44th International Mathematical Olympiad

Short-listed Problems and Solutions

Tokyo Japan

July 2003

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Mathematical Olympiad

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The Problem Selection Committee and the Organising Committee of IMO 2003 thank the following thirty-eight countries for contributing problem proposals.

Armenia	Greece	New Zealand
Australia	Hong Kong	Poland
Austria	India	Puerto Rico
Brazil	Iran	Romania
Bulgaria	Ireland	Russia
Canada	Israel	South Africa
Colombia	Korea	Sweden
Croatia	Lithuania	Taiwan
Czech Republic	Luxembourg	Thailand
Estonia	Mexico	Ukraine
Finland	Mongolia	United Kingdom
France	Morocco	United States
Georgia	Netherlands	

The problems are grouped into four categories: algebra (A), combinatorics (C), geometry (G), and number theory (N). Within each category, the problems are arranged in ascending order of estimated difficulty, although of course it is very hard to judge this accurately.

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Contents

Part I

Problems

Algebra

A1. Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.

Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive, or all zero.

A2. Find all nondecreasing functions $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $f(0) = 0$, $f(1) = 1$;

(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

A3. Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_i = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.

(1) Does there exist a pair $(a_i)_{i \geq 1}$, $(b_i)_{i \geq 1}$ such that the sequences $(A_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$ are unbounded while the sequence $(C_n)_{n \geq 1}$ is bounded?

(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?

Justify your answer.

A4. Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers.

(1) Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

A5. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f \colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

(ii) $f(x) < f(y)$ for all $1 \leq x < y$.

A6. Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that

$$z_{i+j}^2 \geq x_i y_j \qquad \text{for all } 1 \leq i, j \leq n.$$

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Combinatorics

C1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, t_2, $\ldots$, t_{100} in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

C2. Let D_1, $\ldots$, D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

C3. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

C4. Let x_1, $\ldots$, x_n and y_1, $\ldots$, y_n be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \geq 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

C5. Every point with integer coordinates in the plane is the centre of a disc with radius 1/1000.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

C6. Let $f(k)$ be the number of integers n that satisfy the following conditions:

(i) $0 \leq n < 10^k$, so n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

G2. Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

G3. Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

G4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P.

Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

G6. Each pair of opposite sides of a convex hexagon has the following property:

> the distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.

Prove that all the angles of the hexagon are equal.

G7. Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Number Theory

N1. Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, $\ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i, & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \ge m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

N2. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) move the last digit of a to the first position to obtain the number b;

(ii) square b to obtain the number c;

(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.

Find all numbers a for which $d(a) = a^2$.

N3. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

N4. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$:

> there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

N5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property:

> m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

N6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

N7. The sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

N8. Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p - 1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A?

Part II

Solutions

Algebra

A1. Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.

Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

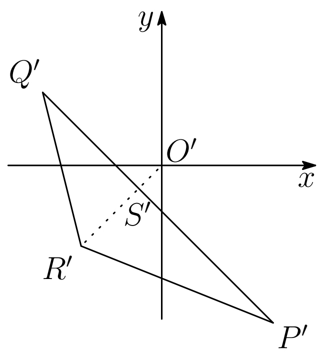
$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive, or all zero.

Solution. Set $O(0,0,0)$, $P(a_{11}, a_{21}, a_{31})$, $Q(a_{12}, a_{22}, a_{32})$, $R(a_{13}, a_{23}, a_{33})$ in the three dimensional Euclidean space. It is enough to find a point in the interior of the triangle PQR whose coordinates are all positive, all negative, or all zero.

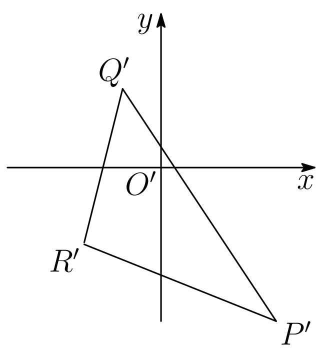
Let O', P', Q', R' be the projections of O, P, Q, R onto the xy-plane. Recall that points P', Q' and R' lie on the fourth, second and third quadrant respectively.

Case 1: O' is in the exterior or on the boundary of the triangle $P'Q'R'$.



Denote by S' the intersection of the segments $P'Q'$ and $O'R'$, and let S be the point on the segment PQ whose projection is S'. Recall that the z-coordinate of the point S is negative, since the z-coordinate of the points P' and Q' are both negative. Thus any point in the interior of the segment SR sufficiently close to S has coordinates all of which are negative, and we are done.

Case 2: O' is in the interior of the triangle $P'Q'R'$.



Let T be the point on the plane PQR whose projection is O'. If $T = O$, we are done again. Suppose T has negative (resp. positive) z-coordinate. Let U be a point in the interior of the triangle PQR, sufficiently close to T, whose x-coordinates and y-coordinates are both negative (resp. positive). Then the coordinates of U are all negative (resp. positive), and we are done.

A2. Find all nondecreasing functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(i) $f(0) = 0$, $f(1) = 1$;

(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

Solution. Let $g(x) = f(x+1) - 1$. Then g is nondecreasing, $g(0) = 0$, $g(-1) = -1$, and $g\big(-(a-1)(b-1)\big) = -g(a-1)g(b-1)$ for $a < 1 < b$. Thus $g(-xy) = -g(x)g(y)$ for $x < 0 < y$, or $g(yz) = -g(y)g(-z)$ for $y, z > 0$. Vice versa, if g satisfies those conditions, then f satisfies the given conditions.

Case 1: If $g(1) = 0$, then $g(z) = 0$ for all $z > 0$. Now let $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ be any nondecreasing function such that $g(-1) = -1$ and $g(x) = 0$ for all $x \geq 0$. Then g satisfies the required conditions.

Case 2: If $g(1) > 0$, putting $y = 1$ yields

$$g(-z) = -\frac{g(z)}{g(1)} \tag{$*$}$$

for all $z > 0$. Hence $g(yz) = g(y)g(z)/g(1)$ for all $y, z > 0$. Let $h(x) = g(x)/g(1)$. Then h is nondecreasing, $h(0) = 0$, $h(1) = 1$, and $h(xy) = h(x)h(y)$. It follows that $h(x^q) = h(x)^q$ for any $x > 0$ and any rational number q. Since h is nondecreasing, there exists a nonnegative number k such that $h(x) = x^k$ for all $x > 0$. Putting $g(1) = c$, we have $g(x) = cx^k$ for all $x > 0$. Furthermore $(*)$ implies $g(-x) = -x^k$ for all $x > 0$. Now let $k \geq 0$, $c > 0$ and

$$g(x) = \begin{cases} cx^k, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -(-x)^k, & \text{if } x < 0. \end{cases}$$

Then g is nondecreasing, $g(0) = 0$, $g(-1) = -1$, and $g(-xy) = -g(x)g(y)$ for $x < 0 < y$. Hence g satisfies the required conditions.

We obtain all solutions for f by the re-substitution $f(x) = g(x-1) + 1$. In Case 1, we have any nondecreasing function f satisfying

$$f(x) = \begin{cases} 1, & \text{if } x \geq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

In Case 2, we obtain

$$f(x) = \begin{cases} c(x-1)^k + 1, & \text{if } x > 1; \\ 1, & \text{if } x = 1; \\ -(1-x)^k + 1, & \text{if } x < 1, \end{cases}$$

where $c > 0$ and $k \geq 0$.

A3. Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_i = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.

(1) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?

(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?

Justify your answer.

Solution. (1) Yes.

Let (c_i) be an arbitrary sequence of positive numbers such that $c_i \geq c_{i+1}$ and $\sum_{i=1}^{\infty} c_i < \infty$. Let (k_m) be a sequence of integers satisfying $1 = k_1 < k_2 < k_3 < \cdots$ and $(k_{m+1} - k_m)c_{k_m} \geq 1$.

Now we define the sequences (a_i) and (b_i) as follows. For n odd and $k_n \leq i < k_{n+1}$, define $a_i = c_{k_n}$ and $b_i = c_i$. Then we have $A_{k_{n+1}-1} \geq A_{k_n - 1} + 1$. For n even and $k_n \leq i < k_{n+1}$, define $a_i = c_i$ and $b_i = c_{k_n}$. Then we have $B_{k_{n+1}-1} \geq B_{k_n-1} + 1$. Thus (A_n) and (B_n) are unbounded and $c_i = \min\{a_i, b_i\}$.

(2) Yes.

Suppose that there is such a pair.

Case 1: $b_i = c_i$ for only finitely many i's.

There exists a sufficiently large I such that $c_i = a_i$ for any $i \geq I$. Therefore

$$\sum_{i \geq I} c_i = \sum_{i \geq I} a_i = \infty,$$

a contradiction.

Case 2: $b_i = c_i$ for infinitely many i's.

Let (k_m) be a sequence of integers satisfying $k_{m+1} \geq 2k_m$ and $b_{k_m} = c_{k_m}$. Then

$$\sum_{k=k_i+1}^{k_{i+1}} c_k \geq (k_{i+1} - k_i)\frac{1}{k_{i+1}} \geq \frac{1}{2}.$$

Thus $\sum_{i=1}^{\infty} c_i = \infty$, a contradiction.

A4. Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers.

(1) Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

Solution. (1) Since both sides of the inequality are invariant under any translation of all x_i's, we may assume without loss of generality that $\sum_{i=1}^n x_i = 0$.

We have

$$\sum_{i,j=1}^{n} |x_i - x_j| = 2\sum_{i<j} (x_j - x_i) = 2\sum_{i=1}^{n} (2i - n - 1)x_i.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le 4\sum_{i=1}^{n} (2i-n-1)^2 \sum_{i=1}^{n} x_i^2 = 4 \cdot \frac{n(n+1)(n-1)}{3} \sum_{i=1}^{n} x_i^2.$$

On the other hand, we have

$$\sum_{i,j=1}^{n} (x_i - x_j)^2 = n\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j + n\sum_{j=1}^{n} x_j^2 = 2n\sum_{i=1}^{n} x_i^2.$$

Therefore

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

(2) If the equality holds, then $x_i = k(2i - n - 1)$ for some k, which means that $x_1, \ldots, x_n$ is an arithmetic sequence.

On the other hand, suppose that $x_1, \ldots, x_{2n}$ is an arithmetic sequence with common difference d. Then we have

$$x_i = \frac{d}{2}(2i - n - 1) + \frac{x_1 + x_n}{2}.$$

Translate x_i's by $-(x_1 + x_n)/2$ to obtain $x_i = d(2i - n - 1)/2$ and $\sum_{i=1}^n x_i = 0$, from which the equality follows.

A5. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

(ii) $f(x) < f(y)$ for all $1 \le x < y$.

Solution 1. We claim that $f(x) = x^\lambda + x^{-\lambda}$, where λ is an arbitrary positive real number.

Lemma. There exists a unique function $g\colon [1, \infty) \longrightarrow [1, \infty)$ such that

$$f(x) = g(x) + \frac{1}{g(x)}.$$

Proof. Put $x = y = z = 1$ in the given functional equation

$$f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$$

to obtain $4f(1) = f(1)^3$. Since $f(1) > 0$, we have $f(1) = 2$.

Define the function $A\colon [1, \infty) \longrightarrow [2, \infty)$ by $A(x) = x + 1/x$. Since f is strictly increasing on $[1, \infty)$ and A is bijective, the function g is uniquely determined. ∎

Since A is strictly increasing, we see that g is also strictly increasing. Since $f(1) = 2$, we have $g(1) = 1$.

We put $(x, y, z) = (t, t, 1/t), (t^2, 1, 1)$ to obtain $f(t) = f(1/t)$ and $f(t^2) = f(t)^2 - 2$. Put $(x, y, z) = (s/t, t/s, st), (s^2, 1/s^2, t^2)$ to obtain

$$f(st) + f\left(\frac{t}{s}\right) = f(s)f(t) \quad \text{and} \quad f(st)f\left(\frac{t}{s}\right) = f(s^2) + f(t^2) = f(s)^2 + f(t)^2 - 4.$$

Let $1 \le x \le y$. We will show that $g(xy) = g(x)g(y)$. We have

$$\begin{aligned} f(xy) + f\left(\frac{y}{x}\right) &= \left(g(x) + \frac{1}{g(x)}\right)\left(g(y) + \frac{1}{g(y)}\right) \\ &= \left(g(x)g(y) + \frac{1}{g(x)g(y)}\right) + \left(\frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right), \end{aligned}$$

and

$$\begin{aligned} f(xy)f\left(\frac{y}{x}\right) &= \left(g(x) + \frac{1}{g(x)}\right)^2 + \left(g(y) + \frac{1}{g(y)}\right)^2 - 4 \\ &= \left(g(x)g(y) + \frac{1}{g(x)g(y)}\right)\left(\frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right). \end{aligned}$$

Thus

$$\left\{f(xy), f\left(\frac{y}{x}\right)\right\} = \left\{g(x)g(y) + \frac{1}{g(x)g(y)}, \frac{g(x)}{g(y)} + \frac{g(y)}{g(x)}\right\} = \left\{A\big(g(x)g(y)\big), A\left(\frac{g(y)}{g(x)}\right)\right\}.$$

Since $f(xy) = A\big(g(xy)\big)$ and A is bijective, it follows that either $g(xy) = g(x)g(y)$ or $g(xy) = g(y)/g(x)$. Since $xy \geq y$ and g is increasing, we have $g(xy) = g(x)g(y)$.

Fix a real number $\varepsilon > 1$ and suppose that $g(\varepsilon) = \varepsilon^\lambda$. Since $g(\varepsilon) > 1$, we have $\lambda > 0$. Using the multiplicity of g, we may easily see that $g(\varepsilon^q) = \varepsilon^{q\lambda}$ for all rationals $q \in [0, \infty)$. Since g is strictly increasing, $g(\varepsilon^t) = \varepsilon^{t\lambda}$ for all $t \in [0, \infty)$, that is, $g(x) = x^\lambda$ for all $x \geq 1$.

For all $x \geq 1$, we have $f(x) = x^\lambda + x^{-\lambda}$. Recalling that $f(t) = f(1/t)$, we have $f(x) = x^\lambda + x^{-\lambda}$ for $0 < x < 1$ as well.

Now we must check that for any $\lambda > 0$, the function $f(x) = x^\lambda + x^{-\lambda}$ satisfies the two given conditions. The condition (i) is satisfied because

$$\begin{aligned} f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx}) &= \big((xy)^{\lambda/2} + (xy)^{-\lambda/2}\big)\big((yz)^{\lambda/2} + (yz)^{-\lambda/2}\big)\big((zx)^{\lambda/2} + (zx)^{-\lambda/2}\big) \\ &= (xyz)^\lambda + x^\lambda + y^\lambda + z^\lambda + x^{-\lambda} + y^{-\lambda} + z^{-\lambda} + (xyz)^{-\lambda} \\ &= f(xyz) + f(x) + f(y) + f(z). \end{aligned}$$

The condition (ii) is also satisfied because $1 \leq x < y$ implies

$$f(y) - f(x) = (y^\lambda - x^\lambda)\left(1 - \frac{1}{(xy)^\lambda}\right) > 0.$$

Solution 2. We can a find positive real number λ such that $f(e) = \exp(\lambda) + \exp(-\lambda)$ since the function $B\colon [0, \infty) \longrightarrow [2, \infty)$ defined by $B(x) = \exp(x) + \exp(-x)$ is bijective.

Since $f(t)^2 = f(t^2) + 2$ and $f(x) > 0$, we have

$$f\left(\exp\left(\frac{1}{2^n}\right)\right) = \exp\left(\frac{\lambda}{2^n}\right) + \exp\left(-\frac{\lambda}{2^n}\right)$$

for all nonnegative integers n.

Since $f(st) = f(s)f(t) - f(t/s)$, we have

$$f\left(\exp\left(\frac{m+1}{2^n}\right)\right) = f\left(\exp\left(\frac{1}{2^n}\right)\right) f\left(\exp\left(\frac{m}{2^n}\right)\right) - f\left(\exp\left(\frac{m-1}{2^n}\right)\right) \qquad (*)$$

for all nonnegative integers m and n.

From $(*)$ and $f(1) = 2$, we obtain by induction that

$$f\left(\exp\left(\frac{m}{2^n}\right)\right) = \exp\left(\frac{m\lambda}{2^n}\right) + \exp\left(-\frac{m\lambda}{2^n}\right)$$

for all nonnegative integers m and n.

Since f is increasing on $[1, \infty)$, we have $f(x) = x^\lambda + x^{-\lambda}$ for $x \geq 1$.

