47$^{\text{th}}$ International Mathematical Olympiad

Slovenia 2006

Shortlisted Problems with Solutions

Contents

Contributing Countries & Problem Selection Committee **5**

Algebra **7**
Problem A1 . . . 7
Problem A2 . . . 9
Problem A3 . . . 10
Problem A4 . . . 13
Problem A5 . . . 15
Problem A6 . . . 17

Combinatorics **19**
Problem C1 . . . 19
Problem C2 . . . 21
Problem C3 . . . 23
Problem C4 . . . 25
Problem C5 . . . 27
Problem C6 . . . 29
Problem C7 . . . 31

Geometry **35**
Problem G1 . . . 35
Problem G2 . . . 36
Problem G3 . . . 38
Problem G4 . . . 39
Problem G5 . . . 40
Problem G6 . . . 42
Problem G7 . . . 45
Problem G8 . . . 46
Problem G9 . . . 48
Problem G10 . . . 51

Number Theory **55**
Problem N1 . . . 55
Problem N2 . . . 56
Problem N3 . . . 57
Problem N4 . . . 58
Problem N5 . . . 59
Problem N6 . . . 60
Problem N7 . . . 63

Contributing Countries

Argentina, Australia, Brazil, Bulgaria, Canada, Colombia, Czech Republic, Estonia, Finland, France, Georgia, Greece, Hong Kong, India, Indonesia, Iran, Ireland, Italy, Japan, Republic of Korea, Luxembourg, Netherlands, Poland, Peru, Romania, Russia, Serbia and Montenegro, Singapore, Slovakia, South Africa, Sweden, Taiwan, Ukraine, United Kingdom, United States of America, Venezuela

Problem Selection Committee

Andrej Bauer
Robert Geretschläger
Géza Kós
Marcin Kuczma
Svetoslav Savchev

Algebra

A1. A sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

Solution. First note that if $a_0 \geq 0$, then all $a_i \geq 0$. For $a_i \geq 1$ we have (in view of $\langle a_i \rangle < 1$ and $\lfloor a_i \rfloor > 0$)

$$\lfloor a_{i+1} \rfloor \leq a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle < \lfloor a_i \rfloor;$$

the sequence $\lfloor a_i \rfloor$ is strictly decreasing as long as its terms are in $[1, \infty)$. Eventually there appears a number from the interval $[0, 1)$ and all subsequent terms are 0.

Now pass to the more interesting situation where $a_0 < 0$; then all $a_i \leq 0$. Suppose the sequence never hits 0. Then we have $\lfloor a_i \rfloor \leq -1$ for all i, and so

$$1 + \lfloor a_{i+1} \rfloor > a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle > \lfloor a_i \rfloor;$$

this means that the sequence $\lfloor a_i \rfloor$ is nondecreasing. And since all its terms are integers from $(-\infty, -1]$, this sequence must be constant from some term on:

$$\lfloor a_i \rfloor = c \qquad \text{for} \quad i \geq i_0\,; \qquad c \ \text{ a negative integer.}$$

The defining formula becomes

$$a_{i+1} = c \cdot \langle a_i \rangle = c(a_i - c) = ca_i - c^2.$$

Consider the sequence

$$b_i = a_i - \frac{c^2}{c-1}\,. \tag{1}$$

It satisfies the recursion rule

$$b_{i+1} = a_{i+1} - \frac{c^2}{c-1} = ca_i - c^2 - \frac{c^2}{c-1} = cb_i,$$

implying

$$b_i = c^{i-i_0} b_{i_0} \qquad \text{for} \quad i \geq i_0. \tag{2}$$

Since all the numbers a_i (for $i \geq i_0$) lie in $[c, c+1)$, the sequence (b_i) is bounded. The equation (2) can be satisfied only if either $b_{i_0} = 0$ or $|c| = 1$, i.e., $c = -1$.

In the first case, $b_i = 0$ for all $i \geq i_0$, so that

$$a_i = \frac{c^2}{c-1} \qquad \text{for} \quad i \geq i_0.$$

In the second case, $c = -1$, equations (1) and (2) say that

$$a_i = -\frac{1}{2} + (-1)^{i-i_0} b_{i_0} = \begin{cases} a_{i_0} & \text{for } \ i = i_0, i_0+2, i_0+4, \dots, \\ 1 - a_{i_0} & \text{for } \ i = i_0+1, i_0+3, i_0+5, \dots. \end{cases}$$

Summarising, we see that (from some point on) the sequence (a_i) either is constant or takes alternately two values from the interval $(-1, 0)$. The result follows.

Comment. There is nothing mysterious in introducing the sequence (b_i). The sequence (a_i) arises by iterating the function $x \mapsto cx - c^2$ whose unique fixed point is $c^2/(c-1)$.

A2. The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for $n \geq 1$.

Solution. The proof goes by induction. For $n = 1$ the formula yields $a_1 = 1/2$. Take $n \geq 1$, assume $a_1, \ldots, a_n > 0$ and write the recurrence formula for n and $n+1$, respectively as

$$\sum_{k=0}^{n} \frac{a_k}{n-k+1} = 0 \qquad \text{and} \qquad \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} = 0.$$

Subtraction yields

$$\begin{aligned} 0 = (n+2) \sum_{k=0}^{n+1} \frac{a_k}{n-k+2} - (n+1) \sum_{k=0}^{n} \frac{a_k}{n-k+1} \\ = (n+2) a_{n+1} + \sum_{k=0}^{n} \left(\frac{n+2}{n-k+2} - \frac{n+1}{n-k+1} \right) a_k. \end{aligned}$$

The coefficient of a_0 vanishes, so

$$a_{n+1} = \frac{1}{n+2} \sum_{k=1}^{n} \left(\frac{n+1}{n-k+1} - \frac{n+2}{n-k+2} \right) a_k = \frac{1}{n+2} \sum_{k=1}^{n} \frac{k}{(n-k+1)(n-k+2)} a_k.$$

The coefficients of $a_1, , \ldots, a_n$ are all positive. Therefore, $a_1, \ldots, a_n > 0$ implies $a_{n+1} > 0$.

Comment. Students familiar with the technique of generating functions will immediately recognise $\sum a_n x^n$ as the power series expansion of $x/\ln(1-x)$ (with value -1 at 0). But this can be a trap; attempts along these lines lead to unpleasant differential equations and integrals hard to handle. Using only tools from real analysis (e.g. computing the coefficients from the derivatives) seems very difficult.

On the other hand, the coefficients can be approached applying complex contour integrals and some other techniques from complex analysis and an attractive formula can be obtained for the coefficients:

$$a_n = \int_1^{\infty} \frac{\mathrm{d}x}{x^n \big(\pi^2 + \log^2(x-1) \big)} \qquad (n \geq 1)$$

which is evidently positive.

A3. The sequence $c_0, c_1, \ldots, c_n, \ldots$ is defined by $c_0 = 1$, $c_1 = 0$ and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j \in J} c_j$, $y = \sum_{j \in J} c_{j-1}$. Prove that there exist real numbers α, β and m, M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

N. B. A sum over the elements of the empty set is assumed to be 0.

Solution. Let $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$ be the roots of the quadratic equation $t^2 - t - 1 = 0$. So $\varphi\psi = -1$, $\varphi + \psi = 1$ and $1 + \psi = \psi^2$. An easy induction shows that the general term c_n of the given sequence satisfies

$$c_n = \frac{\varphi^{n-1} - \psi^{n-1}}{\varphi - \psi} \qquad \text{for } n \geq 0.$$

Suppose that the numbers α and β have the stated property, for appropriately chosen m and M. Since $(c_n, c_{n-1}) \in S$ for each n, the expression

$$\alpha c_n + \beta c_{n-1} = \frac{\alpha}{\sqrt{5}}\left(\varphi^{n-1} - \psi^{n-1}\right) + \frac{\beta}{\sqrt{5}}\left(\varphi^{n-2} - \psi^{n-2}\right) = \frac{1}{\sqrt{5}}\left[(\alpha\varphi + \beta)\varphi^{n-2} - (\alpha\psi + \beta)\psi^{n-2}\right]$$

is bounded as n grows to infinity. Because $\varphi > 1$ and $-1 < \psi < 0$, this implies $\alpha\varphi + \beta = 0$.

To satisfy $\alpha\varphi + \beta = 0$, one can set for instance $\alpha = \psi$, $\beta = 1$. We now find the required m and M for this choice of α and β.

Note first that the above displayed equation gives $c_n\psi + c_{n-1} = \psi^{n-1}$, $n \geq 1$. In the sequel, we denote the pairs in S by (a_J, b_J), where J is a finite subset of the set $\mathbb{N}$ of positive integers and $a_J = \sum_{j \in J} c_j$, $b_J = \sum_{j \in J} c_{j-1}$. Since $\psi a_J + b_J = \sum_{j \in J}(c_j\psi + c_{j-1})$, we obtain

$$\psi a_J + b_J = \sum_{j \in J} \psi^{j-1} \qquad \text{for each } (a_J, b_J) \in S. \tag{1}$$

On the other hand, in view of $-1 < \psi < 0$,

$$-1 = \frac{\psi}{1 - \psi^2} = \sum_{j=0}^{\infty} \psi^{2j+1} < \sum_{j \in J} \psi^{j-1} < \sum_{j=0}^{\infty} \psi^{2j} = \frac{1}{1 - \psi^2} = 1 - \psi = \varphi.$$

Therefore, according to (1),

$$-1 < \psi a_J + b_J < \varphi \qquad \text{for each } (a_J, b_J) \in S.$$

Thus $m = -1$ and $M = \varphi$ is an appropriate choice.

