_1}\cup\cdots\cup C_{i_N}|=\sum_{0\le i_1<\cdots<i_N\le N^2-1}(|S|-|\overline{C}_{i_1}\cap\cdots\cap\overline{C}_{i_N}|)=$$

$$=\binom{N^2}{N}|S|-\binom{N^2-N}{N}|S|=\left(\binom{N^2}{N}-\binom{N^2-N}{N}\right)N^2.$$

Therefore one can find $0\le i_1<\cdots<i_N\le N^2-1$ such that

$$|C_{i_1}\cup\cdots\cup C_{i_N}|\ge\left(1-\binom{N^2-N}{N}\Big/\binom{N^2}{N}\right)N^2>\left(1-\frac{1}{e}\right)N^2.$$

The last inequality follows from

$$\frac{\binom{N^2}{N}}{\binom{N^2-N}{N}}=\frac{N^2(N^2-1)\cdots(N^2-N+1)}{(N^2-N)(N^2-N-1)\cdots(N^2-2N+1)}\ge$$

$$\ge\left(\frac{N^2}{N^2-N}\right)^N=\left(1+\frac{1}{N-1}\right)^N>e.$$

Hence the set $B = \{i_1, \ldots, i_N\}$ satisfies the required inequality:

$$|\text{the set of residues of } A + B| = |C_{i_1} \cup \cdots \cup C_{i_N}| > \left(1 - \frac{1}{e}\right) N^2 > \frac{1}{2} N^2.$$

Remark. It is elementary to prove that $\left(1 + \frac{1}{N-1}\right)^N > 2$, which implies the required conclusion.

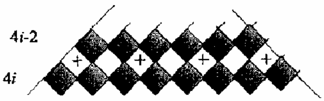
Problem C5

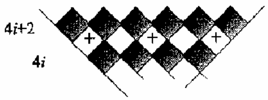
Let n be an even positive integer. We say that two different cells of a $n \times n$ board are **neighboring** *if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.*

Solution. First colour the board black and white like a chessboard. Let $f(n)$ be the number we are looking for and $f_w(n)$ be the minimal number of white cells that must be marked so that any black cell has a white marked neighbor. Define similarly $f_b(n)$. Due to the symmetry of the chessboard ($n = 2k$), we have $f_w(n) = f_b(n)$ and also $f(n) = f_w(n) + f_b(n)$.

It will be more convenient to look at the board along the lines parallel to the longest black diagonal which will be placed horizontally. Thus the 'length' of these black lines are $2, 4, \ldots, 2k, \ldots, 4, 2$.

Cross the 'odd' cells fom the white lines just below the black lines of length $4i - 2$:





In the first case (above the diagonal) there are $2i$ crossed white cells and in the second case (below the diagonal) there are $2i + 1$ crossed white cells. Thus we crossed

$$2 + 4 + \ldots + k + \ldots + 3 + 1 = \frac{k(k+1)}{2} \text{ white cells.}$$

It is easy to see that each black cell has a white crossed neighbor. This implies that

$$f_w(n) \leq \frac{k(k+1)}{2}. \tag{1}$$

Consider the $k(k+1)/2$ crossed white cells: they have no common black neighbor, so we need at least $k(k+1)/2$ black marked cells in order to 'cover' these white cells. Therefore

$$f_b(n) \geq \frac{k(k+1)}{2}. \tag{2}$$

In conclusion we have

$$f_w(n) = f_b(n) = \frac{k(k+1)}{2}$$
$$f(n) = k(k+1).$$

Remark. In a similar way we can prove that

$$f(n) = \begin{cases} 4k^2 - 1 & \text{if } n = 4k - 1 \\ (2k+1)^2 & \text{if } n = 4k + 1. \end{cases}$$

Problem C6

Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x+y$ or $x-y$.

Solution. Suppose that there exists a colouring function $f : \mathbb{Z} \to \{R, B, G, Y\}$ with the property that for any integer a

$$f\{a, a+x, a+y, a+x+y\} = \{R, B, G, Y\}.$$

In particular, colouring the integer lattice by the rule

$$g : \mathbb{Z} \times \mathbb{Z} \to \{R, B, G, Y\}, \quad g(i,j) = f(ix + jy),$$

we obtain that the vertices of any unit square have all four different colours.

Claim 1. If there exists a column $i \times \mathbb{Z}$ such that $g|i \times \mathbb{Z}$ is not periodical with period 2, then there exists a row $\mathbb{Z} \times j$ such that $g|\mathbb{Z} \times j$ is periodical with period 2.

Proof. If $g|i \times \mathbb{Z}$ is not periodical, then we can find the configuration $\begin{matrix} Y \\ B \\ R \end{matrix}$; using the adjacent unit squares, we get $\begin{matrix} RYR \\ GBG \\ YRY \end{matrix}$ and also $\begin{matrix} YRYRY \\ BGBGB \\ RYRYR \end{matrix}$ and so on. Thus we obtain three periodical lines. □

Claim 2. If for one integer i, $g_i = g|\mathbb{Z} \times i$ is periodical with period 2, then for every $j \in \mathbb{Z}$, $g_j = g|\mathbb{Z} \times j$ has period 2. The values of g_i are the values of g_j if $i \equiv j \pmod 2$ and the other two values if $i \not\equiv j \pmod 2$.