We can prove that $f(x) = x^\lambda + x^{-\lambda}$ for $0 < x < 1$ and that this function satisfies the given conditions in the same manner as in the first solution.

A6. Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that

$$z_{i+j}^2 \geq x_i y_j \qquad \text{for all } 1 \leq i, j \leq n.$$

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Solution. Let $X = \max\{x_1, \ldots, x_n\}$ and $Y = \max\{y_1, \ldots, y_n\}$. By replacing x_i by $x_i' = x_i/X$, y_i by $y_i' = y_i/Y$, and z_i by $z_i' = z_i/\sqrt{XY}$, we may assume that $X = Y = 1$. Now we will prove that

$$M + z_2 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n, \tag{$*$}$$

so

$$\frac{M + z_2 + \cdots + z_{2n}}{2n} \geq \frac{1}{2}\left(\frac{x_1 + \cdots + x_n}{n} + \frac{y_1 + \cdots + y_n}{n}\right)$$

which implies the desired result by the AM-GM inequality.

To prove $(*)$, we will show that for any $r \geq 0$, the number of terms greater that r on the left hand side is at least the number of such terms on the right hand side. Then the kth largest term on the left hand side is greater than or equal to the kth largest term on the right hand side for each k, proving $(*)$. If $r \geq 1$, then there are no terms greater than r on the right hand side. So suppose $r < 1$. Let $A = \{1 \leq i \leq n \mid x_i > r\}$, $a = |A|$, $B = \{1 \leq i \leq n \mid y_i > r\}$, $b = |B|$. Since $\max\{x_1, \ldots, x_n\} = \max\{y_1, \ldots, y_n\} = 1$, both a and b are at least 1. Now $x_i > r$ and $y_j > r$ implies $z_{i+j} \geq \sqrt{x_i y_j} > r$, so

$$C = \{2 \leq i \leq 2n \mid z_i > r\} \supset A + B = \{\alpha + \beta \mid \alpha \in A,\ \beta \in B\}.$$

However, we know that $|A + B| \geq |A| + |B| - 1$, because if $A = \{i_1, \ldots, i_a\}$, $i_1 < \cdots < i_a$ and $B = \{j_1, \ldots, j_b\}$, $j_1 < \cdots < j_b$, then the $a + b - 1$ numbers $i_1 + j_1$, $i_1 + j_2$, $\ldots$, $i_1 + j_b$, $i_2 + j_b$, $\ldots$, $i_a + j_b$ are all distinct and belong to $A + B$. Hence $|C| \geq a + b - 1$. In particular, $|C| \geq 1$ so $z_k > r$ for some k. Then $M > r$, so the left hand side of $(*)$ has at least $a + b$ terms greater than r. Since $a + b$ is the number of terms greater than r on the right hand side, we have proved $(*)$.

Combinatorics

C1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, t_2, $\ldots$, t_{100} in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Solution 1. Consider the set $D = \{x - y \mid x, y \in A\}$. There are at most $101 \times 100 + 1 = 10101$ elements in D. Two sets $A + t_i$ and $A + t_j$ have nonempty intersection if and only if $t_i - t_j$ is in D. So we need to choose the 100 elements in such a way that we do not use a difference from D.

Now select these elements by induction. Choose one element arbitrarily. Assume that k elements, $k \leq 99$, are already chosen. An element x that is already chosen prevents us from selecting any element from the set $x + D$. Thus after k elements are chosen, at most $10101k \leq 999999$ elements are forbidden. Hence we can select one more element.

Comment. The size $|S| = 10^6$ is unnecessarily large. The following statement is true:

> If A is a k-element subset of $S = \{1, \ldots, n\}$ and m is a positive integer such that $n > (m-1)\left(\binom{k}{2} + 1\right)$, then there exist $t_1, \ldots, t_m \in S$ such that the sets $A_j = \{x + t_j \mid x \in A\}$, $j = 1, \ldots, m$ are pairwise disjoint.

Solution 2. We give a solution to the generalised version.

Consider the set $B = \big\{|x - y| \bigm| x, y \in A\big\}$. Clearly, $|B| \leq \binom{k}{2} + 1$.

It suffices to prove that there exist $t_1, \ldots, t_m \in S$ such that $|t_i - t_j| \notin B$ for every distinct i and j. We will select t_1, $\ldots$, t_m inductively.

Choose 1 as t_1, and consider the set $C_1 = S \setminus (B + t_1)$. Then we have $|C_1| \geq n - \left(\binom{k}{2} + 1\right) > (m-2)\left(\binom{k}{2} + 1\right)$.

For $1 \leq i < m$, suppose that t_1, $\ldots$, t_i and C_i are already defined and that $|C_i| > (m - i - 1)\left(\binom{k}{2} + 1\right) \geq 0$. Choose the least element in C_i as t_{i+1} and consider the set $C_{i+1} = C_i \setminus (B + t_{i+1})$. Then

$$|C_{i+1}| \geq |C_i| - \left(\binom{k}{2} + 1\right) > (m - i - 2)\left(\binom{k}{2} + 1\right) \geq 0.$$

Clearly, t_1, $\ldots$, t_m satisfy the desired condition.

C2. Let $D_1, \ldots, D_n$ be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

Solution. Pick a disc S with the smallest radius, say s. Subdivide the plane into seven regions as in Figure 1, that is, subdivide the complement of S into six congruent regions T_1, $\ldots$, T_6.

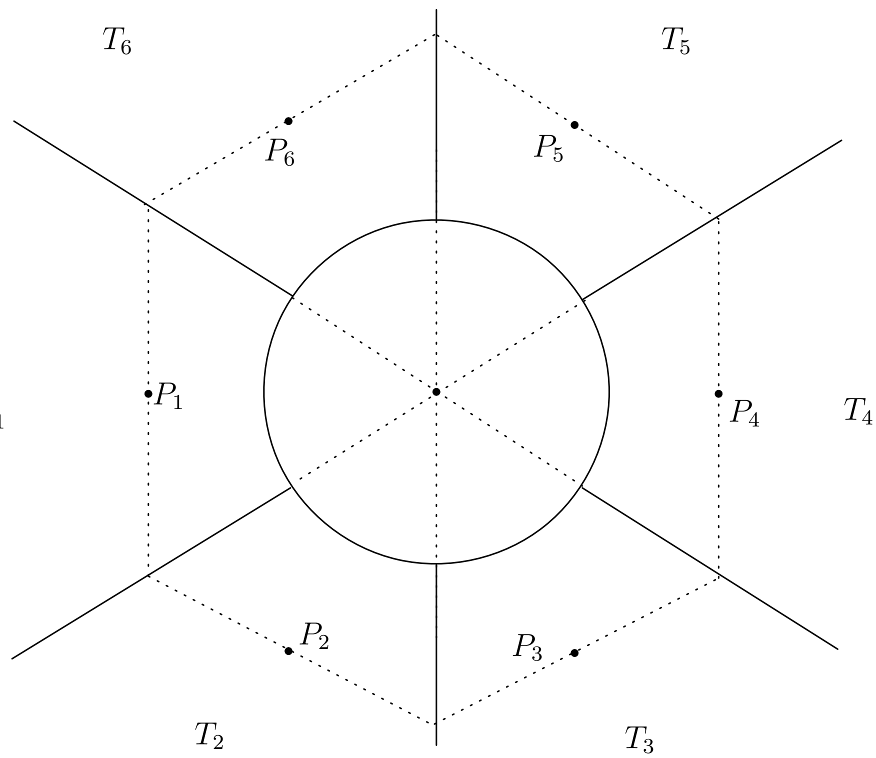


Figure 1

Since s is the smallest radius, any disc different from S whose centre lies inside S contains the centre O of the disc S. Therefore the number of such discs is less than or equal to 2002.

We will show that if a disc D_k has its centre inside T_i and intersects S, then D_k contains P_i, where P_i is the point such that $OP_i = \sqrt{3}\, s$ and OP_i bisects the angle formed by the two half-lines that bound T_i.

Subdivide T_i into U_i and V_i as in Figure 2.

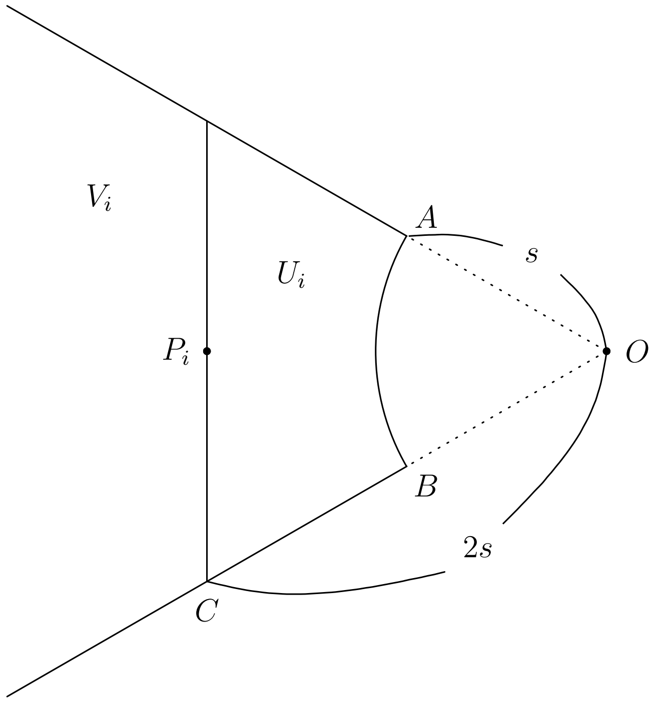


Figure 2

The region U_i is contained in the disc with radius s and centre P_i. Thus, if the centre of D_k is inside U_i, then D_k contains P_i.

Suppose that the centre of D_k is inside V_i. Let Q be the centre of D_k and let R be the intersection of OQ and the boundary of S. Since D_k intersects S, the radius of D_k is greater than QR. Since $\angle QP_iR \geq \angle CP_iB = 60°$ and $\angle P_iRO \geq \angle P_iBO = 120°$, we have $\angle QP_iR \geq \angle P_iRQ$. Hence $QR \geq QP_i$ and so D_k contains P_i.

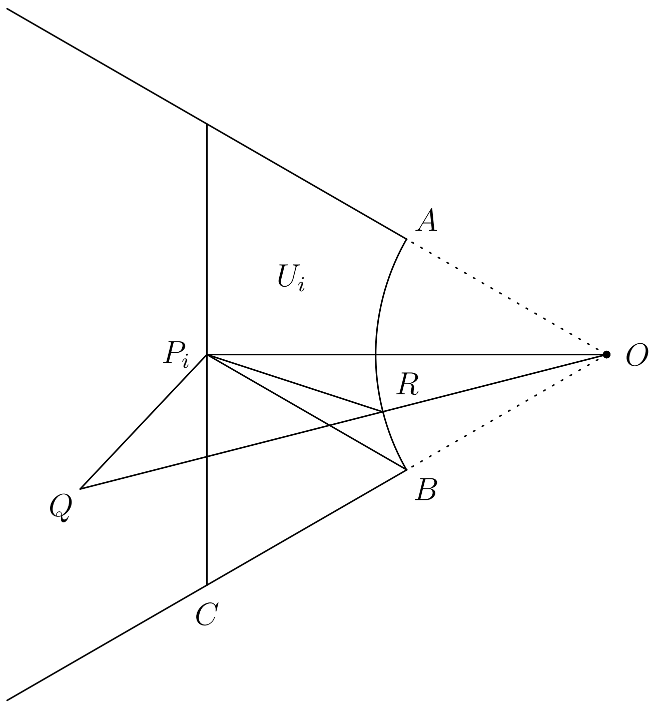


Figure 3

For $i = 1, \ldots, 6$, the number of discs D_k having their centres inside T_i and intersecting S is less than or equal to 2003. Consequently, the number of discs D_k that intersect S is less than or equal to $2002 + 6 \cdot 2003 = 7 \cdot 2003 - 1$.

C3. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

Solution. We will show that the greatest integer k satisfying the given condition is equal to 3 for $n = 5$, and $\lfloor 2n/3 \rfloor + 1$ for $n \geq 6$.

Assume that there exists an n-gon having k internal right angles. Since all other $n - k$ angles are less than $360°$, we have

$$(n - k) \cdot 360° + k \cdot 90° > (n - 2) \cdot 180°,$$

or $k < (2n + 4)/3$. Since k and n are integers, we have $k \leq \lfloor 2n/3 \rfloor + 1$.

If $n = 5$, then $\lfloor 2n/3 \rfloor + 1 = 4$. However, if a pentagon has 4 internal right angles, then the other angle is equal to $180°$, which is not appropriate. Figure 1 gives the pentagon with 3 internal right angles, thus the greatest integer k is equal to 3.

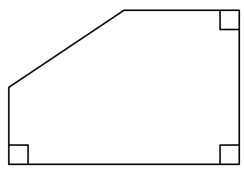


Figure 1

We will construct an n-gon having $\lfloor 2n/3 \rfloor + 1$ internal right angles for each $n \geq 6$. Figure 2 gives the examples for $n = 6, 7, 8$.

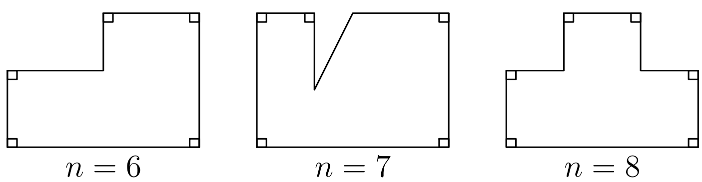


Figure 2

For $n \geq 9$, we will construct examples inductively. Since all internal non-right angles in this construction are greater than $180°$, we can cut off 'a triangle without a vertex' around a non-right angle in order to obtain three more vertices and two more internal right angles as in Figure 3.

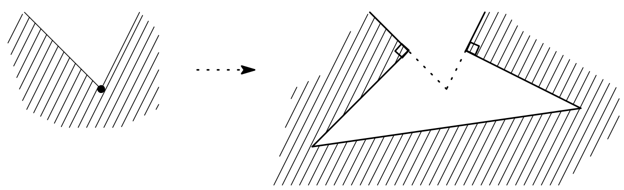


Figure 3

Comment. Here we give two other ways to construct examples.

One way is to add 'a rectangle with a hat' near an internal non-right angle as in Figure 4.

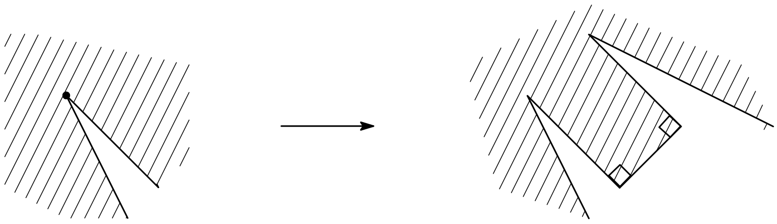
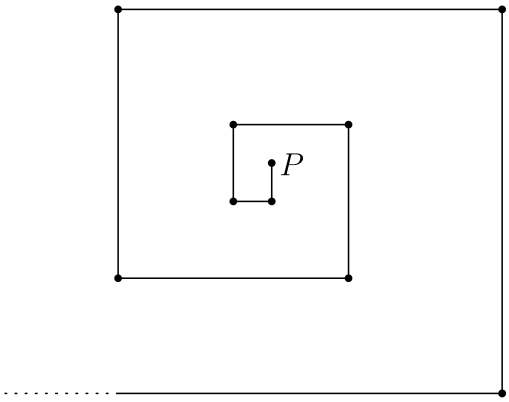
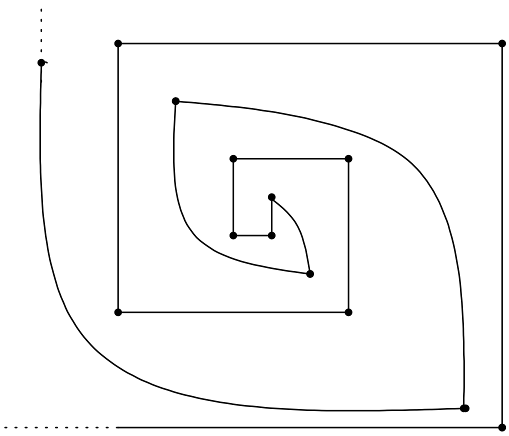


Figure 4

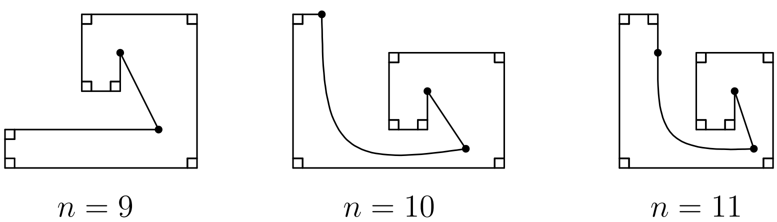
The other way is 'the escaping construction.' First we draw right angles in spiral.