Conversely, we prove that if an ordered pair of nonnegative integers (x, y) satisfies the inequality $-1 < \psi x + y < \varphi$ then $(x, y) \in S$.

Lemma. Let x, y be nonnegative integers such that $-1 < \psi x + y < \varphi$. Then there exists a subset J of $\mathbb{N}$ such that

$$\psi x + y = \sum_{j \in J} \psi^{j-1} \tag{2}$$

Proof. For $x = y = 0$ it suffices to choose the empty subset of $\mathbb{N}$ as J, so let at least one of x, y be nonzero. There exist representations of $\psi x + y$ of the form

$$\psi x + y = \psi^{i_1} + \cdots + \psi^{i_k}$$

where $i_1 \leq \cdots \leq i_k$ is a sequence of nonnegative integers, not necessarily distinct. For instance, we can take x summands $\psi^1 = \psi$ and y summands $\psi^0 = 1$. Consider all such representations of minimum length k and focus on the ones for which i_1 has the minimum possible value j_1. Among them, consider the representations where i_2 has the minimum possible value j_2. Upon choosing $j_3, \ldots, j_k$ analogously, we obtain a sequence $j_1 \leq \cdots \leq j_k$ which clearly satisfies $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$. To prove the lemma, it suffices to show that $j_1, \ldots, j_k$ are pairwise distinct.

Suppose on the contrary that $j_r = j_{r+1}$ for some $r = 1, \ldots, k-1$. Let us consider the case $j_r \geq 2$ first. Observing that $2\psi^2 = 1 + \psi^3$, we replace j_r and j_{r+1} by $j_r - 2$ and $j_r + 1$, respectively. Since

$$\psi^{j_r} + \psi^{j_{r+1}} = 2\psi^{j_r} = \psi^{j_r-2}(1 + \psi^3) = \psi^{j_r-2} + \psi^{j_r+1},$$

the new sequence also represents $\psi x + y$ as needed, and the value of i_r in it contradicts the minimum choice of j_r.

Let $j_r = j_{r+1} = 0$. Then the sum $\psi x + y = \sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^0 = 1$. On the other hand $j_s \neq 1$ for all s, because the equality $1 + \psi = \psi^2$ implies that a representation of minimum length cannot contain consecutive i_r's. It follows that

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} > 2 + \psi^3 + \psi^5 + \psi^7 + \cdots = 2 - \psi^2 = \varphi,$$

contradicting the condition of the lemma.

Let $j_r = j_{r+1} = 1$; then $\sum_{r=1}^{k} \psi^{j_r}$ contains at least two summands equal to $\psi^1 = \psi$. Like in the case $j_r = j_{r+1} = 0$, we also infer that $j_s \neq 0$ and $j_s \neq 2$ for all s. Therefore

$$\psi x + y = \sum_{r=1}^{k} \psi^{j_r} < 2\psi + \psi^4 + \psi^6 + \psi^8 + \cdots = 2\psi - \psi^3 = -1,$$

which is a contradiction again. The conclusion follows. $\square$

Now let the ordered pair (x, y) satisfy $-1 < \psi x + y < \varphi$; hence the lemma applies to (x, y). Let $J \subset \mathbb{N}$ be such that (2) holds. Comparing (1) and (2), we conclude that $\psi x + y = \psi a_J + b_J$. Now, x, y, a_J and b_J are integers, and ψ is irrational. So the last equality implies $x = a_J$ and $y = b_J$. This shows that the numbers $\alpha = \psi$, $\beta = 1$, $m = -1$, $M = \varphi$ meet the requirements.

Comment. We present another way to prove the lemma, constructing the set J inductively. For $x = y = 0$, choose $J = \emptyset$. We induct on $n = 3x + 2y$. Suppose that an appropriate set J exists when $3x + 2y < n$. Now assume $3x + 2y = n > 0$. The current set J should be

$$\text{either} \quad 1 \leq j_1 < j_2 < \cdots < j_k \quad \text{or} \quad j_1 = 0,\ 1 \leq j_2 < \cdots < j_k.$$

These sets fulfil the condition if

$$\frac{\psi x + y}{\psi} = \psi^{i_1-1} + \cdots + \psi^{i_k-1} \quad \text{or} \quad \frac{\psi x + y - 1}{\psi} = \psi^{i_2-1} + \cdots + \psi^{i_k-1},$$

respectively; therefore it suffices to find an appropriate set for $\frac{\psi x+y}{\psi}$ or $\frac{\psi x+y-1}{\psi}$, respectively.

Consider $\frac{\psi x+y}{\psi}$. Knowing that

$$\frac{\psi x + y}{\psi} = x + (\psi - 1)y = \psi y + (x - y),$$

let $x' = y$, $y' = x - y$ and test the induction hypothesis on these numbers. We require $\frac{\psi x+y}{\psi} \in (-1, \varphi)$ which is equivalent to

$$\psi x + y \in (\varphi \cdot \psi, (-1) \cdot \psi) = (-1, -\psi). \tag{3}$$

Relation (3) implies $y' = x - y \geq -\psi x - y > \psi > -1$; therefore $x', y' \geq 0$. Moreover, we have $3x' + 2y' = 2x + y \leq \frac{2}{3}n$; therefore, if (3) holds then the induction applies: the numbers x', y' are represented in the form as needed, hence x, y also.

Now consider $\frac{\psi x+y-1}{\psi}$. Since

$$\frac{\psi x + y - 1}{\psi} = x + (\psi - 1)(y - 1) = \psi(y - 1) + (x - y + 1),$$

we set $x' = y - 1$ and $y' = x - y + 1$. Again we require that $\frac{\psi x+y-1}{\psi} \in (-1, \varphi)$, i.e.

$$\psi x + y \in (\varphi \cdot \psi + 1, (-1) \cdot \psi + 1) = (0, \varphi). \tag{4}$$

If (4) holds then $y - 1 \geq \psi x + y - 1 > -1$ and $x - y + 1 \geq -\psi x - y + 1 > -\varphi + 1 > -1$, therefore $x', y' \geq 0$. Moreover, $3x' + 2y' = 2x + y - 1 < \frac{2}{3}n$ and the induction works.

Finally, $(-1, -\psi) \cup (0, \varphi) = (-1, \varphi)$ so at least one of (3) and (4) holds and the induction step is justified.

A4. Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

for positive real numbers $a_1, a_2, \ldots, a_n$.

Solution 1. Let $S = \sum_i a_i$. Denote by L and R the expressions on the left and right hand side of the proposed inequality. We transform L and R using the identity

$$\sum_{i<j} (a_i + a_j) = (n-1) \sum_i a_i. \tag{1}$$

And thus:

$$L = \sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \sum_{i<j} \frac{1}{4}\Big(a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j}\Big) = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{a_i + a_j}. \tag{2}$$

To represent R we express the sum $\sum\limits_{i<j} a_i a_j$ in two ways; in the second transformation identity (1) will be applied to the squares of the numbers a_i:

$$\sum_{i<j} a_i a_j = \frac{1}{2}\Big(S^2 - \sum_i a_i^2\Big);$$

$$\sum_{i<j} a_i a_j = \frac{1}{2} \sum_{i<j} \Big(a_i^2 + a_j^2 - (a_i - a_j)^2\Big) = \frac{n-1}{2} \cdot \sum_i a_i^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Multiplying the first of these equalities by $n-1$ and adding the second one we obtain

$$n \sum_{i<j} a_i a_j = \frac{n-1}{2} \cdot S^2 - \frac{1}{2} \sum_{i<j} (a_i - a_j)^2.$$

Hence

$$R = \frac{n}{2S} \sum_{i<j} a_i a_j = \frac{n-1}{4} \cdot S - \frac{1}{4} \sum_{i<j} \frac{(a_i - a_j)^2}{S}. \tag{3}$$

Now compare (2) and (3). Since $S \ge a_i + a_j$ for any $i < j$, the claim $L \ge R$ results.

Solution 2. Let $S = a_1 + a_2 + \cdots + a_n$. For any $i \neq j$,

$$4\frac{a_i a_j}{a_i + a_j} = a_i + a_j - \frac{(a_i - a_j)^2}{a_i + a_j} \leq a_i + a_j - \frac{(a_i - a_j)^2}{a_1 + a_2 + \cdots + a_n} = \frac{\sum\limits_{k \neq i} a_i a_k + \sum\limits_{k \neq j} a_j a_k + 2a_i a_j}{S}.$$

The statement is obtained by summing up these inequalities for all pairs i, j:

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} = \frac{1}{2} \sum_i \sum_{j \neq i} \frac{a_i a_j}{a_i + a_j} \leq \frac{1}{8S} \sum_i \sum_{j \neq i} \left(\sum_{k \neq i} a_i a_k + \sum_{k \neq j} a_j a_k + 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} \sum_{j \neq i} a_i a_k + \sum_k \sum_{j \neq k} \sum_{i \neq j} a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{1}{8S} \left(\sum_k \sum_{i \neq k} (n-1) a_i a_k + \sum_k \sum_{j \neq k} (n-1) a_j a_k + \sum_i \sum_{j \neq i} 2a_i a_j \right)$$

$$= \frac{n}{4S} \sum_i \sum_{j \neq i} a_i a_j = \frac{n}{2S} \sum_{i<j} a_i a_j.$$

Comment. Here is an outline of another possible approach. Examine the function $R - L$ subject to constraints $\sum_i a_i = S$, $\sum_{i<j} a_i a_j = U$ for fixed constants $S, U > 0$ (which can jointly occur as values of these symmetric forms). Suppose that among the numbers a_i there are some three, say a_k, a_l, a_m such that $a_k < a_l \leq a_m$. Then it is possible to decrease the value of $R - L$ by perturbing this triple so that in the new triple a'_k, a'_l, a'_m one has $a'_k = a'_l \leq a'_m$, without touching the remaining a_is and without changing the values of S and U; this requires some skill in algebraic manipulations. It follows that the constrained minimum can be only attained for $n - 1$ of the a_is equal and a single one possibly greater. In this case, $R - L \geq 0$ holds almost trivially.