Proof. Applying the square rule to the line $\ldots RBRBRB \ldots$ we get

$$\begin{matrix} \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix} \quad \text{and next} \quad \begin{matrix} \ldots RBRBR \ldots \\ \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix} \quad \text{or} \quad \begin{matrix} \ldots BRBRB \ldots \\ \ldots YGYGY \ldots \\ \ldots RBRBR \ldots \end{matrix}.$$

A similar argument holds for the rows below the line $\mathbb{Z} \times i$. □

Changing between them 'rows' and 'columns' we have similar claims. So we can suppose that the rows are periodical with period 2 and $g(0,0) = R$, $g(1,0) = B$. Therefore $g(y,0) = B$ (y is odd). The row $\mathbb{Z} \times \{x\}$ is odd too; hence $g(\mathbb{Z} \times \{x\}) = \{Y, G\}$. From $g(y,0) = f(xy) = g(0,x)$ we get a contradiction.

Remark. The same result is true for $x = 2^k(2p+1)$, $2^k(2q+1)$; just take

$$g(i,j) = f\left(\frac{ix+jy}{2^k}\right).$$

Problem C7

Let $p > 3$ be a prime number. For each nonempty subset T of $\{0, 1, 2, 3, \dots, p-1\}$, let $E(T)$ be the set of all $(p-1)$-tuples $(x_1, \dots, x_{p-1})$ where each $x_i \in T$ and $x_1 + 2x_2 + \dots + (p-1)x_{p-1}$ is divisible by p and let $|E(T)|$ denote the number of elements in $E(T)$.

Prove that

$$|E(\{0,1,3\})| \geq |E(\{0,1,2\})|$$

with equality if and only if $p = 5$.

Solution. Let $f(x) = (1 + x + x^2)$ and $F(x) = f(x)f(x^2)\cdots f(x^{p-1})$. In expanded form we can write

$$F(x) = \sum_{n=0}^{p(p-1)} a_n x^n.$$

Note here that a_n is the number of $(p-1)$-tuples $(x_1, \dots, x_{p-1})$ with each $x_j \in \{0, 1, 2\}$ such that

$$x_1 + 2x_2 + \dots + (p-1)x_{p-1} = n.$$

Hence $|E(\{0,1,2\})|$ is the sum of those a_n with n divisible by p. Write $w = \cos(2\pi/p) + i\sin(2\pi/p)$ and note that $1, w, w^2, \dots, w^{p-1}$ are the p^{th} roots of unity and that

$$1 + w^j + w^{2j} + \dots + w^{(p-1)j} = \begin{cases} p & \text{if } p \text{ divides } j \\ 0 & \text{if } p \text{ does not divide j.} \end{cases}$$

Substituting $x = 1, w, \dots, w^{p-1}$ into $F(x)$ and adding up the resulting expressions we obtain

$$F(1) + F(w) + \dots + F(w^{p-1}) = pE(\{0,1,2\}).$$

Note that $F(1) = 3^{p-1}$. Note also that for j not divisible by p, $(1, w^j, w^{2j}, \dots, w^{(p-1)j})$ is a permutation of

(∗) $$1, w, w^2, \dots, w^{p-1}.$$

Thus

$$\begin{aligned} F(w) &= F(w^2) = \dots = F(w^{p-1}) = \\ &= (1 + w + w^2)\cdots(1 + w^{p-1} + w^{2(p-1)}) = \\ &= \left(\frac{1-w^3}{1-w}\right)\left(\frac{1-w^6}{1-w^2}\right)\cdots\left(\frac{1-w^{3(p-1)}}{1-w^{p-1}}\right) = \\ &= 1 \end{aligned}$$

using observation $(*)$. Hence

$$|E(\{0,1,2\})| = (3^{p-1} + p - 1)/p.$$

Let $g(x) = 1+x^2+x^3$, $G(x) = g(x)g(x^2)\cdots g(x^{p-1})$. By similar arguments we obtain

$$\begin{aligned} |E(\{0,1,3\})| &= (G(1) + (p-1)G(w))/p = \\ &= (3^{p-1} + (p-1)G(w))/p. \end{aligned}$$

Hence we must show that $G(w) \geq 1$ with equality if and only if $p = 5$. Let $h(x) = x^3 + x + 1$ and write $h(x) = (x-\lambda)(x-\mu)(x-\nu)$ where λ, μ, ν are complex numbers. Since $h(x) > 0$ for $x > 0$ and $\lambda + \mu + \nu = 0$, $h(x)$ has one negative real root, say λ , and $\mu, \nu = \overline{\mu}$, are complex conjugate roots with positive real part.