Then we 'escape' from the point P.



The followings are examples for $n = 9, 10, 11$. The angles around the black points are not right.



$n = 9$ $n = 10$ $n = 11$

The 'escaping lines' are not straight in these figures. However, in fact, we can make them straight when we draw sufficiently large figures.

C4. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \ge 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

Solution 1. Let $B = (b_{ij})_{1\le i,j\le n}$. Define $S = \sum_{1\le i,j\le n}(x_i + y_j)(a_{ij} - b_{ij})$.

On one hand, we have

$$S = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} b_{ij} \right) + \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} b_{ij} \right) = 0.$$

On the other hand, if $x_i + y_j \ge 0$, then $a_{ij} = 1$, which implies $a_{ij} - b_{ij} \ge 0$; if $x_i + y_j < 0$, then $a_{ij} = 0$, which implies $a_{ij} - b_{ij} \le 0$. Therefore $(x_i + y_j)(a_{ij} - b_{ij}) \ge 0$ for every i and j.

Thus we have $(x_i + y_j)(a_{ij} - b_{ij}) = 0$ for every i and j. In particular, if $a_{ij} = 0$, then $x_i + y_j < 0$ and so $a_{ij} - b_{ij} = 0$. This means that $a_{ij} \ge b_{ij}$ for every i and j.

Since the sum of the elements in each row of B is equal to the corresponding sum for A, we have $a_{ij} = b_{ij}$ for every i and j.

Solution 2. Let $B = (b_{ij})_{1\le i,j\le n}$. Suppose that $A \ne B$, that is, there exists (i_0, j_0) such that $a_{i_0 j_0} \ne b_{i_0 j_0}$. We may assume without loss of generality that $a_{i_0 j_0} = 0$ and $b_{i_0 j_0} = 1$.

Since the sum of the elements in the i_0-th row of B is equal to that in A, there exists j_1 such that $a_{i_0 j_1} = 1$ and $b_{i_0 j_1} = 0$. Similarly there exists i_1 such that $a_{i_1 j_1} = 0$ and $b_{i_1 j_1} = 1$. Let us define i_k and j_k inductively in this way so that $a_{i_k j_k} = 0$, $b_{i_k j_k} = 1$, $a_{i_k j_{k+1}} = 1$, $b_{i_k j_{k+1}} = 0$.

Because the size of the matrix is finite, there exist s and t such that $s \ne t$ and $(i_s, j_s) = (i_t, j_t)$.

Since $a_{i_k j_k} = 0$ implies $x_{i_k} + y_{j_k} < 0$ by definition, we have $\sum_{k=s}^{t-1}(x_{i_k} + y_{j_k}) < 0$. Similarly, since $a_{i_k j_{k+1}} = 1$ implies $x_{i_k} + y_{j_{k+1}} \ge 0$, we have $\sum_{k=s}^{t-1}(x_{i_k} + y_{j_{k+1}}) \ge 0$. However, since $j_s = j_t$, we have

$$\sum_{k=s}^{t-1}(x_{i_k} + y_{j_{k+1}}) = \sum_{k=s}^{t-1} x_{i_k} + \sum_{k=s+1}^{t} y_{j_k} = \sum_{k=s}^{t-1} x_{i_k} + \sum_{k=s}^{t-1} y_{j_k} = \sum_{k=s}^{t-1}(x_{i_k} + y_{j_k}).$$

This is a contradiction.

C5. Every point with integer coordinates in the plane is the centre of a disc with radius 1/1000.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

Solution 1. (1) Define $f\colon \mathbb{Z} \longrightarrow [0,1)$ by $f(x) = x\sqrt{3} - \lfloor x\sqrt{3}\rfloor$. By the pigeonhole principle, there exist distinct integers x_1 and x_2 such that $\bigl|f(x_1) - f(x_2)\bigr| < 0.001$. Put $a = |x_1 - x_2|$. Then the distance either between $\bigl(a, a\sqrt{3}\bigr)$ and $\bigl(a, \lfloor a\sqrt{3}\rfloor\bigr)$ or between $\bigl(a, a\sqrt{3}\bigr)$ and $\bigl(a, \lfloor a\sqrt{3}\rfloor + 1\bigr)$ is less than 0.001. Therefore the points $(0,0)$, $(2a,0)$, $\bigl(a, a\sqrt{3}\bigr)$ lie in different discs and form an equilateral triangle.

(2) Suppose that $P'Q'R'$ is a triangle such that $P'Q' = Q'R' = R'P' = l \le 96$ and P', Q', R' lie in discs with centres P, Q, R, respectively. Then

$$l - 0.002 \le PQ, QR, RP \le l + 0.002.$$

Since PQR is not an equilateral triangle, we may assume that $PQ \ne QR$. Therefore

$$\begin{aligned} |PQ^2 - QR^2| &= (PQ + QR)|PQ - QR| \\ &\le \bigl((l+0.002) + (l+0.002)\bigr)\bigl((l+0.002) - (l-0.002)\bigr) \\ &\le 2 \cdot 96.002 \cdot 0.004 \\ &< 1. \end{aligned}$$

However, $PQ^2 - QR^2 \in \mathbb{Z}$. This is a contradiction.

Solution 2. We give another solution to (2).

Lemma. Suppose that ABC and $A'B'C'$ are equilateral triangles and that A, B, C and A', B', C' lie anticlockwise. If $AA', BB' \le r$, then $CC' \le 2r$.

Proof. Let α, β, γ; α', β', γ' be the complex numbers corresponding to A, B, C; A', B', C'. Then

$$\gamma = \omega\beta + (1-\omega)\alpha \quad \text{and} \quad \gamma' = \omega\beta' + (1-\omega)\alpha',$$

where $\omega = \bigl(1 + \sqrt{3}\,i\bigr)/2$. Therefore

$$\begin{aligned} CC' = |\gamma - \gamma'| &= \bigl|\omega(\beta - \beta') + (1-\omega)(\alpha - \alpha')\bigr| \\ &\le |\omega||\beta - \beta'| + |1-\omega||\alpha - \alpha'| = BB' + AA' \\ &\le 2r. \end{aligned}$$

∎

Suppose that P, Q, R lie on discs with radius r and centres P', Q', R', respectively, and that PQR is an equilateral triangle. Let R'' be the point such that $P'Q'R''$ is an equilateral triangle and P', Q', R' lie anticlockwise. It follows from the lemma that $RR'' \le 2r$, and so $R'R'' \le RR' + RR'' \le r + 2r = 3r$ by the triangle inequality.

Put $\overrightarrow{P'Q'} = \begin{pmatrix} m \\ n \end{pmatrix}$ and $\overrightarrow{P'R'} = \begin{pmatrix} s \\ t \end{pmatrix}$, where m, n, s, t are integers. We may suppose that $m, n \ge 0$. Then we have

$$\sqrt{\left(\frac{m - n\sqrt{3}}{2} - s\right)^2 + \left(\frac{n + m\sqrt{3}}{2} - t\right)^2} \le 3r.$$

Setting $a = 2t - n$ and $b = m - 2s$, we obtain

$$\sqrt{\left(a - m\sqrt{3}\right)^2 + \left(b - n\sqrt{3}\right)^2} \le 6r.$$

Since $\left|a - m\sqrt{3}\right| \ge 1/\left|a + m\sqrt{3}\right|$, $\left|b - n\sqrt{3}\right| \ge 1/\left|b + n\sqrt{3}\right|$ and $|a| \le m\sqrt{3} + 6r$, $|b| \le n\sqrt{3} + 6r$, we have

$$\sqrt{\frac{1}{\left(2m\sqrt{3} + 6r\right)^2} + \frac{1}{\left(2n\sqrt{3} + 6r\right)^2}} \le 6r.$$

Since $1/x^2 + 1/y^2 \ge 8/(x+y)^2$ for all positive real numbers x and y, it follows that

$$\frac{2\sqrt{2}}{2\sqrt{3}(m+n) + 12r} \le 6r.$$

As $P'Q' = \sqrt{m^2 + n^2} \ge (m+n)/\sqrt{2}$, we have

$$\frac{2\sqrt{2}}{2\sqrt{6}\,P'Q' + 12r} \le 6r.$$

Therefore

$$P'Q' \ge \frac{1}{6\sqrt{3}\,r} - \sqrt{6}\,r.$$

Finally we obtain

$$PQ \ge P'Q' - 2r \ge \frac{1}{6\sqrt{3}\,r} - \sqrt{6}\,r - 2r.$$

For $r = 1/1000$, we have $PQ \ge 96.22\cdots > 96$.

C6. Let $f(k)$ be the number of integers n that satisfy the following conditions:

(i) $0 \le n < 10^k$, so n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Solution 1. We use the notation $[a_{k-1}a_{k-2}\cdots a_0]$ to indicate the positive integer with digits $a_{k-1}, a_{k-2}, \ldots, a_0$.

The following fact is well-known:

$$[a_{k-1}a_{k-2}\cdots a_0] \equiv i \pmod{11} \iff \sum_{l=0}^{k-1}(-1)^l a_l \equiv i \pmod{11}.$$

Fix $m \in \mathbb{N}$ and define the sets A_i and B_i as follows:

- A_i is the set of all integers n with the following properties:
  (1) $0 \le n < 10^{2m}$, i.e., n has $2m$ digits;
  (2) the right $2m-1$ digits of n can be permuted so that the resulting integer is congruent to i modulo 11.
- B_i is the set of all integers n with the following properties:
  (1) $0 \le n < 10^{2m-1}$, i.e., n has $2m-1$ digits;
  (2) the digits of n can be permuted so that the resulting integer is congruent to i modulo 11.

It is clear that $f(2m) = |A_0|$ and $f(2m-1) = |B_0|$. Since $\underbrace{99\cdots9}_{2m} \equiv 0 \pmod{11}$, we have

$$n \in A_i \iff \underbrace{99\cdots9}_{2m} - n \in A_{-i}.$$

Hence

$$|A_i| = |A_{-i}|. \tag{1}$$

Since $\underbrace{99\cdots9}_{2m-1} \equiv 9 \pmod{11}$, we have

$$n \in B_i \iff \underbrace{99\cdots9}_{2m-1} - n \in B_{9-i}.$$

Thus

$$|B_i| = |B_{9-i}|. \tag{2}$$

For any $2m$-digit integer $n = [ja_{2m-2}\cdots a_0]$, we have

$$n \in A_i \iff [a_{2m-2}\cdots a_0] \in B_{i-j}.$$

Hence

$$|A_i| = |B_i| + |B_{i-1}| + \cdots + |B_{i-9}|.$$

Since $B_i = B_{i+11}$, this can be written as

$$|A_i| = \sum_{k=0}^{10} |B_k| - |B_{i+1}|, \tag{3}$$

hence

$$|A_i| = |A_j| \iff |B_{i+1}| = |B_{j+1}|. \tag{4}$$

From (1), (2), and (4), we obtain $|A_i| = |A_0|$ and $|B_i| = |B_0|$. Substituting this into (3) yields $|A_0| = 10|B_0|$, and so $f(2m) = 10f(2m-1)$.

Comment. This solution works for all even bases b, and the result is $f(2m) = bf(2m-1)$.

Solution 2. We will use the notation in Solution 1. For a $2m$-tuple $(a_0, \ldots, a_{2m-1})$ of integers, we consider the following property:

$$(a_0, \ldots, a_{2m-1}) \text{ can be permuted so that } \sum_{l=0}^{2m-1} (-1)^l a_l \equiv 0 \pmod{11}. \tag{$*$}$$

It is easy to verify that

$$(a_0, \ldots, a_{2m-1}) \text{ satisfies } (*) \iff (a_0 + k, \ldots, a_{2m-1} + k) \text{ satisfies } (*) \tag{1}$$

for all integers k, and that

$$(a_0, \ldots, a_{2m-1}) \text{ satisfies } (*) \iff (ka_0, \ldots, ka_{2m-1}) \text{ satisfies } (*) \tag{2}$$

for all integers $k \not\equiv 0 \pmod{11}$.

For an integer k, denote by $\langle k \rangle$ the nonnegative integer less than 11 congruent to k modulo 11.

For a fixed $j \in \{0, 1, \ldots, 9\}$, let k be the unique integer such that $k \in \{1, 2, \ldots, 10\}$ and $(j+1)k \equiv 1 \pmod{11}$.

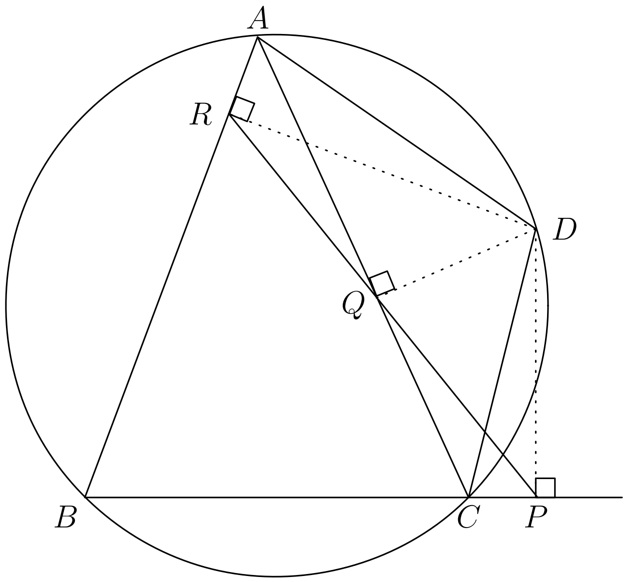
Suppose that $[a_{2m-1} \cdots a_1 j] \in A_0$, that is, $(a_{2m-1}, \ldots, a_1, j)$ satisfies $(*)$. From (1) and (2), it follows that $\big((a_{2m-1} + 1)k - 1, \ldots, (a_1 + 1)k - 1, 0\big)$ also satisfies $(*)$. Putting $b_i = \big\langle (a_i + 1)k \big\rangle - 1$, we have $[b_{2m-1} \cdots b_1] \in B_0$.

For any $j \in \{0, 1, \ldots, 9\}$, we can reconstruct $[a_{2m-1} \ldots a_1 j]$ from $[b_{2m-1} \cdots b_1]$. Hence we have $|A_0| = 10|B_0|$, and so $f(2m) = 10f(2m-1)$.

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Solution 1.



It is well-known that P, Q, R are collinear (Simson's theorem). Moreover, since $\angle DPC$ and $\angle DQC$ are right angles, the points D, P, Q, C are concyclic and so $\angle DCA = \angle DPQ = \angle DPR$. Similarly, since D, Q, R, A are concyclic, we have $\angle DAC = \angle DRP$. Therefore $\triangle DCA \sim \triangle DPR$.

Likewise, $\triangle DAB \sim \triangle DQP$ and $\triangle DBC \sim \triangle DRQ$. Then

$$\frac{DA}{DC} = \frac{DR}{DP} = \frac{DB \cdot \frac{QR}{BC}}{DB \cdot \frac{PQ}{BA}} = \frac{QR}{PQ} \cdot \frac{BA}{BC}.$$

Thus $PQ = QR$ if and only if $DA/DC = BA/BC$.

Now the bisectors of the angles ABC and ADC divide AC in the ratios of BA/BC and DA/DC, respectively. This completes the proof.

Solution 2. Suppose that the bisectors of $\angle ABC$ and $\angle ADC$ meet AC at L and M, respectively. Since $AL/CL = AB/CB$ and $AM/CM = AD/CD$, the bisectors in question meet on AC if and only if $AB/CB = AD/CD$, that is, $AB \cdot CD = CB \cdot AD$. We will prove that $AB \cdot CD = CB \cdot AD$ is equivalent to $PQ = QR$.