A5. Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Solution 1. Note first that the denominators are all positive, e.g. $\sqrt{a}+\sqrt{b}>\sqrt{a+b}>\sqrt{c}$.

Let $x=\sqrt{b}+\sqrt{c}-\sqrt{a}$, $y=\sqrt{c}+\sqrt{a}-\sqrt{b}$ and $z=\sqrt{a}+\sqrt{b}-\sqrt{c}$. Then

$$b+c-a=\left(\frac{z+x}{2}\right)^2+\left(\frac{x+y}{2}\right)^2-\left(\frac{y+z}{2}\right)^2=\frac{x^2+xy+xz-yz}{2}=x^2-\frac{1}{2}(x-y)(x-z)$$

and

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\sqrt{1-\frac{(x-y)(x-z)}{2x^2}} \leq 1-\frac{(x-y)(x-z)}{4x^2},$$

applying $\sqrt{1+2u} \leq 1+u$ in the last step. Similarly we obtain

$$\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \leq 1-\frac{(z-x)(z-y)}{4z^2} \quad \text{and} \quad \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 1-\frac{(y-z)(y-x)}{4y^2}.$$

Substituting these quantities into the statement, it is sufficient to prove that

$$\frac{(x-y)(x-z)}{x^2}+\frac{(y-z)(y-x)}{y^2}+\frac{(z-x)(z-y)}{z^2} \geq 0. \tag{1}$$

By symmetry we can assume $x \leq y \leq z$. Then

$$\frac{(x-y)(x-z)}{x^2}=\frac{(y-x)(z-x)}{x^2} \geq \frac{(y-x)(z-y)}{y^2}=-\frac{(y-z)(y-x)}{y^2},$$

$$\frac{(z-x)(z-y)}{z^2} \geq 0$$

and (1) follows.

Comment 1. Inequality (1) is a special case of the well-known inequality

$$x^t(x-y)(x-z)+y^t(y-z)(y-x)+z^t(z-x)(z-y) \geq 0$$

which holds for all positive numbers x, y, z and real t; in our case $t=-2$. Case $t>0$ is called Schur's inequality. More generally, if $x \leq y \leq z$ are real numbers and p, q, r are nonnegative numbers such that $q \leq p$ or $q \leq r$ then

$$p(x-y)(x-z)+q(y-z)(y-x)+r(z-x)(z-y) \geq 0.$$

Comment 2. One might also start using Cauchy–Schwarz' inequality (or the root mean square vs. arithmetic mean inequality) to the effect that

$$\left(\sum \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}\right)^2 \leq 3 \cdot \sum \frac{b+c-a}{\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)^2}, \tag{2}$$

in cyclic sum notation. There are several ways to prove that the right-hand side of (2) never exceeds 9 (and this is just what we need). One of them is to introduce new variables x, y, z, as in Solution 1, which upon some manipulation brings the problem again to inequality (1).

Alternatively, the claim that right-hand side of (2) is not greater than 9 can be expressed in terms of the symmetric forms $\sigma_1 = \sum x$, $\sigma_2 = \sum xy$, $\sigma_3 = xyz$ equivalently as

$$4\sigma_1\sigma_2\sigma_3 \le \sigma_2^3 + 9\sigma_3^2, \tag{3}$$

which is a known inequality. A yet different method to deal with the right-hand expression in (2) is to consider $\sqrt{a}, \sqrt{b}, \sqrt{c}$ as sides of a triangle. Through standard trigonometric formulas the problem comes down to showing that

$$p^2 \le 4R^2 + 4Rr + 3r^2, \tag{4}$$

p, R and r standing for the semiperimeter, the circumradius and the inradius of that triangle. Again, (4) is another known inequality. Note that the inequalities (1), (3), (4) are equivalent statements about the same mathematical situation.

Solution 2. Due to the symmetry of variables, it can be assumed that $a \ge b \ge c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \le 1 \quad \text{and} \quad \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \le 2.$$

The first inequality follows from

$$\sqrt{a+b-c} - \sqrt{a} = \frac{(a+b-c)-a}{\sqrt{a+b-c}+\sqrt{a}} \le \frac{b-c}{\sqrt{b}+\sqrt{c}} = \sqrt{b}-\sqrt{c}.$$

For proving the second inequality, let $p = \sqrt{a}+\sqrt{b}$ and $q = \sqrt{a}-\sqrt{b}$. Then $a-b = pq$ and the inequality becomes

$$\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q} \le 2.$$

From $a \ge b \ge c$ we have $p \ge 2\sqrt{c}$. Applying the Cauchy-Schwarz inequality,

$$\left(\frac{\sqrt{c-pq}}{\sqrt{c}-q} + \frac{\sqrt{c+pq}}{\sqrt{c}+q}\right)^2 \le \left(\frac{c-pq}{\sqrt{c}-q} + \frac{c+pq}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q} + \frac{1}{\sqrt{c}+q}\right)$$

$$= \frac{2(c\sqrt{c}-pq^2)}{c-q^2} \cdot \frac{2\sqrt{c}}{c-q^2} = 4 \cdot \frac{c^2-\sqrt{c}pq^2}{(c-q^2)^2} \le 4 \cdot \frac{c^2-2cq^2}{(c-q^2)^2} \le 4.$$

A6. Determine the smallest number M such that the inequality

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b, c.

Solution. We first consider the cubic polynomial

$$P(t) = tb(t^2-b^2)+bc(b^2-c^2)+ct(c^2-t^2).$$

It is easy to check that $P(b) = P(c) = P(-b-c) = 0$, and therefore

$$P(t) = (b-c)(t-b)(t-c)(t+b+c),$$

since the cubic coefficient is $b-c$. The left-hand side of the proposed inequality can therefore be written in the form

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| = |P(a)| = |(b-c)(a-b)(a-c)(a+b+c)|.$$

The problem comes down to finding the smallest number M that satisfies the inequality

$$|(b-c)(a-b)(a-c)(a+b+c)| \le M\cdot(a^2+b^2+c^2)^2. \qquad (1)$$

Note that this expression is symmetric, and we can therefore assume $a \le b \le c$ without loss of generality. With this assumption,

$$|(a-b)(b-c)| = (b-a)(c-b) \le \left(\frac{(b-a)+(c-b)}{2}\right)^2 = \frac{(c-a)^2}{4}, \qquad (2)$$

with equality if and only if $b-a = c-b$, i.e. $2b = a+c$. Also

$$\left(\frac{(c-b)+(b-a)}{2}\right)^2 \le \frac{(c-b)^2+(b-a)^2}{2},$$

or equivalently,

$$3(c-a)^2 \le 2\cdot[(b-a)^2+(c-b)^2+(c-a)^2], \qquad (3)$$

again with equality only for $2b = a+c$. From (2) and (3) we get

$$\begin{aligned}
&|(b-c)(a-b)(a-c)(a+b+c)| \\
\le\ & \frac{1}{4}\cdot|(c-a)^3(a+b+c)| \\
=\ & \frac{1}{4}\cdot\sqrt{(c-a)^6(a+b+c)^2} \\
\le\ & \frac{1}{4}\cdot\sqrt{\left(\frac{2\cdot[(b-a)^2+(c-b)^2+(c-a)^2]}{3}\right)^3\cdot(a+b+c)^2} \\
=\ & \frac{\sqrt{2}}{2}\cdot\left(\sqrt[4]{\left(\frac{(b-a)^2+(c-b)^2+(c-a)^2}{3}\right)^3\cdot(a+b+c)^2}\right)^2.
\end{aligned}$$

By the weighted AM-GM inequality this estimate continues as follows:

$$
\begin{aligned}
& |(b-c)(a-b)(a-c)(a+b+c)| \\
\leq\ & \frac{\sqrt{2}}{2} \cdot \left(\frac{(b-a)^2 + (c-b)^2 + (c-a)^2 + (a+b+c)^2}{4} \right)^2 \\
=\ & \frac{9\sqrt{2}}{32} \cdot (a^2+b^2+c^2)^2.
\end{aligned}
$$

We see that the inequality (1) is satisfied for $M = \frac{9}{32}\sqrt{2}$, with equality if and only if $2b = a+c$ and

$$\frac{(b-a)^2 + (c-b)^2 + (c-a)^2}{3} = (a+b+c)^2.$$

Plugging $b = (a+c)/2$ into the last equation, we bring it to the equivalent form

$$2(c-a)^2 = 9(a+c)^2.$$

The conditions for equality can now be restated as

$$2b = a+c \qquad \text{and} \qquad (c-a)^2 = 18b^2.$$

Setting $b = 1$ yields $a = 1 - \frac{3}{2}\sqrt{2}$ and $c = 1 + \frac{3}{2}\sqrt{2}$. We see that $M = \frac{9}{32}\sqrt{2}$ is indeed the smallest constant satisfying the inequality, with equality for any triple (a, b, c) proportional to $\left(1 - \frac{3}{2}\sqrt{2}, 1, 1 + \frac{3}{2}\sqrt{2}\right)$, up to permutation.