Note that

$$\begin{aligned} G(w) &= \prod_{j=1}^{p-1}(1 + w^j + w^{3j}) = \\ &= \prod_{j=1}^{p-1}(w^j - \lambda)(w^j - \mu)(w^j - \nu) = \\ &= \left(\frac{\lambda^p - 1}{\lambda - 1}\right)\left(\frac{\mu^p - 1}{\mu - 1}\right)\left(\frac{\nu^p - 1}{\nu - 1}\right) \end{aligned}$$

since $\prod_{j=1}^{p-1}(\lambda - w^j) = u(\lambda)$ etc., where

$$u(x) = \prod_{j=1}^{p-1}(x - w^j) = \frac{x^p - 1}{x - 1}.$$

Note that $(\lambda - 1)(\mu - 1)(\nu - 1) = -h(1) = -3$.

Next

$$\lambda^3 + \lambda + 1 = 0$$

so for each positive integer k,

$$\lambda^{k+3} + \lambda^{k+1} + \lambda^k = 0$$

with similar equations for μ, ν. Adding these we deduce, using induction, that $\lambda^r + \mu^r + \nu^r$ is an integer for all positive integers r.

Suppose now that $G(w) = 1$, so

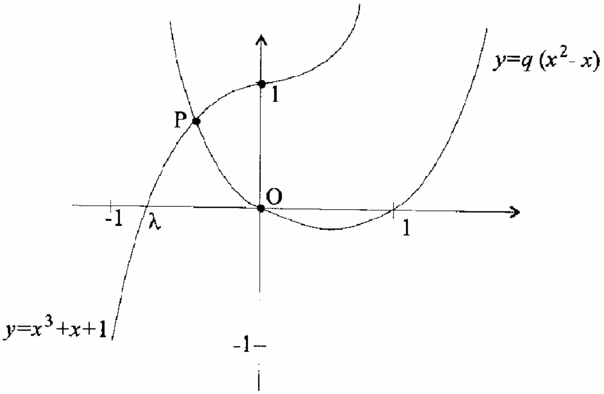
$$(\lambda^p - 1)(\mu^p - 1)(\nu^p - 1) = -3$$

and λ^p, μ^p and ν^p are the roots of the cubic equation

$$m(x) := x^3 - qx^2 + (1+q)x + 1 = 0$$

and $q = \lambda^p + \mu^p + \nu^p$ is an integer. Since λ is the real root of $x^3 + x + 1 = 0$, $-1 < \lambda < -\frac{1}{2}$, so if $q < 0$, $x^3 - qx^2 > 0$, $(1 + q)x \geq 0$ for $x = \lambda^p$ and $m(\lambda^p) \neq 0$. If $q = 0$, $\lambda^p = \lambda$, so $p = 1$, again yielding a contradiction.

Hence $q \geq 1$. For $-1 \leq x \leq 0$, $q(x^2 - x)$ is positive and $x^3 + x + 1$ is strictly increasing, and $\lambda^3 + \lambda + 1 = 0$.



λ^p is the x-coordinate of the intersection P of $y = x^3 + x + 1$ and $y = q(x^2 - x)$ in the interval $[-1, 0]$ and thus as p increases (and λ^p gets closer to 0), q increases.

Note that if $p = 5$, $g(w) = 1 + w + w^3 = -w^2(1 + w^2)$ and thus

$$G(w) = \prod_{j=1}^{p-1}(1 + w^{2j}) = \mu(-1) = 1.$$

Note that

$$\lambda^5 = -\lambda^2(\lambda + 1) = -\lambda^2 + \lambda + 1$$

and hence that

$$q = -(\lambda^2 + \mu^2 + \nu^2) + (\lambda + \mu + \nu) + 3 = 5$$

since

$$\begin{aligned}\lambda^2 + \mu^2 + \nu^2 &= (\lambda + \mu + \nu)^2 - 2(\lambda\mu + \mu\nu + \nu\lambda) \\ &= -2(\lambda\mu + \mu\nu + \nu\lambda) \\ &= -2,\end{aligned}$$

as λ, μ, ν are the roots of $x^3 + x + 1 = 0$. Hence $q \geq 5$ and if $q = 5$, $p = 5$.

Suppose $q \geq 6$. Consider $m(x) = x^3 + x + 1 - q(x^2 - x)$. Note that

$$m(-1) < 0, \quad m(0) > 0, \quad m(2) < 0,$$

and that $m(x) > 0$ for all sufficiently large $x > 0$. Hence $m(x) = 0$ has three distinct real roots. But the roots of $m(x) = 0$ are λ^p, μ^p and $\nu^p = \overline{\mu}^p$, so if μ^p is real, $m(x)$ must have repeated real roots. This is a contradiction. So $q < 6$. Thus $q = 5$ and thus $p = 5$ is the only case where $G(w) = 1$. The formula

$$|E(\{0,1,3\})| = (3^{p-1} + (p-1)G(w))/p$$

and the fact that $G(w) \geq 0$ (which follows from $\overline{g(w^j)} = g(w^{p-j})$) implies that $G(w) \geq 1$. Hence

$$|E(\{0,1,3\})| \geq |E(\{0,1,2\})|$$

with equality only if $p = 5$.