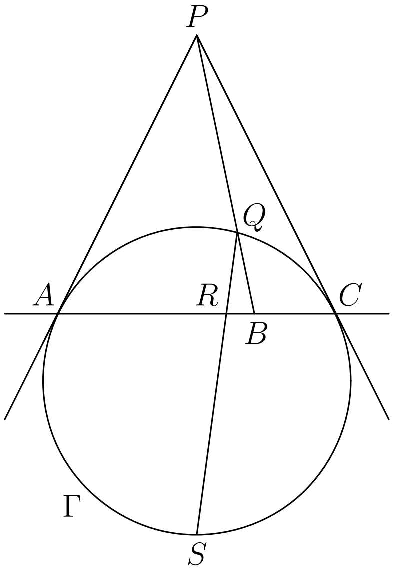
Because $DP \perp BC$, $DQ \perp AC$, $DR \perp AB$, the circles with diameters DC and DA contain the pairs of points P, Q and Q, R, respectively. It follows that $\angle PDQ$ is equal to γ or $180^\circ - \gamma$, where $\gamma = \angle ACB$. Likewise, $\angle QDR$ is equal to α or $180^\circ - \alpha$, where $\alpha = \angle CAB$. Then, by the law of sines, we have $PQ = CD \sin\gamma$ and $QR = AD \sin\alpha$. Hence the condition $PQ = QR$ is equivalent to $CD/AD = \sin\alpha/\sin\gamma$.

On the other hand, $\sin\alpha/\sin\gamma = CB/AB$ by the law of sines again. Thus $PQ = QR$ if and only if $CD/AD = CB/AB$, which is the same as $AB \cdot CD = CB \cdot AD$.

Comment. Solution 2 shows that this problem can be solved without the knowledge of Simson's theorem.

G2. Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

Solution 1.



Suppose that the bisector of $\angle AQC$ intersects the line AC and the circle Γ at R and S, respectively, where S is not equal to Q.

Since $\triangle APC$ is an isosceles triangle, we have $AB : BC = \sin\angle APB : \sin\angle CPB$. Likewise, since $\triangle ASC$ is an isosceles triangle, we have $AR : RC = \sin\angle ASQ : \sin\angle CSQ$.

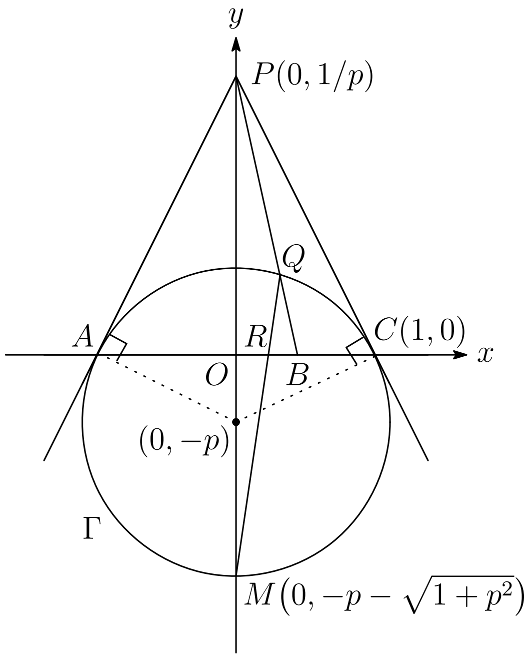
Applying the sine version of Ceva's theorem to the triangle PAC and Q, we obtain

$$\sin\angle APB : \sin\angle CPB = \sin\angle PAQ \sin\angle QCA : \sin\angle PCQ \sin\angle QAC.$$

The tangent theorem shows that $\angle PAQ = \angle ASQ = \angle QCA$ and $\angle PCQ = \angle CSQ = \angle QAC$.

Hence $AB : BC = AR^2 : RC^2$, and so R does not depend on Γ.

Solution 2.



Let R be the intersection of the bisector of the angle AQC and the line AC.

We may assume that $A(-1,0)$, $B(b,0)$, $C(1,0)$, and $\Gamma\colon x^2+(y+p)^2 = 1+p^2$. Then $P(0,1/p)$.

Let M be the midpoint of the largest arc AC. Then $M\bigl(0,-p-\sqrt{1+p^2}\bigr)$. The points Q, R, M are collinear, since $\angle AQR = \angle CQR$.

Because $PB\colon y = -x/pb + 1/p$, computation shows that

$$Q\left(\frac{(1+p^2)b - pb\sqrt{(1+p^2)(1-b^2)}}{1+p^2b^2}, \frac{-p(1-b^2)+\sqrt{(1+p^2)(1-b^2)}}{1+p^2b^2}\right),$$

so we have

$$\frac{QP}{BQ} = \frac{\sqrt{1+p^2}}{p\sqrt{1-b^2}}.$$

Since

$$\frac{MO}{PM} = \frac{p+\sqrt{1+p^2}}{\frac{1}{p}+p+\sqrt{1+p^2}} = \frac{p}{\sqrt{1+p^2}},$$

we obtain

$$\frac{OR}{RB} = \frac{MO}{PM}\cdot\frac{QP}{BQ} = \frac{p}{\sqrt{1+p^2}}\cdot\frac{\sqrt{1+p^2}}{p\sqrt{1-b^2}} = \frac{1}{\sqrt{1-b^2}}.$$

Therefore R does not depend on p or Γ.

G3. Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_AI_BI_C$.

Solution. Since the given condition implies

$$0 = (BP^2 + PE^2) - (CP^2 + PF^2) = (BP^2 - PF^2) - (CP^2 - PE^2) = BF^2 - CE^2,$$

we may put $x = BF = CE$. Similarly we may put $y = CD = AF$ and $z = AE = BD$.

If one of three points D, E, F does not lie on the sides of the triangle ABC, then this contradicts the triangle inequality. Indeed, if, for example, B, C, D lie in this order, we have $AB + BC = (x + y) + (z - y) = x + z = AC$, a contradiction. Thus all three points lie on the sides of the triangle ABC.

Putting $a = BC$, $b = CA$, $c = AB$ and $s = (a + b + c)/2$, we have $x = s - a$, $y = s - b$, $z = s - c$. Since $BD = s - c$ and $CD = s - b$, we see that D is the point at which the excircle of the triangle ABC opposite to A meets BC. Similarly E and F are the points at which the excircle opposite to B and C meet CA and AB, respectively. Since both PD and I_AD are perpendicular to BC, the three points P, D, I_A are collinear. Analogously P, E, I_B are collinear and P, F, I_C are collinear.

The three points I_A, C, I_B are collinear and the triangle PI_AI_B is isosceles because $\angle PI_AC = \angle PI_BC = \angle C/2$. Likewise we have $PI_A = PI_C$ and so $PI_A = PI_B = PI_C$. Thus P is the circumcentre of the triangle $I_AI_BI_C$.

Comment 1. The conclusion is true even if the point P lies outside the triangle ABC.

Comment 2. In fact, the common value of $AP^2 + PD^2$, $BP^2 + PE^2$, $CP^2 + PF^2$ is equal to $8R^2 - s^2$, where R is the circumradius of the triangle ABC and $s = (BC + CA + AB)/2$. We can prove this as follows:

Observe that the circumradius of the triangle $I_AI_BI_C$ is equal to $2R$ since its orthic triangle is ABC. It follows that $PD = PI_A - DI_A = 2R - r_A$, where r_A is the radius of the excircle of the triangle ABC opposite to A. Putting r_B and r_C in a similar manner, we have $PE = 2R - r_B$ and $PF = 2R - r_C$. Now we have

$$AP^2 + PD^2 = AE^2 + PE^2 + PD^2 = (s - c)^2 + (2R - r_B)^2 + (2R - r_A)^2.$$

Since

$$\begin{aligned}
(2R - r_A)^2 &= 4R^2 - 4Rr_A + r_A^2 \\
&= 4R^2 - 4 \cdot \frac{abc}{4\,\text{area}(\triangle ABC)} \cdot \frac{\text{area}(\triangle ABC)}{s - a} + \left(\frac{\text{area}(\triangle ABC)}{s - a}\right)^2 \\
&= 4R^2 + \frac{s(s - b)(s - c) - abc}{s - a} \\
&= 4R^2 + bc - s^2
\end{aligned}$$

and we can obtain $(2R - r_B)^2 = 4R^2 + ca - s^2$ in a similar way, it follows that

$$AP^2 + PD^2 = (s - c)^2 + (4R^2 + ca - s^2) + (4R^2 + bc - s^2) = 8R^2 - s^2.$$

G4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P.

Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Solution 1.

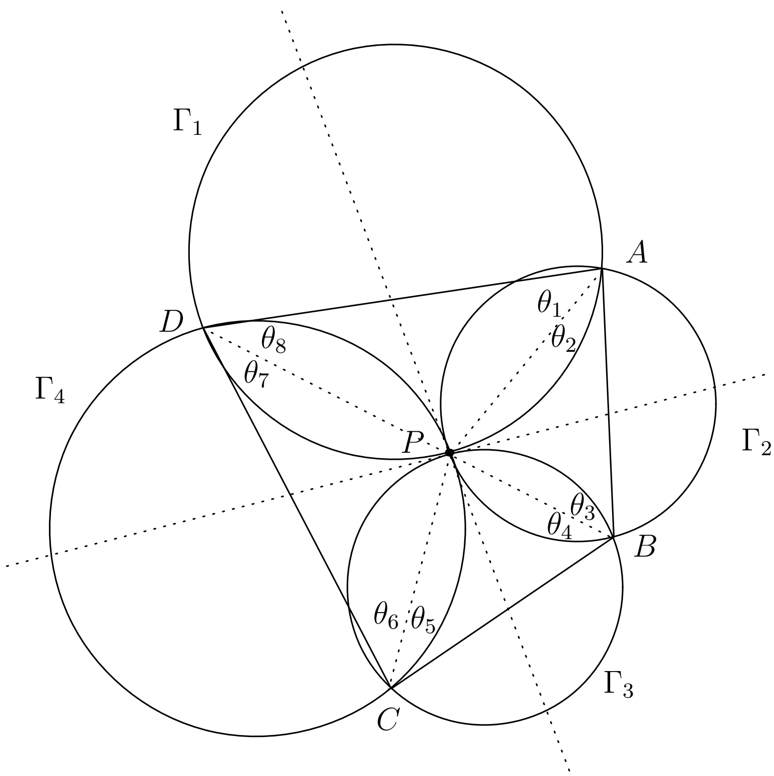


Figure 1

Let Q be the intersection of the line AB and the common tangent of Γ_1 and Γ_3. Then

$$\angle APB = \angle APQ + \angle BPQ = \angle PDA + \angle PCB.$$

Define $\theta_1, \ldots, \theta_8$ as in Figure 1. Then

$$\theta_2 + \theta_3 + \angle APB = \theta_2 + \theta_3 + \theta_5 + \theta_8 = 180°. \tag{1}$$

Similarly, $\angle BPC = \angle PAB + \angle PDC$ and

$$\theta_4 + \theta_5 + \theta_2 + \theta_7 = 180°. \tag{2}$$

Multiply the side-lengths of the triangles PAB, PBC, PCD, PAD by $PC \cdot PD$, $PD \cdot PA$, $PA \cdot PB$, $PB \cdot PC$, respectively, to get the new quadrilateral $A'B'C'D'$ as in Figure 2.

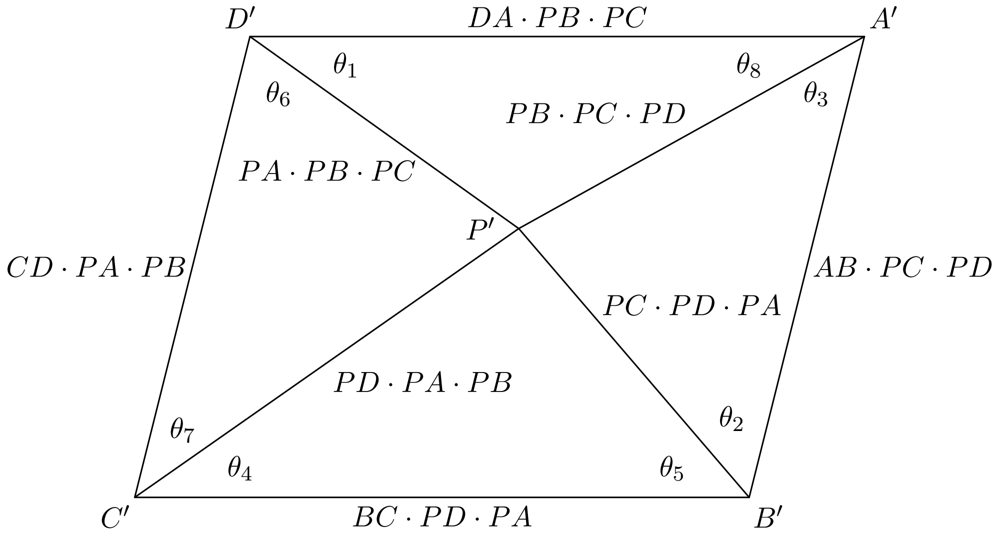
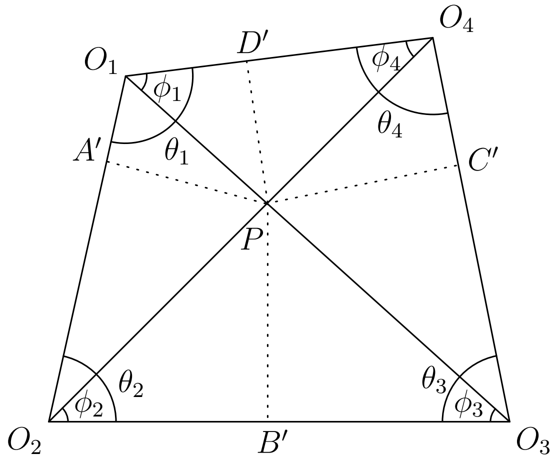


Figure 2

(1) and (2) show that $A'D' \parallel B'C'$ and $A'B' \parallel C'D'$. Thus the quadrilateral $A'B'C'D'$ is a parallelogram. It follows that $A'B' = C'D'$ and $A'D' = C'B'$, that is, $AB \cdot PC \cdot PD = CD \cdot PA \cdot PB$ and $AD \cdot PB \cdot PC = BC \cdot PA \cdot PD$, from which we see that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Solution 2. Let O_1, O_2, O_3, O_4 be the centres of Γ_1, Γ_2, Γ_3, Γ_4, respectively, and let A', B', C', D' be the midpoints of PA, PB, PC, PD, respectively. Since Γ_1, Γ_3 are externally tangent at P, it follows that O_1, O_3, P are collinear. Similarly we see that O_2, O_4, P are collinear.



Put $\theta_1 = \angle O_4O_1O_2$, $\theta_2 = \angle O_1O_2O_3$, $\theta_3 = \angle O_2O_3O_4$, $\theta_4 = \angle O_3O_4O_1$ and $\phi_1 = \angle PO_1O_4$, $\phi_2 = \angle PO_2O_3$, $\phi_3 = \angle PO_3O_2$, $\phi_4 = \angle PO_4O_1$. By the law of sines, we have

$$O_1O_2 : O_1O_3 = \sin\phi_3 : \sin\theta_2, \qquad O_3O_4 : O_2O_4 = \sin\phi_2 : \sin\theta_3,$$
$$O_3O_4 : O_1O_3 = \sin\phi_1 : \sin\theta_4, \qquad O_1O_2 : O_2O_4 = \sin\phi_4 : \sin\theta_1.$$

Since the segment PA is the common chord of Γ_1 and Γ_2, the segment PA' is the altitude from P to O_1O_2. Similarly PB', PC', PD' are the altitudes from P to O_2O_3, O_3O_4, O_4O_1, respectively. Then O_1, A', P, D' are concyclic. So again by the law of sines, we have

$$D'A' : PD' = \sin\theta_1 : \sin\phi_1.$$

Likewise we have

$$A'B' : PB' = \sin\theta_2 : \sin\phi_2, \quad B'C' : PB' = \sin\theta_3 : \sin\phi_3, \quad C'D' : PD' = \sin\theta_4 : \sin\phi_4.$$

Since $A'B' = AB/2$, $B'C' = BC/2$, $C'D' = CD/2$, $D'A' = DA/2$, $PB' = PB/2$, $PD' = PD/2$, we have

$$\begin{aligned}\frac{AB \cdot BC}{AD \cdot DC} \cdot \frac{PD^2}{PB^2} &= \frac{A'B' \cdot B'C'}{A'D' \cdot D'C'} \cdot \frac{PD'^2}{PB'^2} = \frac{\sin\theta_2 \sin\theta_3 \sin\phi_4 \sin\phi_1}{\sin\phi_2 \sin\phi_3 \sin\theta_4 \sin\theta_1} \\ &= \frac{O_1O_3}{O_1O_2} \cdot \frac{O_2O_4}{O_3O_4} \cdot \frac{O_1O_2}{O_2O_4} \cdot \frac{O_3O_4}{O_1O_3} = 1\end{aligned}$$

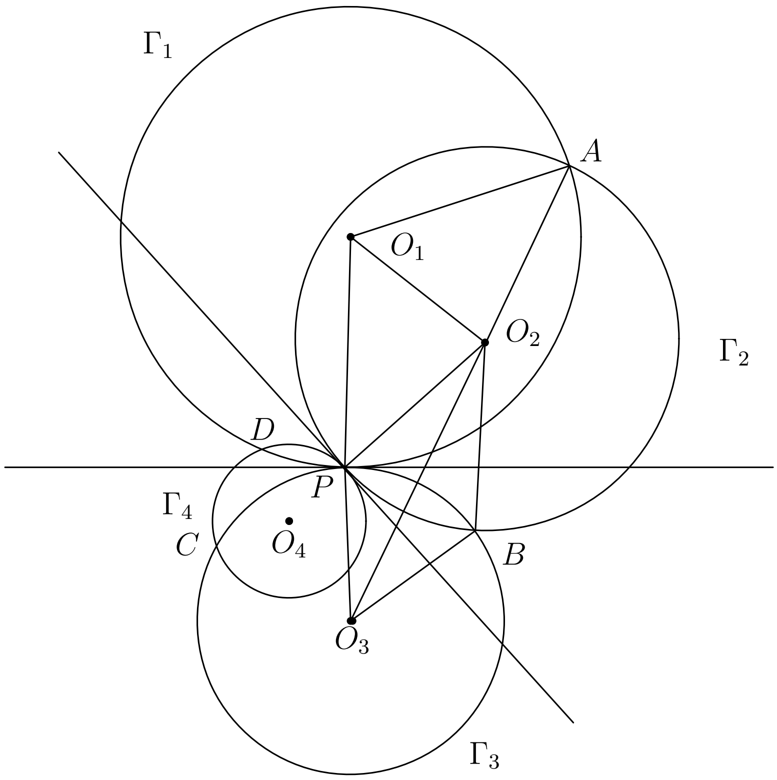
and the conclusion follows.