Comment. With the notation $x = b - a$, $y = c - b$, $z = a - c$, $s = a + b + c$ and $r^2 = a^2 + b^2 + c^2$, the inequality (1) becomes just $|sxyz| \leq Mr^4$ (with suitable constraints on s and r). The original asymmetric inequality turns into a standard symmetric one; from this point on the solution can be completed in many ways. One can e.g. use the fact that, for fixed values of $\sum x$ and $\sum x^2$, the product xyz is a maximum/minimum only if some of x, y, z are equal, thus reducing one degree of freedom, etc.

As observed by the proposer, a specific attraction of the problem is that the maximum is attained at a point (a, b, c) with all coordinates distinct.

Combinatorics

C1. We have $n \geq 2$ lamps $L_1, \ldots, L_n$ in a row, each of them being either *on* or *off*. Every second we simultaneously modify the state of each lamp as follows:
— if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
— otherwise, L_i is switched on.
Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.

(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

Solution. (a) Experiments with small n lead to the guess that every n of the form 2^k should be good. This is indeed the case, and more precisely: let A_k be the $2^k \times 2^k$ matrix whose rows represent the evolution of the system, with entries $0, 1$ (for *off* and *on* respectively). The top row shows the initial state $[1, 0, 0, \ldots, 0]$; the bottom row shows the state after $2^k - 1$ steps. The claim is that:

The bottom row of A_k is $[1, 1, 1, \ldots, 1]$.

This will of course suffice because one more move then produces $[0, 0, 0, \ldots, 0]$, as required.

The proof is by induction on k. The base $k = 1$ is obvious. Assume the claim to be true for a $k \geq 1$ and write the matrix A_{k+1} in the block form $\begin{pmatrix} A_k & O_k \\ B_k & C_k \end{pmatrix}$ with four $2^k \times 2^k$ matrices. After m steps, the last 1 in a row is at position $m + 1$. Therefore O_k is the zero matrix. According to the induction hypothesis, the bottom row of $[A_k \; O_k]$ is $[1, \ldots, 1, 0, \ldots, 0]$, with 2^k ones and 2^k zeros. The next row is thus

$$[\underbrace{0, \ldots, 0}_{2^k - 1}, 1, 1, \underbrace{0, \ldots, 0}_{2^k - 1}]$$

It is symmetric about its midpoint, and this symmetry is preserved in all subsequent rows because the procedure described in the problem statement is left/right symmetric. Thus B_k is the mirror image of C_k. In particular, the rightmost column of B_k is identical with the leftmost column of C_k.

Imagine the matrix C_k in isolation from the rest of A_{k+1}. Suppose it is subject to evolution as defined in the problem: the first (leftmost) term in a row depends only on the two first terms in the preceding row, according as they are equal or not. Now embed C_k again in A_k. The 'leftmost' terms in the rows of C_k now have neighbours on their left side—but these neighbours are their exact copies. Consequently the actual evolution within C_k is the same, whether or not C_k is considered as a piece of A_{k+1} or in isolation. And since the top row of C_k is $[1, 0, \ldots, 0]$, it follows that C_k is identical with A_k.

The bottom row of A_k is $[1, 1, \dots, 1]$; the same is the bottom row of C_k, hence also of B_k, which mirrors C_k. So the bottom row of A_{k+1} consists of ones only and the induction is complete.

(b) There are many ways to produce an infinite sequence of those n for which the state $[0, 0, \dots, 0]$ will never be achieved. As an example, consider $n = 2^k + 1$ (for $k \geq 1$). The evolution of the system can be represented by a matrix $\mathcal{A}$ of width $2^k + 1$ with infinitely many rows. The top 2^k rows form the matrix A_k discussed above, with one column of zeros attached at its right.

In the next row we then have the vector $[0, 0, \dots, 0, 1, 1]$. But this is just the second row of $\mathcal{A}$ reversed. Subsequent rows will be mirror copies of the foregoing ones, starting from the second one. So the configuration $[1, 1, 0, \dots, 0, 0]$, i.e. the second row of $\mathcal{A}$, will reappear. Further rows will periodically repeat this pattern and there will be no row of zeros.

C2. A diagonal of a regular 2006-gon is called *odd* if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals.

Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

Solution 1. Call an isosceles triangle *odd* if it has two odd sides. Suppose we are given a dissection as in the problem statement. A triangle in the dissection which is odd and isosceles will be called *iso-odd* for brevity.

Lemma. Let AB be one of dissecting diagonals and let $\mathcal{L}$ be the shorter part of the boundary of the 2006-gon with endpoints A, B. Suppose that $\mathcal{L}$ consists of n segments. Then the number of iso-odd triangles with vertices on $\mathcal{L}$ does not exceed $n/2$.

Proof. This is obvious for $n = 2$. Take n with $2 < n \leq 1003$ and assume the claim to be true for every $\mathcal{L}$ of length less than n. Let now $\mathcal{L}$ (endpoints A, B) consist of n segments. Let PQ be the longest diagonal which is a side of an iso-odd triangle PQS with all vertices on $\mathcal{L}$ (if there is no such triangle, there is nothing to prove). Every triangle whose vertices lie on $\mathcal{L}$ is obtuse or right-angled; thus S is the summit of PQS. We may assume that the five points A, P, S, Q, B lie on $\mathcal{L}$ in this order and partition $\mathcal{L}$ into four pieces $\mathcal{L}_{AP}$, $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$, $\mathcal{L}_{QB}$ (the outer ones possibly reducing to a point).

By the definition of PQ, an iso-odd triangle cannot have vertices on both $\mathcal{L}_{AP}$ and $\mathcal{L}_{QB}$. Therefore every iso-odd triangle within $\mathcal{L}$ has all its vertices on just one of the four pieces. Applying to each of these pieces the induction hypothesis and adding the four inequalities we get that the number of iso-odd triangles within $\mathcal{L}$ other than PQS does not exceed $n/2$. And since each of $\mathcal{L}_{PS}$, $\mathcal{L}_{SQ}$ consists of an odd number of sides, the inequalities for these two pieces are actually strict, leaving a $1/2 + 1/2$ in excess. Hence the triangle PSQ is also covered by the estimate $n/2$. This concludes the induction step and proves the lemma. □

The remaining part of the solution in fact repeats the argument from the above proof. Consider the longest dissecting diagonal XY. Let $\mathcal{L}_{XY}$ be the shorter of the two parts of the boundary with endpoints X, Y and let XYZ be the triangle in the dissection with vertex Z not on $\mathcal{L}_{XY}$. Notice that XYZ is acute or right-angled, otherwise one of the segments XZ, YZ would be longer than XY. Denoting by $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ the two pieces defined by Z and applying the lemma to each of $\mathcal{L}_{XY}$, $\mathcal{L}_{XZ}$, $\mathcal{L}_{YZ}$ we infer that there are no more than 2006/2 iso-odd triangles in all, unless XYZ is one of them. But in that case XZ and YZ are odd diagonals and the corresponding inequalities are strict. This shows that also in this case the total number of iso-odd triangles in the dissection, including XYZ, is not greater than 1003.

This bound can be achieved. For this to happen, it just suffices to select a vertex of the 2006-gon and draw a broken line joining every second vertex, starting from the selected one. Since 2006 is even, the line closes. This already gives us the required 1003 iso-odd triangles. Then we can complete the triangulation in an arbitrary fashion.

Solution 2. Let the terms *odd triangle* and *iso-odd triangle* have the same meaning as in the first solution.

Let ABC be an iso-odd triangle, with AB and BC odd sides. This means that there are an odd number of sides of the 2006-gon between A and B and also between B and C. We say that these sides *belong* to the iso-odd triangle ABC.

At least one side in each of these groups does not belong to any other iso-odd triangle. This is so because any odd triangle whose vertices are among the points between A and B has two sides of equal length and therefore has an even number of sides belonging to it in total. Eliminating all sides belonging to any other iso-odd triangle in this area must therefore leave one side that belongs to no other iso-odd triangle. Let us *assign* these two sides (one in each group) to the triangle ABC.

To each iso-odd triangle we have thus assigned a pair of sides, with no two triangles sharing an assigned side. It follows that at most 1003 iso-odd triangles can appear in the dissection.

This value can be attained, as shows the example from the first solution.

C3. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

NB. A line segment, a point and the empty set are considered as convex polygons of 2, 1 and 0 vertices, respectively.

Solution 1. For each convex polygon P whose vertices are in S, let $c(P)$ be the number of points of S which are inside P, so that $a(P)+b(P)+c(P)=n$, the total number of points in S. Denoting $1-x$ by y,

$$\sum_P x^{a(P)}y^{b(P)} = \sum_P x^{a(P)}y^{b(P)}(x+y)^{c(P)} = \sum_P \sum_{i=0}^{c(P)} \binom{c(P)}{i} x^{a(P)+i}y^{b(P)+c(P)-i}\,.$$

View this expression as a homogeneous polynomial of degree n in two independent variables x, y. In the expanded form, it is the sum of terms $x^r y^{n-r}$ $(0 \le r \le n)$ multiplied by some nonnegative integer coefficients.