Comment. It is not necessary to assume that Γ_1, Γ_3 and Γ_2, Γ_4 are *externally* tangent. We may change the first sentence in the problem to the following:

> Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are tangent at P, and Γ_2, Γ_4 are tangent at the same point P.

The following two solutions are valid for the changed version.

Solution 3.



Let O_i and r_i be the centre and the signed radius of Γ_i, $i = 1, 2, 3, 4$. We may assume that $r_1 > 0$. If O_1, O_3 are in the same side of the common tangent, then we have $r_3 > 0$; otherwise we have $r_3 < 0$.

Put $\theta = \angle O_1PO_2$. We have $\angle O_iPO_{i+1} = \theta$ or $180^\circ - \theta$, which shows that

$$\sin\angle O_iPO_{i+1} = \sin\theta. \tag{1}$$

Since $PB \perp O_2O_3$ and $\triangle PO_2O_3 \equiv \triangle BO_2O_3$, we have

$$\frac{1}{2}\cdot\frac{1}{2}\cdot O_2O_3\cdot PB = \text{area}(\triangle PO_2O_3) = \frac{1}{2}\cdot PO_2\cdot PO_3\cdot \sin\theta = \frac{1}{2}|r_2||r_3|\sin\theta.$$

It follows that

$$PB = \frac{2|r_2||r_3|\sin\theta}{O_2O_3}. \tag{2}$$

Because the triangle O_2AB is isosceles, we have

$$AB = 2|r_2|\sin\frac{\angle AO_2B}{2}. \tag{3}$$

Since $\angle O_1O_2P = \angle O_1O_2A$ and $\angle O_3O_2P = \angle O_3O_2B$, we have

$$\sin(\angle AO_2B/2) = \sin\angle O_1O_2O_3.$$

Therefore, keeping in mind that

$$\begin{aligned}\frac{1}{2}\cdot O_1O_2\cdot O_2O_3\cdot\sin\angle O_1O_2O_3 = \text{area}(\triangle O_1O_2O_3) &= \frac{1}{2}\cdot O_1O_3\cdot PO_2\cdot\sin\theta\\ &= \frac{1}{2}|r_1-r_3||r_2|\sin\theta,\end{aligned}$$

we have

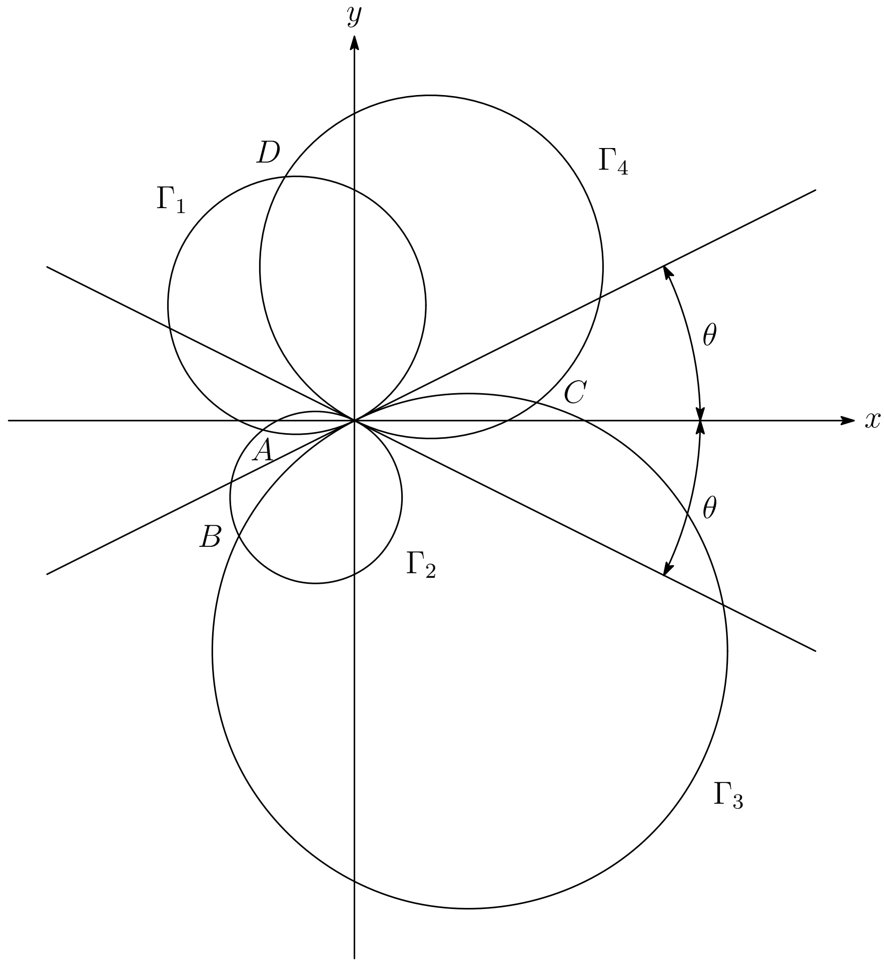
$$AB = 2|r_2|\frac{|r_1-r_3||r_2|\sin\theta}{O_1O_2\cdot O_2O_3}$$

by (3).

Likewise, by (1), (2), (4), we can obtain the lengths of PD, BC, CD, DA and compute as follows:

$$\begin{aligned}\frac{AB\cdot BC}{CD\cdot DA} &= \frac{2|r_1-r_3|r_2^2\sin\theta}{O_1O_2\cdot O_2O_3}\cdot\frac{2|r_2-r_4|r_3^2\sin\theta}{O_2O_3\cdot O_3O_4}\cdot\frac{O_3O_4\cdot O_4O_1}{2|r_1-r_3|r_4^2\sin\theta}\cdot\frac{O_4O_1\cdot O_1O_2}{2|r_2-r_4|r_1^2\sin\theta}\\ &= \left(\frac{2|r_2||r_3|\sin\theta}{O_2O_3}\right)^2\left(\frac{O_4O_1}{2|r_4||r_1|\sin\theta}\right)^2\\ &= \frac{PB^2}{PD^2}.\end{aligned}$$

Solution 4. Let l_1 be the common tangent of the circles Γ_1 and Γ_3 and let l_2 be that of Γ_2 and Γ_4. Set the coordinate system as in the following figure.



We may assume that

$$\Gamma_1\colon x^2+y^2+2ax\sin\theta-2ay\cos\theta=0,\qquad \Gamma_2\colon x^2+y^2+2bx\sin\theta+2by\cos\theta=0,$$
$$\Gamma_3\colon x^2+y^2-2cx\sin\theta+2cy\cos\theta=0,\qquad \Gamma_4\colon x^2+y^2-2dx\sin\theta-2dy\cos\theta=0.$$

Simple computation shows that

$$A\left(-\frac{4ab(a+b)\sin\theta\cos^2\theta}{a^2+b^2+2ab\cos2\theta},-\frac{4ab(a-b)\sin^2\theta\cos\theta}{a^2+b^2+2ab\cos2\theta}\right),$$
$$B\left(\frac{4bc(b-c)\sin\theta\cos^2\theta}{b^2+c^2-2bc\cos2\theta},-\frac{4bc(b+c)\sin^2\theta\cos\theta}{b^2+c^2-2bc\cos2\theta}\right),$$
$$C\left(\frac{4cd(c+d)\sin\theta\cos^2\theta}{c^2+d^2+2cd\cos2\theta},\frac{4cd(c-d)\sin^2\theta\cos\theta}{c^2+d^2+2cd\cos2\theta}\right),$$
$$D\left(-\frac{4da(d-a)\sin\theta\cos^2\theta}{d^2+a^2-2da\cos2\theta},\frac{4da(d+a)\sin^2\theta\cos\theta}{d^2+a^2-2da\cos2\theta}\right).$$

Slightly long computation shows that

$$\begin{aligned}
AB &= \frac{4b^2|a+c|\sin\theta\cos\theta}{\sqrt{(a^2+b^2+2ab\cos 2\theta)(b^2+c^2-2bc\cos 2\theta)}},\\
BC &= \frac{4c^2|b+d|\sin\theta\cos\theta}{\sqrt{(b^2+c^2-2bc\cos 2\theta)(c^2+d^2+2cd\cos 2\theta)}},\\
CD &= \frac{4d^2|c+a|\sin\theta\cos\theta}{\sqrt{(c^2+d^2+2cd\cos 2\theta)(d^2+a^2-2da\cos 2\theta)}},\\
DA &= \frac{4a^2|d+b|\sin\theta\cos\theta}{\sqrt{(d^2+a^2-2da\cos 2\theta)(a^2+b^2+2ab\cos 2\theta)}},
\end{aligned}$$

which implies

$$\frac{AB\cdot BC}{AD\cdot DC} = \frac{b^2c^2(d^2+a^2-2da\cos 2\theta)}{d^2a^2(b^2+c^2-2bc\cos 2\theta)}.$$

On the other hand, we have

$$MB = \frac{4|b||c|\sin\theta\cos\theta}{\sqrt{b^2+c^2-2bc\cos 2\theta}} \quad \text{and} \quad MD = \frac{4|d||a|\sin\theta\cos\theta}{\sqrt{d^2+a^2-2da\cos 2\theta}},$$

which implies

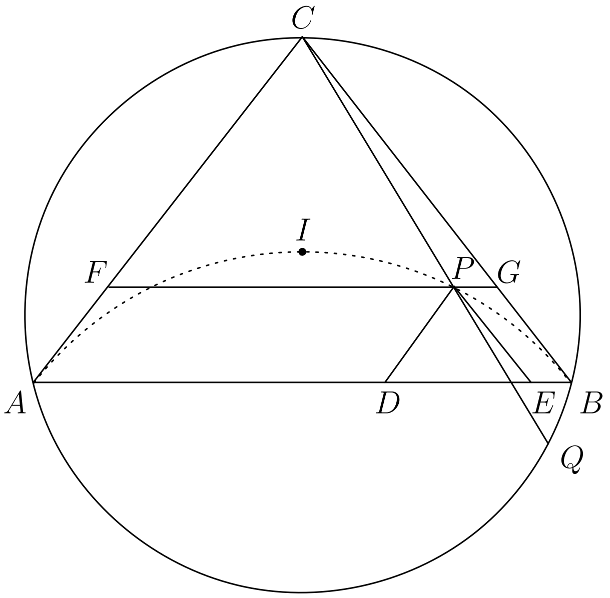
$$\frac{MB^2}{MD^2} = \frac{b^2c^2(d^2+a^2-2da\cos 2\theta)}{d^2a^2(b^2+c^2-2bc\cos 2\theta)}.$$

Hence we obtain

$$\frac{AB\cdot BC}{AD\cdot DC} = \frac{MB^2}{MD^2}.$$

G5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Solution 1.



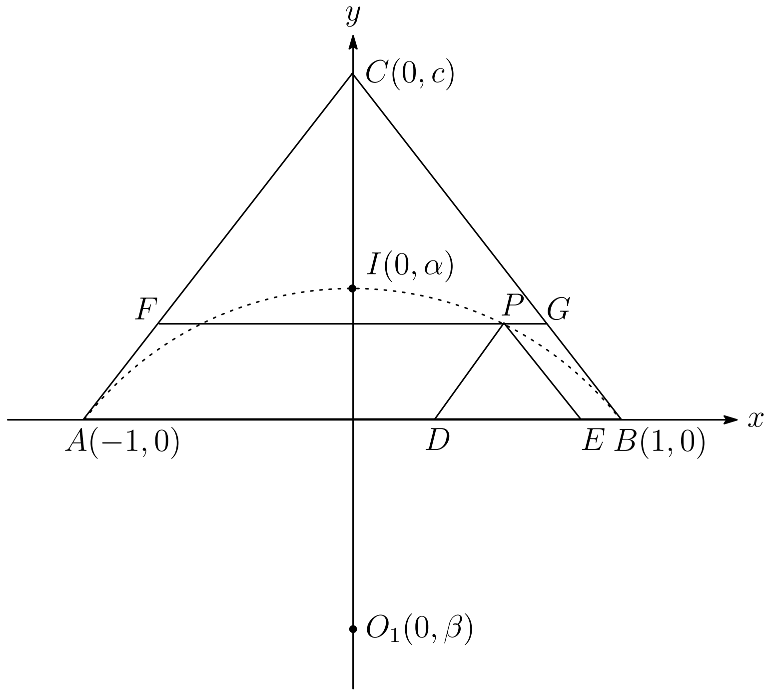
The corresponding sides of the triangles PDE and CFG are parallel. Therefore, if DF and EG are not parallel, then they are homothetic, and so DF, EG, CP are concurrent at the centre of the homothety. This observation leads to the following claim:

Claim. Suppose that CP meets again the circumcircle of the triangle ABC at Q. Then Q is the intersection of DF and EG.

Proof. Since $\angle AQP = \angle ABC = \angle BAC = \angle PFC$, it follows that the quadrilateral $AQPF$ is cyclic, and so $\angle FQP = \angle PAF$. Since $\angle IBA = \angle CBA/2 = \angle CAB/2 = \angle IAC$, the circumcircle of the triangle AIB is tangent to CA at A, which implies that $\angle PAF = \angle DBP$. Since $\angle QBD = \angle QCA = \angle QPD$, it follows that the quadrilateral $DQBP$ is cyclic, and so $\angle DBP = \angle DQP$. Thus $\angle FQP = \angle PAF = \angle DBP = \angle DQP$, which implies that F, D, Q are collinear. Analogously we obtain that G, E, Q are collinear. ∎

Hence the lines DF, EG, CP meet the circumcircle of the triangle ABC at the same point.

Solution 2.



Set the coordinate system so that $A(-1,0)$, $B(1,0)$, $C(0,c)$. Suppose that $I(0,\alpha)$.

Since

$$\operatorname{area}(\triangle ABC) = \frac{1}{2}(AB+BC+CA)\alpha,$$

we obtain

$$\alpha = \frac{c}{1+\sqrt{1+c^2}}.$$

Suppose that $O_1(0,\beta)$ is the centre of the circumcircle Γ_1 of the triangle AIB. Since

$$(\beta-\alpha)^2 = O_1I^2 = O_1A^2 = 1+\beta^2,$$

we have $\beta = -1/c$ and so $\Gamma_1\colon x^2+(y+1/c)^2 = 1+(1/c)^2$.