For a fixed r, the coefficient of $x^r y^{n-r}$ represents the number of ways of choosing a convex polygon P and then choosing some of the points of S inside P so that the number of vertices of P and the number of chosen points inside P jointly add up to r.

This corresponds to just choosing an r-element subset of S. The correspondence is bijective because every set T of points from S splits in exactly one way into the union of two disjoint subsets, of which the first is the set of vertices of a convex polygon — namely, the convex hull of T — and the second consists of some points inside that polygon.

So the coefficient of $x^r y^{n-r}$ equals $\binom{n}{r}$. The desired result follows:

$$\sum_P x^{a(P)}y^{b(P)} = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = (x+y)^n = 1.$$

Solution 2. Apply induction on the number n of points. The case $n=0$ is trivial. Let $n>0$ and assume the statement for less than n points. Take a set S of n points.

Let C be the set of vertices of the convex hull of S, let $m=|C|$.

Let $X\subset C$ be an arbitrary nonempty set. For any convex polygon P with vertices in the set $S\setminus X$, we have $b(P)$ points of S outside P. Excluding the points of X — all outside P — the set $S\setminus X$ contains exactly $b(P)-|X|$ of them. Writing $1-x=y$, by the induction hypothesis

$$\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)-|X|}=1$$

(where $P\subset S\setminus X$ means that the vertices of P belong to the set $S\setminus X$). Therefore

$$\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)}=y^{|X|}.$$

All convex polygons appear at least once, except the convex hull C itself. The convex hull adds x^m. We can use the inclusion-exclusion principle to compute the sum of the other terms:

$$\sum_{P\neq C} x^{a(P)}y^{b(P)}=\sum_{k=1}^{m}(-1)^{k-1}\sum_{|X|=k}\sum_{P\subset S\setminus X} x^{a(P)}y^{b(P)}=\sum_{k=1}^{m}(-1)^{k-1}\sum_{|X|=k} y^k$$

$$=\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}y^k=-\big((1-y)^m-1\big)=1-x^m$$

and then

$$\sum_{P} x^{a(P)}y^{b(P)}=\sum_{P=C}+\sum_{P\neq C}=x^m+(1-x^m)=1.$$

C4. A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:

A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

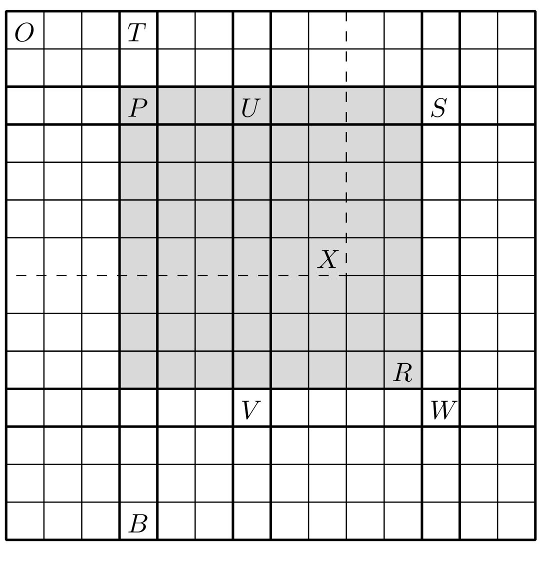
Solution. We use capital letters to denote unit squares; O is the top left corner square. For any two squares X and Y let $[XY]$ be the smallest grid rectangle containing these two squares. Strawberries lie on some squares in arrangement $\mathcal{A}$. Put a plum on each square of the target configuration $\mathcal{B}$. For a square X denote by $a(X)$ and $b(X)$ respectively the number of strawberries and the number of plums in $[OX]$. By hypothesis $a(X) \le b(X)$ for each X, with strict inequality for some X (otherwise the two arrangements coincide and there is nothing to prove).

The idea is to show that by a legitimate switch one can obtain an arrangement $\mathcal{A}'$ such that

$$a(X) \le a'(X) \le b(X) \quad \text{for each } X; \qquad \sum_X a(X) < \sum_X a'(X) \tag{1}$$

(with $a'(X)$ defined analogously to $a(X)$; the sums range over all unit squares X). This will be enough because the same reasoning then applies to $\mathcal{A}'$, giving rise to a new arrangement $\mathcal{A}''$, and so on (induction). Since $\sum a(X) < \sum a'(X) < \sum a''(X) < \dots$ and all these sums do not exceed $\sum b(X)$, we eventually obtain a sum with all summands equal to the respective $b(X)$s; all strawberries will meet with plums.

Consider the uppermost row in which the plum and the strawberry lie on different squares P and S (respectively); clearly P must be situated left to S. In the column passing through P, let T be the top square and B the bottom square. The strawberry in that column lies below the plum (because there is no plum in that column above P, and the positions of strawberries and plums coincide everywhere above the row of P). Hence there is at least one strawberry in the region $[BS]$ below $[PS]$. Let V be the position of the uppermost strawberry in that region.



Denote by W the square at the intersection of the row through V with the column through S and let R be the square vertex-adjacent to W up-left. We claim that

$$a(X) < b(X) \qquad \text{for all} \quad X \in [PR]. \tag{2}$$

This is so because if $X \in [PR]$ then the portion of $[OX]$ left to column $[TB]$ contains at least as many plums as strawberries (the hypothesis of the problem); in the portion above the row through P and S we have perfect balance; and in the remaining portion, i.e. rectangle $[PX]$ we have a plum on square P and no strawberry at all.

Now we are able to perform the required switch. Let U be the square at the intersection of the row through P with the column through V (some of P, U, R can coincide). We move strawberries from squares S and V to squares U and W. Then

$$a'(X) = a(X) + 1 \quad \text{for} \quad X \in [UR]; \qquad a'(X) = a(X) \quad \text{for other } X.$$

And since the rectangle $[UR]$ is contained in $[PR]$, we still have $a'(X) \le b(X)$ for all S, in view of (2); conditions (1) are satisfied and the proof is complete.

C5. An (n, k)-tournament is a contest with n players held in k rounds such that:

(i) Each player plays in each round, and every two players meet at most once.

(ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n, k) for which there exists an (n, k)-tournament.

Solution. For each k, denote by t_k the unique integer such that $2^{t_k-1} < k+1 \le 2^{t_k}$. We show that an (n, k)-tournament exists if and only if 2^{t_k} divides n.

First we prove that if $n = 2^t$ for some t then there is an (n, k)-tournament for all $k \le 2^t - 1$. Let S be the set of $0-1$ sequences with length t. We label the 2^t players with the elements of S in an arbitrary fashion (which is possible as there are exactly 2^t sequences in S). Players are identified with their labels in the construction below. If $\alpha, \beta \in S$, let $\alpha + \beta \in S$ be the result of the modulo 2 term-by-term addition of α and β (with rules $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 0$; there is no carryover). For each $i = 1, \ldots, 2^t - 1$ let $\omega(i) \in S$ be the sequence of base 2 digits of i, completed with leading zeros if necessary to achieve length t.

Now define a tournament with $n = 2^t$ players in $k \le 2^t - 1$ rounds as follows: For all $i = 1, \ldots, k$, let player α meet player $\alpha + \omega(i)$ in round i. The tournament is well-defined as $\alpha + \omega(i) \in S$ and $\alpha + \omega(i) = \beta + \omega(i)$ implies $\alpha = \beta$; also $[\alpha + \omega(i)] + \omega(i) = \alpha$ for each $\alpha \in S$ (meaning that player $\alpha + \omega(i)$ meets player α in round i, as needed). Each player plays in each round. Next, every two players meet at most once (exactly once if $k = 2^t - 1$), since $\omega(i) \ne \omega(j)$ if $i \ne j$. Thus condition (i) holds true, and condition (ii) is also easy to check.

Let player α meet player β in round i, player γ meet player δ in round i, and player α meet player γ in round j. Then $\beta = \alpha + \omega(i)$, $\delta = \gamma + \omega(i)$ and $\gamma = \alpha + \omega(j)$. By definition, β will play in round j with

$$\beta + \omega(j) = [\alpha + \omega(i)] + \omega(j) = [\alpha + \omega(j)] + \omega(i) = \gamma + \omega(i) = \delta,$$

as required by (ii).

So there exists an (n, k)-tournament for pairs (n, k) such that $n = 2^t$ and $k \le 2^t - 1$. The same conclusion is straightforward for n of the form $n = 2^t s$ and $k \le 2^t - 1$. Indeed, consider s different $(2^t, k)$-tournaments $T_1, \ldots, T_s$, no two of them having players in common. Their union can be regarded as a $(2^t s, k)$-tournament T where each round is the union of the respective rounds in $T_1, \ldots, T_s$.

In summary, the condition that 2^{t_k} divides n is sufficient for an (n, k)-tournament to exist. We prove that it is also necessary.

Consider an arbitrary (n, k)-tournament. Represent each player by a point and after each round, join by an edge every two players who played in this round. Thus to a round $i = 1, \ldots, k$ there corresponds a graph G_i. We say that player Q is an *i-neighbour* of player P if there is a path of edges in G_i from P to Q; in other words, if there are players $P = X_1, X_2, \ldots, X_m = Q$ such that player X_j meets player X_{j+1} in one of the first i rounds, $j = 1, 2 \ldots, m-1$. The set of i-neighbours of a player will be called its *i-component*. Clearly two i-components are either disjoint or coincide.

Hence after each round i the set of players is partitioned into pairwise disjoint i-components. So, to achieve our goal, it suffices to show that all k-components have size divisible by 2^{t_k}.

To this end, let us see how the i-component Γ of a player A changes after round $i+1$. Suppose that A meets player B with i-component Δ in round $i+1$ (components Γ and Δ are not necessarily distinct). We claim that then *in round $i+1$ each player from Γ meets a player from Δ, and vice versa.*

Indeed, let C be any player in Γ, and let C meet D in round $i{+}1$. Since C is an i-neighbour of A, there is a sequence of players $A = X_1, X_2, \ldots, X_m = C$ such that X_j meets X_{j+1} in one of the first i rounds, $j = 1, 2 \ldots, m{-}1$. Let X_j meet Y_j in round $i{+}1$, for $j = 1, 2 \ldots, m$; in particular $Y_1 = B$ and $Y_m = D$. Players Y_j exists in view of condition (i). Suppose that X_j and X_{j+1} met in round r, where $r \leq i$. Then condition (ii) implies that and Y_j and Y_{j+1} met in round r, too. Hence $B = Y_1, Y_2, \ldots, Y_m = D$ is a path in G_i from B to D. This is to say, D is in the i-component Δ of B, as claimed. By symmetry, each player from Δ meets a player from Γ in round $i{+}1$. It follows in particular that Γ and Δ have the same cardinality.

It is straightforward now that the $(i{+}1)$-component of A is $\Gamma \cup \Delta$, the union of two sets with the same size. Since Γ and Δ are either disjoint or coincide, we have either $|\Gamma \cup \Delta| = 2|\Gamma|$ or $|\Gamma \cup \Delta| = |\Gamma|$; as usual, $|\cdots|$ denotes the cardinality of a finite set.

Let $\Gamma_1, \ldots, \Gamma_k$ be the consecutive components of a given player A. We obtained that either $|\Gamma_{i+1}| = 2|\Gamma_i|$ or $|\Gamma_{i+1}| = |\Gamma_i|$ for $i = 1, \ldots, k{-}1$. Because $|\Gamma_1| = 2$, each $|\Gamma_i|$ is a power of 2, $i = 1, \ldots, k{-}1$. In particular $|\Gamma_k| = 2^u$ for some u.

On the other hand, player A has played with k different opponents by (i). All of them belong to Γ_k, therefore $|\Gamma_k| \geq k{+}1$.

Thus $2^u \geq k{+}1$, and since t_k is the least integer satisfying $2^{t_k} \geq k{+}1$, we conclude that $u \geq t_k$. So the size of each k-component is divisible by 2^{t_k}, which completes the argument.

C6. A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

Solution. Let T be a holey triangle. The unit triangles in it will be called *cells.* We say simply "triangle" instead of "upward equilateral triangle" and "size" instead of "side length."

The necessity will be proven first. Assume that a holey triangle T can be tiled with diamonds and consider such a tiling. Let T' be a triangle of size k in T containing h holes. Focus on the diamonds which cover (one or two) cells in T'. Let them form a figure R. The boundary of T' consists of upward cells, so R is a triangle of size k with h upward holes cut out and possibly some downward cells sticking out. Hence there are exactly $(k^2+k)/2 - h$ upward cells in R, and at least $(k^2-k)/2$ downward cells (not counting those sticking out). On the other hand each diamond covers one upward and one downward cell, which implies $(k^2+k)/2 - h \geq (k^2-k)/2$. It follows that $h \leq k$, as needed.

We pass on to the sufficiency. For brevity, let us say that a set of holes in a given triangle T is *spread out* if every triangle of size k in T contains at most k holes. For any set S of spread out holes, a triangle of size k will be called *full of* S if it contains exactly k holes of S. The proof is based on the following observation.

Lemma. Let S be a set of spread out holes in T. Suppose that two triangles T' and T'' are full of S, and that they touch or intersect. Let $T'+T''$ denote the smallest triangle in T containing them. Then $T'+T''$ is also full of S.

Proof. Let triangles T', T'', $T' \cap T''$ and $T'+T''$ have sizes a, b, c and d, and let them contain a, b, x and y holes of S, respectively. (Note that $T' \cap T''$ could be a point, in which case $c = 0$.) Since S is spread out, we have $x \leq c$ and $y \leq d$. The geometric configuration of triangles clearly satisfies $a+b = c+d$. Furthermore, $a+b \leq x+y$, since $a+b$ counts twice the holes in $T' \cap T''$. These conclusions imply $x = c$ and $y = d$, as we wished to show. □

Now let T_n be a holey triangle of size n, and let the set H of its holes be spread out. We show by induction on n that T_n can be tiled with diamonds. The base $n = 1$ is trivial. Suppose that $n \geq 2$ and that the claim holds for holey triangles of size less than n.

Denote by B the bottom row of T_n and by T' the triangle formed by its top $n-1$ rows. There is at least one hole in B as T' contains at most $n-1$ holes. If this hole is only one, there is a unique way to tile B with diamonds. Also, T' contains exactly $n-1$ holes, making it a holey triangle of size $n-1$, and these holes are spread out. Hence it remains to apply the induction hypothesis.

So suppose that there are $m \geq 2$ holes in B and label them $a_1, \ldots, a_m$ from left to right. Let ℓ be the line separating B from T'. For each $i = 1, \ldots, m-1$, pick an upward cell b_i between a_i and a_{i+1}, with base on ℓ. Place a diamond to cover b_i and its lower neighbour, a downward cell in B. The remaining part of B can be tiled uniquely with diamonds. Remove from T_n row B and the cells $b_1, \ldots, b_{m-1}$ to obtain a holey triangle T_{n-1} of size $n-1$. The conclusion will follow by induction if the choice of $b_1, \ldots, b_{m-1}$ guarantees that the following condition is satisfied: If the holes $a_1, \ldots, a_{m-1}$ are replaced by $b_1, \ldots, b_{m-1}$ then the new set of holes is spread out again.

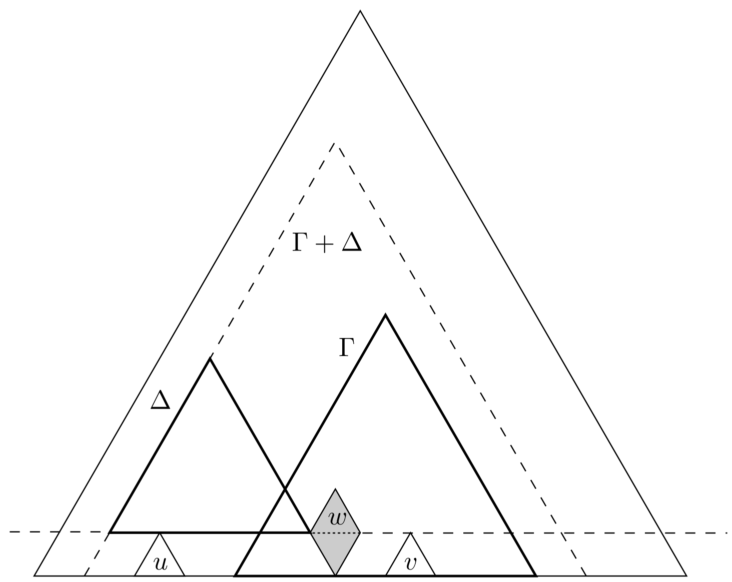
We show that such a choice is possible. The cells $b_1, \ldots, b_{m-1}$ can be defined one at a time in this order, making sure that the above condition holds at each step. Thus it suffices to prove that there is an appropriate choice for b_1, and we set $a_1 = u$, $a_2 = v$ for clarity.

Let Δ be the triangle of maximum size which is full of H, contains the top vertex of the hole u, and has base on line ℓ. Call Δ the *associate of* u. Observe that Δ does not touch v.

Indeed, if Δ has size r then it contains r holes of T_n. Extending its slanted sides downwards produces a triangle Δ' of size $r+1$ containing at least one more hole, namely u. Since there are at most $r+1$ holes in Δ', it cannot contain v. Consequently, Δ does not contain the top vertex of v.

Let w be the upward cell with base on ℓ which is to the right of Δ and shares a common vertex with it. The observation above shows that w is to the left of v. Note that w is not a hole, or else Δ could be extended to a larger triangle full of H.

We prove that if the hole u is replaced by w then the new set of holes is spread out again. To verify this, we only need to check that if a triangle Γ in T_n contains w but not u then Γ is not full of H. Suppose to the contrary that Γ is full of H. Consider the minimum triangle $\Gamma+\Delta$ containing Γ and the associate Δ of u. Clearly $\Gamma+\Delta$ is larger than Δ, because Γ contains w but Δ does not. Next, $\Gamma+\Delta$ is full of $H \setminus \{u\}$ by the lemma, since Γ and Δ have a common point and neither of them contains u.



If Γ is above line ℓ then so is $\Gamma+\Delta$, which contradicts the maximum choice of Δ. If Γ contains cells from row B, observe that $\Gamma+\Delta$ contains u. Let s be the size of $\Gamma+\Delta$. Being full of $H \setminus \{u\}$, $\Gamma+\Delta$ contains s holes other than u. But it also contains u, contradicting the assumption that H is spread out.