Let $P(p,q)$. Since $D(p-q/c,0)$, $E(p+q/c,0)$, $F(q/c-1,q)$, $G(-q/c+1,q)$, it follows that the equations of the lines DF and EG are

$$y = \frac{q}{\frac{2q}{c}-p-1}\left(x-\left(p-\frac{q}{c}\right)\right) \quad \text{and} \quad y = \frac{q}{-\frac{2q}{c}-p+1}\left(x-\left(p+\frac{q}{c}\right)\right),$$

respectively. Therefore the intersection Q of these lines is $\big((q-c)p/(2q-c), q^2/(2q-c)\big)$.

Let $O_2(0,\gamma)$ be the circumcentre of the triangle ABC. Then $\gamma = (c^2-1)/2c$ since $1+\gamma^2 = O_2A^2 = O_2C^2 = (\gamma-c)^2$.

Note that $p^2+(q+1/c)^2 = 1+(1/c)^2$ since $P(p,q)$ is on the circle Γ_1. It follows that

$$O_2Q^2 = \left(\frac{q-c}{2q-c}\right)^2 p^2 + \left(\frac{q^2}{2q-c} - \frac{c^2-1}{2c}\right)^2 = \left(\frac{c^2+1}{2c}\right)^2 = O_2C^2,$$

which shows that Q is on the circumcircle of the triangle ABC.

Comment. The point P can be any point on the circumcircle of the triangle AIB other than A and B; that is, P need not lie inside the triangle ABC.

G6. Each pair of opposite sides of a convex hexagon has the following property:

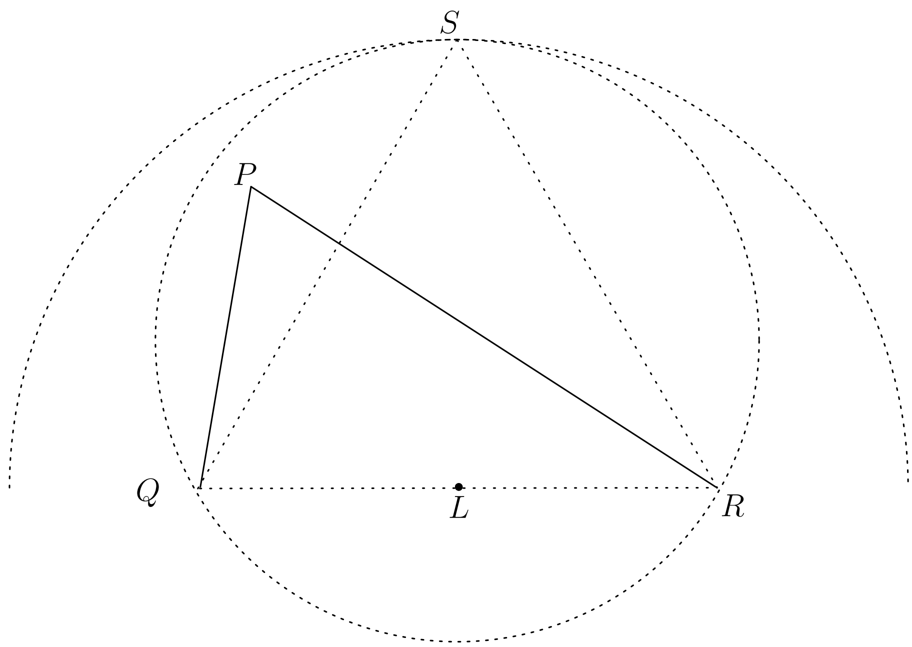
> the distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths.

Prove that all the angles of the hexagon are equal.

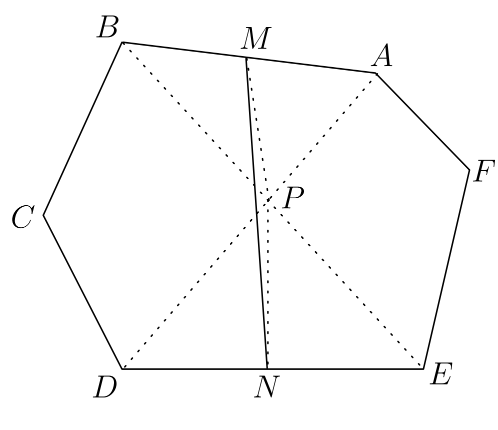
Solution 1. We first prove the following lemma:

Lemma. Consider a triangle PQR with $\angle QPR \geq 60^\circ$. Let L be the midpoint of QR. Then $PL \leq \sqrt{3}\,QR/2$, with equality if and only if the triangle PQR is equilateral.

Proof.



Let S be the point such that the triangle QRS is equilateral, where the points P and S lie in the same half-plane bounded by the line QR. Then the point P lies inside the circumcircle of the triangle QRS, which lies inside the circle with centre L and radius $\sqrt{3}\,QR/2$. This completes the proof of the lemma. ∎



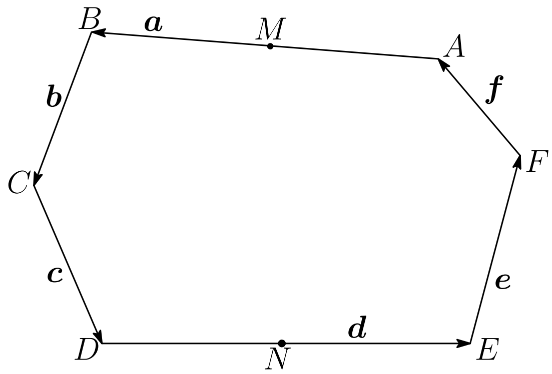
The main diagonals of a convex hexagon form a triangle though the triangle can be degenerated. Thus we may choose two of these three diagonals that form an angle greater than or equal to 60°. Without loss of generality, we may assume that the diagonals AD and BE of the given hexagon $ABCDEF$ satisfy $\angle APB \geq 60°$, where P is the intersection of these diagonals. Then, using the lemma, we obtain

$$MN = \frac{\sqrt{3}}{2}(AB + DE) \geq PM + PN \geq MN,$$

where M and N are the midpoints of AB and DE, respectively. Thus it follows from the lemma that the triangles ABP and DEP are equilateral.

Therefore the diagonal CF forms an angle greater than or equal to 60° with one of the diagonals AD and BE. Without loss of generality, we may assume that $\angle AQF \geq 60°$, where Q is the intersection of AD and CF. Arguing in the same way as above, we infer that the triangles AQF and CQD are equilateral. This implies that $\angle BRC = 60°$, where R is the intersection of BE and CF. Using the same argument as above for the third time, we obtain that the triangles BCR and EFR are equilateral. This completes the solution.

Solution 2. Let $ABCDEF$ be the given hexagon and let $\boldsymbol{a} = \overrightarrow{AB}$, $\boldsymbol{b} = \overrightarrow{BC}$, $\ldots$, $\boldsymbol{f} = \overrightarrow{FA}$.



Let M and N be the midpoints of the sides AB and DE, respectively. We have

$$\overrightarrow{MN} = \frac{1}{2}\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \frac{1}{2}\boldsymbol{d} \quad \text{and} \quad \overrightarrow{MN} = -\frac{1}{2}\boldsymbol{a} - \boldsymbol{f} - \boldsymbol{e} - \frac{1}{2}\boldsymbol{d}.$$

Thus we obtain

$$\overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{e} - \boldsymbol{f}). \tag{1}$$

From the given property, we have

$$\overrightarrow{MN} = \frac{\sqrt{3}}{2}\big(|\boldsymbol{a}| + |\boldsymbol{d}|\big) \geq \frac{\sqrt{3}}{2}|\boldsymbol{a} - \boldsymbol{d}|. \tag{2}$$

Set $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{d}$, $\boldsymbol{y} = \boldsymbol{c} - \boldsymbol{f}$, $\boldsymbol{z} = \boldsymbol{e} - \boldsymbol{b}$. From (1) and (2), we obtain

$$|\boldsymbol{y} - \boldsymbol{z}| \geq \sqrt{3}\,|\boldsymbol{x}|. \tag{3}$$

Similarly we see that

$$|\boldsymbol{z} - \boldsymbol{x}| \geq \sqrt{3}\,|\boldsymbol{y}|, \tag{4}$$
$$|\boldsymbol{x} - \boldsymbol{y}| \geq \sqrt{3}\,|\boldsymbol{z}|. \tag{5}$$

Note that

$$\begin{aligned} (3) &\iff |\boldsymbol{y}|^2 - 2\boldsymbol{y}\cdot\boldsymbol{z} + |\boldsymbol{z}|^2 \geq 3|\boldsymbol{x}|^2,\\ (4) &\iff |\boldsymbol{z}|^2 - 2\boldsymbol{z}\cdot\boldsymbol{x} + |\boldsymbol{x}|^2 \geq 3|\boldsymbol{y}|^2,\\ (5) &\iff |\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + |\boldsymbol{y}|^2 \geq 3|\boldsymbol{z}|^2. \end{aligned}$$

By adding up the last three inequalities, we obtain

$$-|\boldsymbol{x}|^2 - |\boldsymbol{y}|^2 - |\boldsymbol{z}|^2 - 2\boldsymbol{y}\cdot\boldsymbol{z} - 2\boldsymbol{z}\cdot\boldsymbol{x} - 2\boldsymbol{x}\cdot\boldsymbol{y} \geq 0,$$

or $-|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}|^2 \geq 0$. Thus $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z} = \boldsymbol{0}$ and the equalities hold in all inequalities above. Hence we conclude that

$$\begin{aligned} &\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z} = \boldsymbol{0},\\ |\boldsymbol{y}-\boldsymbol{z}| &= \sqrt{3}\,|\boldsymbol{x}|, \quad \boldsymbol{a} \parallel \boldsymbol{d} \parallel \boldsymbol{x},\\ |\boldsymbol{z}-\boldsymbol{x}| &= \sqrt{3}\,|\boldsymbol{y}|, \quad \boldsymbol{c} \parallel \boldsymbol{f} \parallel \boldsymbol{y},\\ |\boldsymbol{x}-\boldsymbol{y}| &= \sqrt{3}\,|\boldsymbol{z}|, \quad \boldsymbol{e} \parallel \boldsymbol{b} \parallel \boldsymbol{z}. \end{aligned}$$

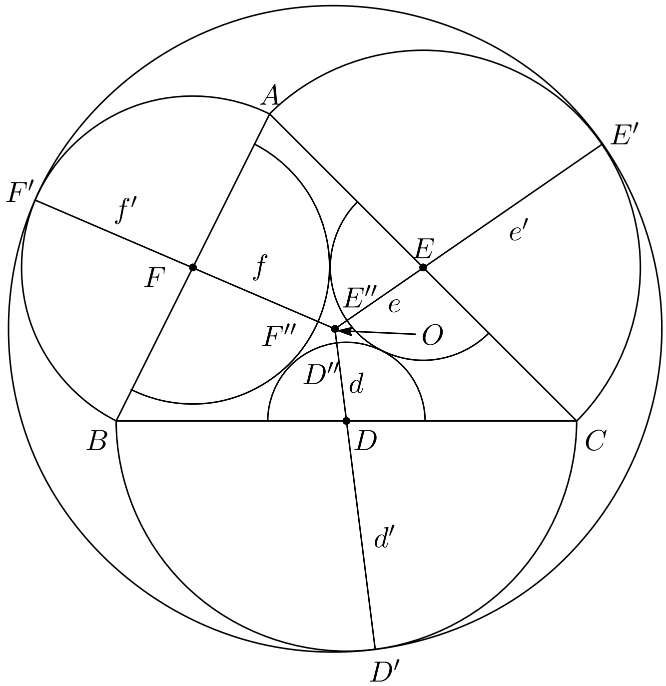
Suppose that PQR is the triangle such that $\overrightarrow{PQ} = \boldsymbol{x}$, $\overrightarrow{QR} = \boldsymbol{y}$, $\overrightarrow{RP} = \boldsymbol{z}$. We may assume $\angle QPR \geq 60^\circ$, without loss of generality. Let L be the midpoint of QR, then $PL = |\boldsymbol{z}-\boldsymbol{x}|/2 = \sqrt{3}\,|\boldsymbol{y}|/2 = \sqrt{3}\,QR/2$. It follows from the lemma in Solution 1 that the triangle PQR is equilateral. Thus we have $\angle ABC = \angle BCD = \cdots = \angle FAB = 120^\circ$.

Comment. We have obtained the complete characterisation of the hexagons satisfying the given property. They are all obtained from an equilateral triangle by cutting its 'corners' at the same height.

G7. Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all three semicircles has radius t. Prove that

$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Solution 1.



Let O be the centre of the circle and let D, E, F be the midpoints of BC, CA, AB, respectively. Denote by D', E', F' the points at which the circle is tangent to the semicircles. Let d', e', f' be the radii of the semicircles. Then all of DD', EE', FF' pass through O, and $s = d' + e' + f'$.

Put

$$d = \frac{s}{2} - d' = \frac{-d' + e' + f'}{2}, \quad e = \frac{s}{2} - e' = \frac{d' - e' + f'}{2}, \quad f = \frac{s}{2} - f' = \frac{d' + e' - f'}{2}.$$

Note that $d + e + f = s/2$. Construct smaller semicircles inside the triangle ABC with radii d, e, f and centres D, E, F. Then the smaller semicircles touch each other, since $d + e = f' = DE$, $e + f = d' = EF$, $f + d = e' = FD$. In fact, the points of tangency are the points where the incircle of the triangle DEF touches its sides.

Suppose that the smaller semicircles cut DD', EE', FF' at D'', E'', F'', respectively. Since these semicircles do not overlap, the point O is outside the semicircles. Therefore $D'O > D'D''$, and so $t > s/2$. Put $g = t - s/2$.

Clearly, $OD'' = OE'' = OF'' = g$. Therefore the circle with centre O and radius g touches all of the three mutually tangent semicircles.

Claim. We have

$$\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2} = \frac{1}{2}\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{g}\right)^2.$$

Proof. Consider a triangle PQR and let $p = QR$, $q = RP$, $r = PQ$. Then

$$\cos\angle QPR = \frac{-p^2+q^2+r^2}{2qr}$$

and

$$\sin\angle QPR = \frac{\sqrt{(p+q+r)(-p+q+r)(p-q+r)(p+q-r)}}{2qr}.$$

Since

$$\cos\angle EDF = \cos(\angle ODE + \angle ODF) = \cos\angle ODE\cos\angle ODF - \sin\angle ODE\sin\angle ODF,$$

we have

$$\begin{aligned}\frac{d^2+de+df-ef}{(d+e)(d+f)} = {} & \frac{(d^2+de+dg-eg)(d^2+df+dg-fg)}{(d+g)^2(d+e)(d+f)} \\ & - \frac{4dg\sqrt{(d+e+g)(d+f+g)ef}}{(d+g)^2(d+e)(d+f)},\end{aligned}$$

which simplifies to

$$(d+g)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right) - 2\left(\frac{d}{g}+1+\frac{g}{d}\right) = -2\sqrt{\frac{(d+e+g)(d+f+g)}{ef}}.$$

Squaring and simplifying, we obtain

$$\begin{aligned}\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right)^2 &= 4\left(\frac{1}{de}+\frac{1}{df}+\frac{1}{dg}+\frac{1}{ef}+\frac{1}{eg}+\frac{1}{fg}\right) \\ &= 2\left(\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\frac{1}{g}\right)^2 - \left(\frac{1}{d^2}+\frac{1}{e^2}+\frac{1}{f^2}+\frac{1}{g^2}\right)\right),\end{aligned}$$

from which the conclusion follows. ∎

Solving for the smaller value of g, i.e., the larger value of $1/g$, we obtain

$$\begin{aligned}\frac{1}{g} &= \frac{1}{d}+\frac{1}{e}+\frac{1}{f}+\sqrt{2\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)^2 - 2\left(\frac{1}{d^2}+\frac{1}{e^2}+\frac{1}{f^2}\right)} \\ &= \frac{1}{d}+\frac{1}{e}+\frac{1}{f}+2\sqrt{\frac{d+e+f}{def}}.\end{aligned}$$

Comparing the formulas $\text{area}(\triangle DEF) = \text{area}(\triangle ABC)/4 = rs/4$ and $\text{area}(\triangle DEF) = \sqrt{(d+e+f)def}$, we have

$$\frac{r}{2} = \frac{2}{s}\sqrt{(d+e+f)def} = \sqrt{\frac{def}{d+e+f}}.$$

All we have to prove is that

$$\frac{r}{2g} \geq \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

Since

$$\frac{r}{2g} = \sqrt{\frac{def}{d+e+f}}\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}+2\sqrt{\frac{d+e+f}{def}}\right) = \frac{x+y+z}{\sqrt{xy+yz+zx}}+2,$$