The claim follows, showing that $b_1 = w$ is an appropriate choice for $a_1 = u$ and $a_2 = v$. As explained above, this is enough to complete the induction.

C7. Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it.

Call a pair of points of the polyhedron *antipodal* if there exist two parallel planes passing through these points and such that the polyhedron is contained between these planes.

Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoints of edges. Determine the difference $A-B$ in terms of the numbers of vertices, edges and faces.

Solution 1. Denote the polyhedron by Γ; let its vertices, edges and faces be $V_1, V_2, \ldots, V_n$, $E_1, E_2, \ldots, E_m$ and $F_1, F_2, \ldots, F_\ell$, respectively. Denote by Q_i the midpoint of edge E_i.

Let S be the unit sphere, the set of all unit vectors in three-dimensional space. Map the boundary elements of Γ to some objects on S as follows.

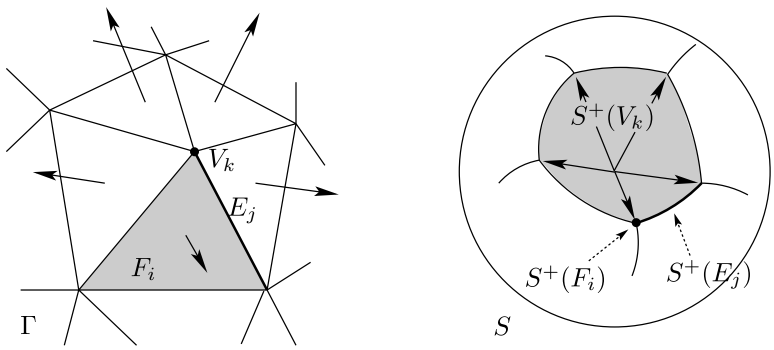
For a face F_i, let $S^+(F_i)$ and $S^-(F_i)$ be the unit normal vectors of face F_i, pointing outwards from Γ and inwards to Γ, respectively. These points are diametrically opposite.

For an edge E_j, with neighbouring faces F_{i_1} and F_{i_2}, take all support planes of Γ (planes which have a common point with Γ but do not intersect it) containing edge E_j, and let $S^+(E_j)$ be the set of their outward normal vectors. The set $S^+(E_j)$ is an arc of a great circle on S. Arc $S^+(E_j)$ is perpendicular to edge E_j and it connects points $S^+(F_{i_1})$ and $S^+(F_{i_2})$.

Define also the set of inward normal vectors $S^-(E_i)$ which is the reflection of $S^+(E_i)$ across the origin.

For a vertex V_k, which is the common endpoint of edges $E_{j_1}, \ldots, E_{j_h}$ and shared by faces $F_{i_1}, \ldots, F_{i_h}$, take all support planes of Γ through point V_k and let $S^+(V_k)$ be the set of their outward normal vectors. This is a region on S, a spherical polygon with vertices $S^+(F_{i_1}), \ldots, S^+(F_{i_h})$ bounded by arcs $S^+(E_{j_1}), \ldots, S^+(E_{j_h})$. Let $S^-(V_k)$ be the reflection of $S^+(V_k)$, the set of inward normal vectors.

Note that region $S^+(V_k)$ is convex in the sense that it is the intersection of several half spheres.



Now translate the conditions on Γ to the language of these objects.

(a) Polyhedron Γ has no parallel edges — the great circles of arcs $S^+(E_i)$ and $S^-(E_j)$ are different for all $i \neq j$.

(b) If an edge E_i does not belong to a face F_j then they are not parallel — the great circle which contains arcs $S^+(E_i)$ and $S^-(E_i)$ does not pass through points $S^+(F_j)$ and $S^-(F_j)$.

(c) Polyhedron Γ has no parallel faces — points $S^+(F_i)$ and $S^-(F_j)$ are pairwise distinct.

The regions $S^+(V_k)$, arcs $S^+(E_j)$ and points $S^+(F_i)$ provide a decomposition of the surface of the sphere. Regions $S^-(V_k)$, arcs $S^-(E_j)$ and points $S^-(F_i)$ provide the reflection of this decomposition. These decompositions are closely related to the problem.

Lemma 1. For any $1 \leq i, j \leq n$, regions $S^-(V_i)$ and $S^+(V_j)$ overlap if and only if vertices V_i and V_j are antipodal.

Lemma 2. For any $1 \le i, j \le m$, arcs $S^-(E_i)$ and $S^+(E_j)$ intersect if and only if the midpoints Q_i and Q_j of edges E_i and E_j are antipodal.

Proof of lemma 1. First note that by properties (a,b,c) above, the two regions cannot share only a single point or an arc. They are either disjoint or they overlap.

Assume that the two regions have a common interior point u. Let P_1 and P_2 be two parallel support planes of Γ through points V_i and V_j, respectively, with normal vector u. By the definition of regions $S^-(V_i)$ and $S^+(V_j)$, u is the inward normal vector of P_1 and the outward normal vector of P_2. Therefore polyhedron Γ lies between the two planes; vertices V_i and V_j are antipodal.

To prove the opposite direction, assume that V_i and V_j are antipodal. Then there exist two parallel support planes P_1 and P_2 through V_i and V_j, respectively, such that Γ is between them. Let u be the inward normal vector of P_1; then u is the outward normal vector of P_2, therefore $u \in S^-(V_i) \cap S^+(V_j)$. The two regions have a common point, so they overlap. □

Proof of lemma 2. Again, by properties (a,b) above, the endpoints of arc $S^-(E_i)$ cannot belong to $S^+(E_j)$ and vice versa. The two arcs are either disjoint or intersecting.

Assume that arcs $S^-(E_i)$ and $S^+(E_j)$ intersect at point u. Let P_1 and P_2 be the two support planes through edges E_i and E_j, respectively, with normal vector u. By the definition of arcs $S^-(E_i)$ and $S^+(E_j)$, vector u points inwards from P_1 and outwards from P_2. Therefore Γ is between the planes. Since planes P_1 and P_2 pass through Q_i and Q_j, these points are antipodal.

For the opposite direction, assume that points Q_i and Q_j are antipodal. Let P_1 and P_2 be two support planes through these points, respectively. An edge cannot intersect a support plane, therefore E_i and E_j lie in the planes P_1 and P_2, respectively. Let u be the inward normal vector of P_1, which is also the outward normal vector of P_2. Then $u \in S^-(E_i) \cap S^+(E_j)$. So the two arcs are not disjoint; they therefore intersect. □

Now create a new decomposition of sphere S. Draw all arcs $S^+(E_i)$ and $S^-(E_j)$ on sphere S and put a knot at each point where two arcs meet. We have ℓ knots at points $S^+(F_i)$ and another ℓ knots at points $S^-(F_i)$, corresponding to the faces of Γ; by property (c) they are different. We also have some pairs $1 \le i, j \le m$ where arcs $S^-(E_i)$ and $S^+(E_j)$ intersect. By Lemma 2, each antipodal pair (Q_i, Q_j) gives rise to two such intersections; hence, the number of all intersections is $2B$ and we have $2\ell + 2B$ knots in all.

Each intersection knot splits two arcs, increasing the number of arcs by 2. Since we started with $2m$ arcs, corresponding the edges of Γ, the number of the resulting curve segments is $2m + 4B$.

The network of these curve segments divides the sphere into some "new" regions. Each new region is the intersection of some overlapping sets $S^-(V_i)$ and $S^+(V_j)$. Due to the convexity, the intersection of two overlapping regions is convex and thus contiguous. By Lemma 1, each pair of overlapping regions corresponds to an antipodal vertex pair and each antipodal vertex pair gives rise to two different overlaps, which are symmetric with respect to the origin. So the number of new regions is $2A$.

The result now follows from Euler's polyhedron theorem. We have $n + l = m + 2$ and

$$(2\ell + 2B) + 2A = (2m + 4B) + 2,$$

therefore

$$A - B = m - \ell + 1 = n - 1.$$

Therefore $A - B$ is by one less than the number of vertices of Γ.

Solution 2. Use the same notations for the polyhedron and its vertices, edges and faces as in Solution 1. We regard points as vectors starting from the origin. Polyhedron Γ is regarded as a closed convex set, including its interior. In some cases the edges and faces of Γ are also regarded as sets of points. The symbol ∂ denotes the boundary of the certain set; e.g. $\partial\Gamma$ is the surface of Γ.

Let $\Delta = \Gamma - \Gamma = \{U - V : U, V \in \Gamma\}$ be the set of vectors between arbitrary points of Γ. Then Δ, being the sum of two bounded convex sets, is also a bounded convex set and, by construction, it is also centrally symmetric with respect to the origin. We will prove that Δ is also a polyhedron and express the numbers of its faces, edges and vertices in terms n, m, ℓ, A and B.

Lemma 1. For points $U, V \in \Gamma$, point $W = U - V$ is a boundary point of Δ if and only if U and V are antipodal. Moreover, for each boundary point $W \in \partial\Delta$ there exists exactly one pair of points $U, V \in \Gamma$ such that $W = U - V$.