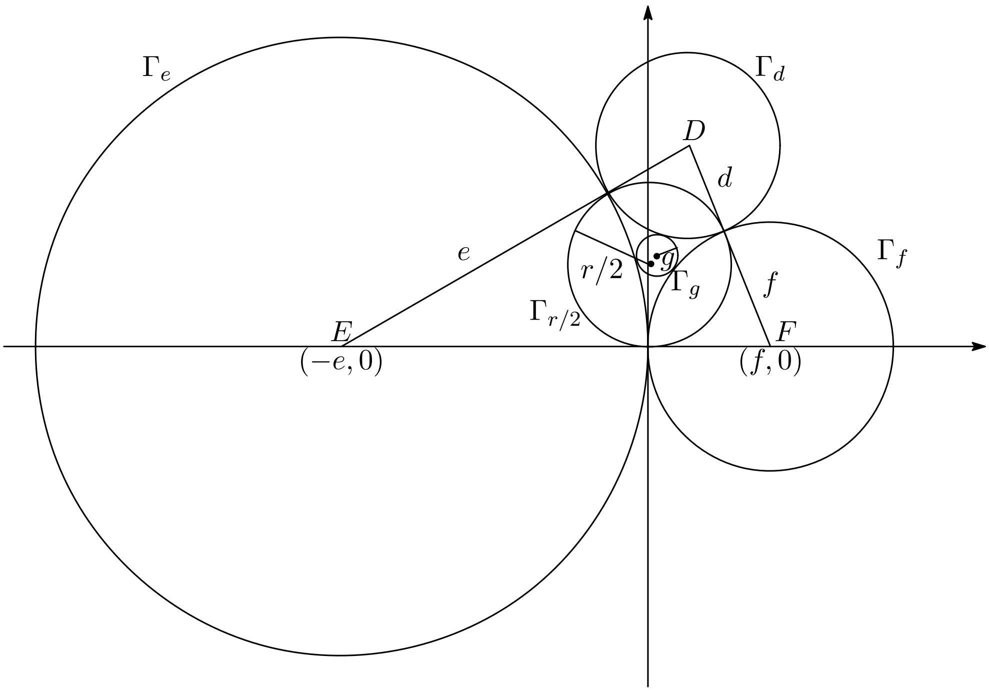
where $x = 1/d$, $y = 1/e$, $z = 1/f$, it suffices to prove that

$$\frac{(x+y+z)^2}{xy+yz+zx} \geq 3.$$

This inequality is true because

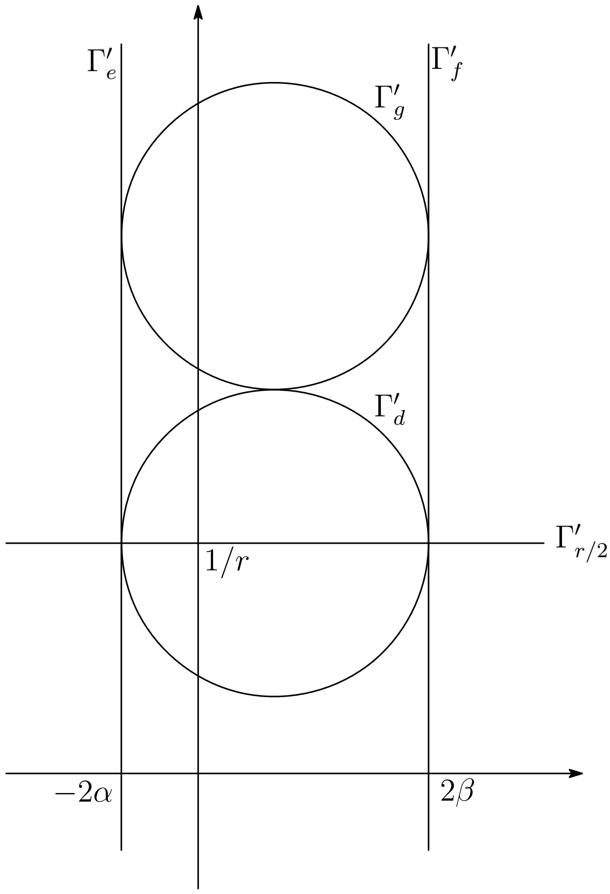
$$(x+y+z)^2 - 3(xy+yz+zx) = \frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right) \geq 0.$$

Solution 2. We prove that $t > s/2$ in the same way as in Solution 1. Put $g = t - s/2$.



Now set the coordinate system so that $E(-e,0)$, $F(f,0)$, and the y-coordinate of D is positive. Let Γ_d, Γ_e, Γ_f, Γ_g be the circles with radii d, e, f, g and centres D, E, F, O, respectively. Let $\Gamma_{r/2}$ be the incircle of the triangle DEF. Note that the radius of $\Gamma_{r/2}$ is $r/2$.

Now consider the inversion with respect to the circle with radius 1 and centre $(0,0)$.



Let Γ'_d, Γ'_e, Γ'_f, Γ'_g, $\Gamma'_{r/2}$ be the images of Γ_d, Γ_e, Γ_f, Γ_g, $\Gamma_{r/2}$, respectively. Set $\alpha = 1/4e$, $\beta = 1/4f$ and $R = \alpha + \beta$. The equations of the lines Γ'_e, Γ'_f and $\Gamma'_{r/2}$ are $x = -2\alpha$, $x = 2\beta$ and $y = 1/r$, respectively. Both of the radii of the circles Γ'_d and Γ'_g are R, and their centres are $(-\alpha + \beta, 1/r)$ and $(-\alpha + \beta, 1/r + 2R)$, respectively.

Let D be the distance between $(0, 0)$ and the centre of Γ'_g. Then we have

$$2g = \frac{1}{D - R} - \frac{1}{D + R} = \frac{2R}{D^2 - R^2},$$

which shows $g = R/(D^2 - R^2)$.

What we have to show is $g \leq \left(1 - \sqrt{3}/2\right)r$, that is $\left(4 + 2\sqrt{3}\right)g \leq r$. This is verified by the following computation:

$$\begin{aligned}
r - \left(4 + 2\sqrt{3}\right)g = r - \left(4 + 2\sqrt{3}\right)\frac{R}{D^2 - R^2} &= \frac{r}{D^2 - R^2}\left((D^2 - R^2) - \left(4 + 2\sqrt{3}\right)\frac{1}{r}R\right) \\
&= \frac{r}{D^2 - R^2}\left(\left(\frac{1}{r} + 2R\right)^2 + (\alpha - \beta)^2 - R^2 - \left(4 + 2\sqrt{3}\right)\frac{1}{r}R\right) \\
&= \frac{r}{D^2 - R^2}\left(3\left(R - \frac{1}{\sqrt{3}\,r}\right)^2 + (\alpha - \beta)^2\right) \\
&\geq 0.
\end{aligned}$$

Number Theory

N1. Let m be a fixed integer greater than 1. The sequence $x_0,\ x_1,\ x_2,\ \ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i, & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j}, & \text{if } i \ge m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

Solution. Let r_i be the remainder of $x_i \bmod m$. Then there are at most m^m types of m-consecutive blocks in the sequence (r_i). So, by the pigeonhole principle, some type reappears. Since the definition formula works forward and backward, the sequence (r_i) is purely periodic.

Now the definition formula backward $x_i = x_{i+m} - \sum_{j=1}^{m-1} x_{i+j}$ applied to the block $(r_0, \ldots, r_{m-1})$ produces the m-consecutive block $\underbrace{0, \ldots, 0}_{m-1}, 1$. Together with the pure periodicity, we see that $\max k \ge m-1$.

On the other hand, if there are m-consecutive zeroes in (r_i), then the definition formula and the pure periodicity force $r_i = 0$ for any $i \ge 0$, a contradiction. Thus $\max k = m-1$.

N2. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) move the last digit of a to the first position to obtain the number b;

(ii) square b to obtain the number c;

(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.

Find all numbers a for which $d(a) = a^2$.

Solution. Let a be a positive integer for which the procedure yields $d = d(a) = a^2$. Further assume that a has $n + 1$ digits, $n \geq 0$.

Let s be the last digit of a and f the first digit of c. Since $(* \cdots * s)^2 = a^2 = d = * \cdots * f$ and $(s * \cdots *)^2 = b^2 = c = f * \cdots *$, where the stars represent digits that are unimportant at the moment, f is both the last digit of the square of a number that ends in s and the first digit of the square of a number that starts in s.

The square $a^2 = d$ must have either $2n+1$ or $2n+2$ digits. If $s = 0$, then $n \neq 0$, b has n digits, its square c has at most $2n$ digits, and so does d, a contradiction. Thus the last digit of a is not 0.

Consider now, for example, the case $s = 4$. Then f must be 6, but this is impossible, since the squares of numbers that start in 4 can only start in 1 or 2, which is easily seen from

$$160 \cdots 0 = (40 \cdots 0)^2 \leq (4 * \cdots *)^2 < (50 \cdots 0)^2 = 250 \cdots 0.$$

Thus s cannot be 4.

The following table gives all possibilities:

s	1	2	3	4	5	6	7	8	9
$f =$ last digit of $(\cdots s)^2$	1	4	9	6	5	6	9	4	1
$f =$ first digit of $(s \cdots)^2$	1, 2, 3	4, 5, 6, 7, 8	9, 1	1, 2	2, 3	3, 4	4, 5, 6	6, 7, 8	8, 9

Thus $s = 1$, $s = 2$, or $s = 3$ and in each case $f = s^2$. When s is 1 or 2, the square $c = b^2$ of the $(n+1)$-digit number b which starts in s has $2n+1$ digits. Moreover, when $s = 3$, the square $c = b^2$ either has $2n+1$ digits and starts in 9 or has $2n+2$ digits and starts in 1. However the latter is impossible since $f = s^2 = 9$. Thus c must have $2n+1$ digits.

Let $a = 10x + s$, where x is an n-digit number (in case $x = 0$ we set $n = 0$). Then

$$\begin{aligned}
b &= 10^n s + x, \\
c &= 10^{2n} s^2 + 2 \cdot 10^n sx + x^2, \\
d &= 10(c - 10^{m-1} f) + f = 10^{2n+1} s^2 + 20 \cdot 10^n sx + 10x^2 - 10^m f + f,
\end{aligned}$$

where m is the number of digits of c. However, we already know that m must be $2n+1$ and $f = s^2$, so

$$d = 20 \cdot 10^n sx + 10x^2 + s^2$$

and the equality $a^2 = d$ yields

$$x = 2s \cdot \frac{10^n - 1}{9},$$

i.e.,

$$a = \underbrace{6 \cdots 6}_{n} 3 \quad \text{or} \quad a = \underbrace{4 \cdots 4}_{n} 2 \quad \text{or} \quad a = \underbrace{2 \cdots 2}_{n} 1,$$

for $n \geq 0$. The first two possibilities must be rejected for $n \geq 1$, since $a^2 = d$ would have $2n + 2$ digits, which means that c would have to have at least $2n + 2$ digits, but we already know that c must have $2n + 1$ digits. Thus the only remaining possibilities are

$$a = 3 \quad \text{or} \quad a = 2 \quad \text{or} \quad a = \underbrace{2 \cdots 2}_{n} 1,$$

for $n \geq 0$. It is easily seen that they all satisfy the requirements of the problem.

N3. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Solution. Let (a, b) be a pair of positive integers satisfying the condition. Because $k = a^2/(2ab^2 - b^3 + 1) > 0$, we have $2ab^2 - b^3 + 1 > 0$, $a > b/2 - 1/2b^2$, and hence $a \geq b/2$. Using this, we infer from $k \geq 1$, or $a^2 \geq b^2(2a - b) + 1$, that $a^2 > b^2(2a - b) \geq 0$. Hence

$$a > b \quad \text{or} \quad 2a = b. \tag{$*$}$$

Now consider the two solutions a_1, a_2 to the equation

$$a^2 - 2kb^2 a + k(b^3 - 1) = 0 \tag{$\sharp$}$$

for fixed positive integers k and b, and assume that one of them is an integer. Then the other is also an integer because $a_1 + a_2 = 2kb^2$. We may assume that $a_1 \geq a_2$, and we have $a_1 \geq kb^2 > 0$. Furthermore, since $a_1 a_2 = k(b^3 - 1)$, we get

$$0 \leq a_2 = \frac{k(b^3 - 1)}{a_1} \leq \frac{k(b^3 - 1)}{kb^2} < b.$$

Together with $(*)$, we conclude that $a_2 = 0$ or $a_2 = b/2$ (in the latter case b must be even).

If $a_2 = 0$, then $b^3 - 1 = 0$, and hence $a_1 = 2k$, $b = 1$.

If $a_2 = b/2$, then $k = b^2/4$ and $a_1 = b^4/2 - b/2$.

Therefore the only possibilities are

$$(a, b) = (2l, 1) \quad \text{or} \quad (l, 2l) \quad \text{or} \quad (8l^4 - l, 2l)$$

for some positive integer l. All of these pairs satisfy the given condition.

Comment 1. An alternative way to see $(*)$ is as follows: Fix $a \geq 1$ and consider the function $f_a(b) = 2ab^2 - b^3 + 1$. Then f_a is increasing on $[0, 4a/3]$ and decreasing on $[4a/3, \infty)$. We have

$$\begin{aligned} f_a(a) &= a^3 + 1 > a^2, \\ f_a(2a - 1) &= 4a^2 - 4a + 2 > a^2, \\ f_a(2a + 1) &= -4a^2 - 4a < 0. \end{aligned}$$

Hence if $b \geq a$ and $a^2/f_a(b)$ is a positive integer, then $b = 2a$.

Indeed, if $a \leq b \leq 4a/3$, then $f_a(b) \geq f_a(a) > a^2$, and so $a^2/f_a(b)$ is not an integer, a contradiction, and if $b > 4a/3$, then

(i) if $b \geq 2a + 1$, then $f_a(b) \leq f_a(2a + 1) < 0$, a contradiction;

(ii) if $b \leq 2a - 1$, then $f_a(b) \geq f_a(2a - 1) > a^2$, and so $a^2/f_a(b)$ is not an integer, a contradiction.

Comment 2. There are several alternative solutions to this problem. Here we sketch three of them.

1. The discriminant D of the equation ($\sharp$) is the square of some integer $d \geq 0$: $D = (2b^2k - b)^2 + 4k - b^2 = d^2$. If $e = 2b^2k - b = d$, we have $4k = b^2$ and $a = 2b^2k - b/2, b/2$. Otherwise, the clear estimation $|d^2 - e^2| \geq 2e - 1$ for $d \neq e$ implies $|4k - b^2| \geq 4b^2k - 2b - 1$. If $4k - b^2 > 0$, this implies $b = 1$. The other case yields no solutions.

2. Assume that $b \neq 1$ and let $s = \gcd(2a, b^3 - 1)$, $2a = su$, $b^3 - 1 = st'$, and $2ab^2 - b^3 + 1 = st$. Then $t + t' = ub^2$ and $\gcd(u, t) = 1$. Together with $st \mid a^2$, we have $t \mid s$. Let $s = rt$. Then the problem reduces to the following lemma:

> **Lemma.** Let b, r, t, t', u be positive integers satisfying $b^3 - 1 = rtt'$ and $t + t' = ub^2$. Then $r = 1$. Furthermore, either one of t or t' or u is 1.

The lemma is proved as follows. We have $b^3 - 1 = rt(ub^2 - t) = rt'(ub^2 - t')$. Since $rt^2 \equiv rt'^2 \equiv 1 \pmod{b^2}$, if $rt^2 \neq 1$ and $rt'^2 \neq 1$, then $t, t' > b/\sqrt{r}$. It is easy to see that

$$r\frac{b}{\sqrt{r}}\left(ub^2 - \frac{b}{\sqrt{r}}\right) \geq b^3 - 1,$$

unless $r = u = 1$.

3. With the same notation as in the previous solution, since $rt^2 \mid (b^3 - 1)^2$, it suffices to prove the following lemma:

> **Lemma.** Let $b \geq 2$. If a positive integer $x \equiv 1 \pmod{b^2}$ divides $(b^3 - 1)^2$, then $x = 1$ or $x = (b^3 - 1)^2$ or $(b, x) = (4, 49)$ or $(4, 81)$.

To prove this lemma, let p, q be positive integers with $p > q > 0$ satisfying $(b^3 - 1)^2 = (pb^2 + 1)(qb^2 + 1)$. Then

$$b^4 = 2b + p + q + pqb^2. \tag{1}$$

A natural observation leads us to multiply (1) by $qb^2 - 1$. We get

$$\big(q(pq - b^2) + 1\big)b^4 = p - (q + 2b)(qb^2 - 1).$$

Together with the simple estimation

$$-3 < \frac{p - (q + 2b)(qb^2 - 1)}{b^4} < 1,$$

the conclusion of the lemma follows.

Comment 3. The problem was originally proposed in the following form:

> Let a, b be relatively prime positive integers. Suppose that $a^2/(2ab^2 - b^3 + 1)$ is a positive integer greater than 1. Prove that $b = 1$.

N4. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$:

> there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

Solution. For $b = 6, 7, 8, 9$, the number 5 is congruent to no square numbers modulo b, and hence x_n is not a square. For $b = 10$, we have $x_n = \big((10^n + 5)/3\big)^2$ for all n. By algebraic calculation, it is easy to see that $x_n = (b^{2n} + b^{n+1} + 3b - 5)/(b - 1)$.

Consider now the case $b \geq 11$ and put $y_n = (b - 1)x_n$. Assume that the condition in the problem is satisfied. Then it follows that $y_n y_{n+1}$ is a perfect square for $n > M$. Since $b^{2n} + b^{n+1} + 3b - 5 < (b^n + b/2)^2$, we infer

$$y_n y_{n+1} < \left(b^n + \frac{b}{2}\right)^2 \left(b^{n+1} + \frac{b}{2}\right)^2 = \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} + \frac{b^2}{4}\right)^2. \tag{1}$$

On the other hand, we can prove by computation that

$$y_n y_{n+1} > \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} - b^3\right)^2. \tag{2}$$

From (1) and (2), we conclude that for all integers $n > M$, there is an integer a_n such that

$$y_n y_{n+1} = \left(b^{2n+1} + \frac{b^{n+1}(b+1)}{2} + a_n\right)^2 \quad \text{and} \quad -b^3 < a_n < \frac{b^2}{4}. \tag{3}$$

It follows that $b^n \mid \big(a_n^2 - (3b-5)^2\big)$, and thus $a_n = \pm(3b - 5)$ for all sufficiently large n. Substituting in (3), we obtain $a_n = 3b - 5$ and

$$8(3b-5)b + b^2(b+1)^2 = 4b^3 + 4(3b-5)(b^2+1). \tag{4}$$

The left hand side of the equation (4) is divisible by b. The other side is a polynomial in b with integral coefficients and its constant term is -20. Hence b must divide 20. Since $b \geq 11$, we conclude that $b = 20$, but then $x_n \equiv 5 \pmod 8$ and hence x_n is not a square.

Comment. Here is a shorter solution using a limit argument:

Assume that x_n is a square for all $n > M$, where M is a positive integer.

For $n > M$, take $y_n = \sqrt{x_n} \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} \frac{\frac{b^{2n}}{b-1}}{x_n} = 1.$$

Hence

$$\lim_{n\to\infty} \frac{\frac{b^n}{\sqrt{b-1}}}{y_n} = 1.$$

On the other hand,

$$(by_n + y_{n+1})(by_n - y_{n+1}) = b^2 x_n - x_{n+1} = b^{n+2} + 3b^2 - 2b - 5. \tag{$*$}$$

These equations imply

$$\lim_{n\to\infty} (by_n - y_{n+1}) = \frac{b\sqrt{b-1}}{2}.$$

As $by_n - y_{n+1}$ is an integer, there exists $N > M$ such that $by_n - y_{n+1} = b\sqrt{b-1}/2$ for any $n > N$. This means that $b - 1$ is a perfect square.

If b is odd, then $\sqrt{b-1}/2$ is an integer and so b divides $b\sqrt{b-1}/2$. Hence using $(*)$, we obtain $b \mid 5$. This is a contradiction.

If b is even, then $b/2$ divides 5. Hence $b = 10$.

In the case $b = 10$, we have $x_n = \big((10^n + 5)/3\big)^2$ for $n \geq 1$.

N5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property:

> m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

Solution. Assume that m is expressed as $m = u + v + w$ and uvw is an odd perfect square. Then u, v, w are odd and because $uvw \equiv 1 \pmod 4$, exactly two or none of them are congruent to 3 modulo 4. In both cases, we have $m = u + v + w \equiv 3 \pmod 4$.

Conversely, we prove that $4k + 3$ has the required property. To prove this, we look for representations of the form

$$4k + 3 = xy + yz + zx.$$

In any such representations, the product of the three summands is a perfect square. Setting $x = 1 + 2l$ and $y = 1 - 2l$, we have $z = 2l^2 + 2k + 1$ from above. Then

$$\begin{aligned} xy &= 1 - 4l^2 = f(l), \\ yz &= -4l^3 + 2l^2 - (4k+2)l + 2k + 1 = g(l), \\ zx &= 4l^3 + 2l^2 + (4k+2)l + 2k + 1 = h(l). \end{aligned}$$

The numbers $f(l)$, $g(l)$, $h(l)$ are odd for each integer l and their product is a perfect square, as noted above. They are distinct, except for finitely many l. It remains to note that $|g(l)|$ and $|h(l)|$ are not perfect squares for infinitely many l (note that $|f(l)|$ is not a perfect square, unless $l = 0$).

Choose distinct prime numbers p, q such that $p, q > 4k + 3$ and pick l such that

$$\begin{aligned} 1 + 2l &\equiv 0 \pmod p, & 1 + 2l &\not\equiv 0 \pmod{p^2}, \\ 1 - 2l &\equiv 0 \pmod q, & 1 - 2l &\not\equiv 0 \pmod{q^2}. \end{aligned}$$

We can choose such l by the Chinese remainder theorem. Then $2l^2 + 2k + 1$ is not divisible by p, because $p > 4k + 3$. Hence $|h(l)|$ is not a perfect square. Similarly, $|g(l)|$ is not a perfect square.

N6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Solution. Since $(p^p - 1)/(p-1) = 1 + p + p^2 + \cdots + p^{p-1} \equiv p + 1 \pmod{p^2}$, we can get at least one prime divisor of $(p^p - 1)/(p-1)$ which is not congruent to 1 modulo p^2. Denote such a prime divisor by q. This q is what we wanted. The proof is as follows. Assume that there exists an integer n such that $n^p \equiv p \pmod{q}$. Then we have $n^{p^2} \equiv p^p \equiv 1 \pmod{q}$ by the definition of q. On the other hand, from Fermat's little theorem, $n^{q-1} \equiv 1 \pmod{q}$, because q is a prime. Since $p^2 \nmid q-1$, we have $(p^2, q-1) \mid p$, which leads to $n^p \equiv 1 \pmod{q}$. Hence we have $p \equiv 1 \pmod{q}$. However, this implies $1 + p + \cdots + p^{p-1} \equiv p \pmod{q}$. From the definition of q, this leads to $p \equiv 0 \pmod{q}$, a contradiction.

Comment 1. First, students will come up, perhaps, with the idea that q has to be of the form $pk+1$. Then,

$$\exists n \quad n^p \equiv p \pmod{q} \iff p^k \equiv 1 \pmod{q},$$

i.e.,

$$\forall n \quad n^p \not\equiv p \pmod{q} \iff p^k \not\equiv 1 \pmod{q}.$$

So, we have to find such q. These observations will take you quite naturally to the idea of taking a prime divisor of $p^p - 1$. Therefore the idea of the solution is not so tricky or technical.

Comment 2. The prime q satisfies the required condition if and only if q remains a prime in $k = \mathbb{Q}(\sqrt[p]{p})$. By applying Chebotarev's density theorem to the Galois closure of k, we see that the set of such q has the density $1/p$. In particular, there are infinitely many q satisfying the required condition. This gives an alternative solution to the problem.

N7. The sequence a_0, a_1, a_2, $\ldots$ is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

Solution. By induction, we show that

$$a_n = \frac{\left(2+\sqrt{3}\right)^{2^n} + \left(2-\sqrt{3}\right)^{2^n}}{2}.$$

Case 1: $x^2 \equiv 3 \pmod{p}$ has an integer solution

Let m be an integer such that $m^2 \equiv 3 \pmod{p}$. Then $(2+m)^{2^n} + (2-m)^{2^n} \equiv 0 \pmod{p}$. Therefore $(2+m)(2-m) \equiv 1 \pmod{p}$ shows that $(2+m)^{2^{n+1}} \equiv -1 \pmod{p}$ and that $2+m$ has the order 2^{n+2} modulo p. This implies $2^{n+2} \mid (p-1)$ and so $2^{n+3} \mid (p^2-1)$.

Case 2: otherwise

Similarly, we see that there exist integers a, b satisfying $\left(2+\sqrt{3}\right)^{2^{n+1}} = -1 + pa + pb\sqrt{3}$. Furthermore, since $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^2 = (a_n+1)(2+\sqrt{3})$, there exist integers a', b' satisfying $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{2^{n+2}} = -1 + pa' + pb'\sqrt{3}$.

Let us consider the set $S = \{i + j\sqrt{3} \mid 0 \leq i, j \leq p-1,\ (i,j) \neq (0,0)\}$. Let $I = \left\{a + b\sqrt{3} \mid a \equiv b \equiv 0 \pmod{p}\right\}$. We claim that for each $i + j\sqrt{3} \in S$, there exists an $i' + j'\sqrt{3} \in S$ satisfying $\left(i+j\sqrt{3}\right)\left(i'+j'\sqrt{3}\right) - 1 \in I$. In fact, since $i^2 - 3j^2 \not\equiv 0 \pmod{p}$ (otherwise 3 is a square mod p), we can take an integer k satisfying $k(i^2-3j^2) - 1 \in I$. Then $i' + j'\sqrt{3}$ with $i' + j'\sqrt{3} - k\left(i - j\sqrt{3}\right) \in I$ will do. Now the claim together with the previous observation implies that the minimal r with $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^r - 1 \in I$ is equal to 2^{n+3}. The claim also implies that a map $f\colon S \longrightarrow S$ satisfying $\left(i+j\sqrt{3}\right)\left(1+\sqrt{3}\right)a_{n-1} - f\left(i+j\sqrt{3}\right) \in I$ for any $i + j\sqrt{3} \in S$ exists and is bijective. Thus $\prod_{x\in S} x = \prod_{x\in S} f(x)$, so

$$\left(\prod_{x\in S} x\right)\left(\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{p^2-1} - 1\right) \in I.$$

Again, by the claim, we have $\left(\left(1+\sqrt{3}\right)a_{n-1}\right)^{p^2-1} - 1 \in I$. Hence $2^{n+3} \mid (p^2-1)$.

Comment 1. Not only Case 2 but also Case 1 can be treated by using $\left(1+\sqrt{3}\right)a_{n-1}$. In fact, we need not divide into cases: in any case, the element $\left(1+\sqrt{3}\right)a_{n-1} = \left(1+\sqrt{3}\right)/\sqrt{2}$ of the multiplicative group $\mathbb{F}_{p^2}^{\times}$ of the finite field $\mathbb{F}_{p^2}$ having p^2 elements has the order 2^{n+3}, which suffices (in Case 1, the number $\left(1+\sqrt{3}\right)a_{n-1}$ even belongs to the subgroup $\mathbb{F}_p^{\times}$ of $\mathbb{F}_{p^2}^{\times}$, so $2^{n+3} \mid (p-1)$).

Comment 2. The numbers a_k are the numerators of the approximation to $\sqrt{3}$ obtained by using the Newton method with $f(x) = x^2 - 3$, $x_0 = 2$. More precisely,

$$x_{k+1} = \frac{x_k + \frac{3}{x_k}}{2}, \qquad x_k = \frac{a_k}{d_k},$$

where

$$d_k = \frac{\left(2+\sqrt{3}\right)^{2^k} - \left(2-\sqrt{3}\right)^{2^k}}{2\sqrt{3}}.$$

Comment 3. Define $f_n(x)$ inductively by

$$f_0(x) = x, \qquad f_{k+1}(x) = f_k(x)^2 - 2 \quad \text{for } k \geq 0.$$

Then the condition $p \mid a_n$ can be read that the $\bmod p$ reduction of the minimal polynomial f_n of the algebraic integer $\alpha = \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$ over $\mathbb{Q}$ has the root $2a_0$ in $\mathbb{F}_p$, where $\zeta_{2^{n+2}}$ is a primitive 2^{n+2}-th root of 1. Thus the conclusion $(p^2-1) \mid 2^{n+3}$ of the problem is a part of the decomposition theorem in the class field theory applied to the abelian extension $\mathbb{Q}(\alpha)$, which asserts that a prime p is completely decomposed in $\mathbb{Q}(\alpha)$ (equivalently, f_n has a root $\bmod p$) if and only if the class of p in $(\mathbb{Z}/2^{n+2}\mathbb{Z})^\times$ belongs to its subgroup $\{1, -1\}$. Thus the problem illustrates a result in the class field theory.

N8. Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A?

Solution. The answer is $(p-1)^2$. For simplicity, let $r = p-1$. Suppose that the prime numbers $p_1, \ldots, p_r$ are distinct. Define

$$B_i = \{p_i, p_i^{p+1}, p_i^{2p+1}, \ldots, p_i^{(r-1)p+1}\},$$

and let $B = \bigcup_{i=1}^{r} B_i$. Then B has r^2 elements and clearly satisfies (i) and (ii).

Now suppose that $|A| \geq r^2 + 1$ and that A satisfies (i) and (ii). We will show that a (nonempty) product of elements in A is a perfect p-th power. This will complete the proof.

Let $p_1, \ldots, p_r$ be distinct prime numbers for which each $t \in A$ can be written as $t = p_1^{a_1} \cdots p_r^{a_r}$. Take $t_1, \ldots, t_{r^2+1} \in A$, and for each i, let $v_i = (a_{i1}, a_{i2}, \ldots, a_{ir})$ denote the vector of exponents of prime divisors of t_i. We would like to show that a (nonempty) sum of v_i is the zero vector modulo p.

We shall show that the following system of congruence equations has a nonzero solution:

$$F_1 = \sum_{i=1}^{r^2+1} a_{i1} x_i^r \equiv 0 \pmod{p},$$
$$F_2 = \sum_{i=1}^{r^2+1} a_{i2} x_i^r \equiv 0 \pmod{p},$$
$$\vdots$$
$$F_r = \sum_{i=1}^{r^2+1} a_{ir} x_i^r \equiv 0 \pmod{p}.$$

If $(x_1, \ldots, x_{r^2+1})$ is a nonzero solution to the above system, then, since $x_i^r \equiv 0$ or $1 \pmod{p}$, a sum of vectors v_i is the zero vector modulo p.

In order to find a nonzero solution to the above system, it is enough to show that the following congruence equation has a nonzero solution:

$$F = F_1^r + F_2^r + \cdots + F_r^r \equiv 0 \pmod{p}. \qquad (*)$$

In fact, because each F_i^r is 0 or 1 modulo p, the nonzero solution to this equation $(*)$ has to satisfy $F_i^r \equiv 0$ for $1 \leq i \leq r$.

We will show that the number of the solutions to the equation $(*)$ is divisible by p. Then since $(0, 0, \ldots, 0)$ is a trivial solution, there exists a nonzero solution to $(*)$ and we are done.

We claim that

$$\sum F^r(x_1, \ldots, x_{r^2+1}) \equiv 0 \pmod{p},$$

where the sum is over the set of all vectors $(x_1, \ldots, x_{r^2+1})$ in the vector space $\mathbb{F}_p^{r^2+1}$ over the finite field $\mathbb{F}_p$. By Fermat's little theorem, this claim evidently implies that the number of solutions to the equation $(*)$ is divisible by p.

We prove the claim. In each monomial in F^r, there are at most r^2 variables, and therefore at least one of the variables is absent. Suppose that the monomial is of the form $bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, where $1 \le k \le r^2$. Then $\sum bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, where the sum is over the same set as above, is equal to $p^{r^2+1-k}\sum_{x_{i_1},\ldots,x_{i_k}} bx_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$, which is divisible by p. This proves the claim.

Comment. In general, if we replace $p-1$ in (i) with any positive integer d, the answer is $(p-1)d$. In fact, if $k > (p-1)d$, then the constant term of the element $(1-g_1)\cdots(1-g_k)$ of the group algebra $\mathbb{Q}_p(\zeta_p)\left[(\mathbb{Z}/p\mathbb{Z})^d\right]$ can be evaluated p-adically so we see that it is not equal to 1. Here $g_1, \ldots, g_k \in (\mathbb{Z}/p\mathbb{Z})^d$, $\mathbb{Q}_p$ is the p-adic number field, and ζ_p is a primitive p-th root of 1. This also gives an alternative solution to the problem.