Proof. Assume first that U and V are antipodal points of Γ. Let parallel support planes P_1 and P_2 pass through them such that Γ is in between. Consider plane $P = P_1 - U = P_2 - V$. This plane separates the interiors of $\Gamma - U$ and $\Gamma - V$. After reflecting one of the sets, e.g. $\Gamma - V$, the sets $\Gamma - U$ and $-\Gamma + V$ lie in the same half space bounded by P. Then $(\Gamma - U) + (-\Gamma + V) = \Delta - W$ lies in that half space, so $0 \in P$ is a boundary point of the set $\Delta - W$. Translating by W we obtain that W is a boundary point of Δ.

To prove the opposite direction, let $W = U - V$ be a boundary point of Δ, and let $\Psi = (\Gamma - U) \cap (\Gamma - V)$. We claim that $\Psi = \{0\}$. Clearly Ψ is a bounded convex set and $0 \in \Psi$. For any two points $X, Y \in \Psi$, we have $U+X, V+Y \in \Gamma$ and $W+(X-Y) = (U+X)-(V+Y) \in \Delta$. Since W is a boundary point of Δ, the vector $X-Y$ cannot have the same direction as W. This implies that the interior of Ψ is empty. Now suppose that Ψ contains a line segment S. Then $S+U$ and $S+V$ are subsets of some faces or edges of Γ and these faces/edges are parallel to S. In all cases, we find two faces, two edges, or a face and an edge which are parallel, contradicting the conditions of the problem. Therefore, $\Psi = \{0\}$ indeed.

Since $\Psi = (\Gamma - U) \cap (\Gamma - V)$ consists of a single point, the interiors of bodies $\Gamma - U$ and $\Gamma - V$ are disjoint and there exists a plane P which separates them. Let u be the normal vector of P pointing into that half space bounded by P which contains $\Gamma - U$. Consider the planes $P + U$ and $P + V$; they are support planes of Γ, passing through U and V, respectively. From plane $P + U$, the vector u points into that half space which contains Γ. From plane $P + V$, vector u points into the opposite half space containing Γ. Therefore, we found two proper support through points U and V such that Γ is in between.

For the uniqueness part, assume that there exist points $U_1, V_1 \in \Gamma$ such that $U_1 - V_1 = U - V$. The points $U_1 - U$ and $V_1 - V$ lie in the sets $\Gamma - U$ and $\Gamma - V$ separated by P. Since $U_1 - U = V_1 - V$, this can happen only if both are in P; but the only such point is 0. Therefore, $U_1 - V_1 = U - V$ implies $U_1 = U$ and $V_1 = V$. The lemma is complete. □

Lemma 2. Let U and V be two antipodal points and assume that plane P, passing through 0, separates the interiors of $\Gamma - U$ and $\Gamma - V$. Let $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$. Then $\Delta \cap (P + U - V) = \Psi_1 - \Psi_2 + U - V$.

Proof. The sets $\Gamma - U$ and $-\Gamma + V$ lie in the same closed half space bounded by P. Therefore, for any points $X \in (\Gamma - U)$ and $Y \in (-\Gamma + V)$, we have $X + Y \in P$ if and only if $X, Y \in P$. Then

$$(\Delta - (U - V)) \cap P = \big((\Gamma - U) + (-\Gamma + V)\big) \cap P = \big((\Gamma - U) \cap P\big) + \big((-\Gamma + V) \cap P\big) = \Psi_1 - \Psi_2.$$

Now a translation by $(U - V)$ completes the lemma. □

Now classify the boundary points $W = U - V$ of Δ, according to the types of points U and V. In all cases we choose a plane P through 0 which separates the interiors of $\Gamma - U$ and $\Gamma - V$. We will use the notation $\Psi_1 = (\Gamma - U) \cap P$ and $\Psi_2 = (\Gamma - V) \cap P$ as well.

Case 1: Both U and V are vertices of Γ. Bodies $\Gamma - U$ and $\Gamma - V$ have a common vertex which is 0. Choose plane P in such a way that $\Psi_1 = \Psi_2 = \{0\}$. Then Lemma 2 yields $\Delta \cap (P + W) = \{W\}$. Therefore $P + W$ is a support plane of Δ such that they have only one common point so no line segment exists on $\partial\Delta$ which would contain W in its interior.

Since this case occurs for antipodal vertex pairs and each pair is counted twice, the number of such boundary points on Δ is $2A$.

Case 2: Point U is an interior point of an edge E_i and V is a vertex of Γ. Choose plane P such that $\Psi_1 = E_i - U$ and $\Psi_2 = \{0\}$. By Lemma 2, $\Delta \cap (P + W) = E_i - V$. Hence there exists a line segment in $\partial\Delta$ which contains W in its interior, but there is no planar region in $\partial\Delta$ with the same property.

We obtain a similar result if V belongs to an edge of Γ and U is a vertex.

Case 3: Points U and V are interior points of edges E_i and E_j, respectively. Let P be the plane of $E_i - U$ and $E_j - V$. Then $\Psi_1 = E_i - U$, $\Psi_2 = E_j - V$ and $\Delta \cap (P + W) = E_i - E_j$. Therefore point W belongs to a parallelogram face on $\partial\Delta$.

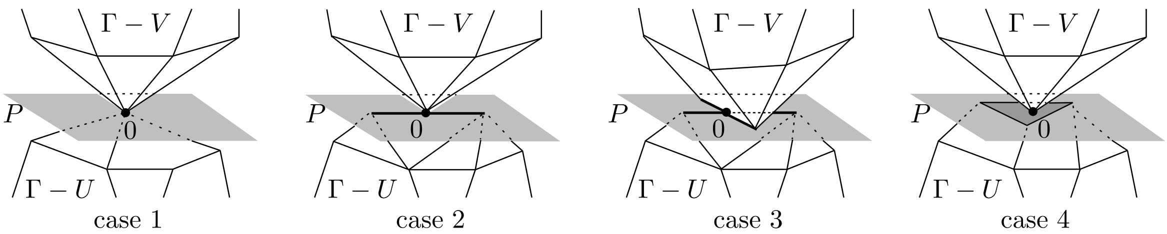
The centre of the parallelogram is $Q_i - Q_j$, the vector between the midpoints. Therefore an edge pair (E_i, E_j) occurs if and only if Q_i and Q_j are antipodal which happens $2B$ times.

Case 4: Point U lies in the interior of a face F_i and V is a vertex of Γ. The only choice for P is the plane of $F_i - U$. Then we have $\Psi_1 = F_i - U$, $\Psi_2 = \{0\}$ and $\Delta \cap (P + W) = F_i - V$. This is a planar face of $\partial\Delta$ which is congruent to F_i.

For each face F_i, the only possible vertex V is the farthest one from the plane of F_i.

If U is a vertex and V belongs to face F_i then we obtain the same way that W belongs to a face $-F_i + U$ which is also congruent to F_i. Therefore, each face of Γ has two copies on $\partial\Delta$, a translated and a reflected copy.

Case 5: Point U belongs to a face F_i of Γ and point V belongs to an edge or a face G. In this case objects F_i and G must be parallel which is not allowed.



Now all points in $\partial\Delta$ belong to some planar polygons (cases 3 and 4), finitely many line segments (case 2) and points (case 1). Therefore Δ is indeed a polyhedron. Now compute the numbers of its vertices, edges and faces.

The vertices are obtained in case 1, their number is $2A$.

Faces are obtained in cases 3 and 4. Case 3 generates $2B$ parallelogram faces. Case 4 generates 2ℓ faces.

We compute the number of edges of Δ from the degrees (number of sides) of faces of Γ. Let d_i be the the degree of face F_i. The sum of degrees is twice as much as the number of edges, so $d_1 + d_2 + \ldots + d_l = 2m$. The sum of degrees of faces of Δ is $2B \cdot 4 + 2(d_1 + d_2 + \cdots + d_l) = 8B + 4m$, so the number of edges on Δ is $4B + 2m$.

Applying Euler's polyhedron theorem on Γ and Δ, we have $n + l = m + 2$ and $2A + (2B + 2\ell) = (4B + 2m) + 2$. Then the conclusion follows:

$$A - B = m - \ell + 1 = n - 1.$$

Geometry

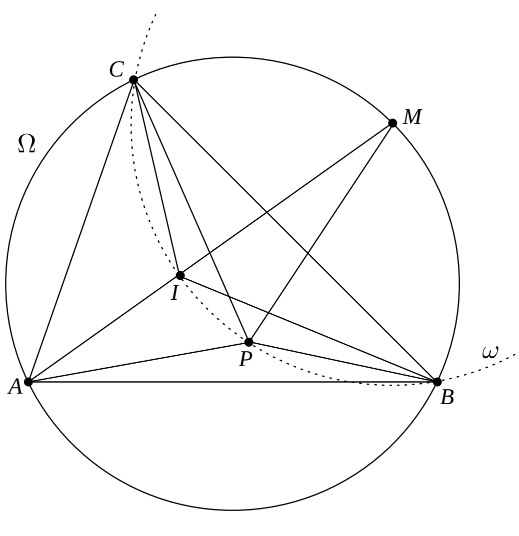
G1. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$ and that equality holds if and only if P coincides with I.

Solution. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. Since $\angle PBA + \angle PCA + \angle PBC + \angle PCB = \beta + \gamma$, the condition from the problem statement is equivalent to $\angle PBC + \angle PCB = (\beta + \gamma)/2$, i. e. $\angle BPC = 90° + \alpha/2$.

On the other hand $\angle BIC = 180° - (\beta + \gamma)/2 = 90° + \alpha/2$. Hence $\angle BPC = \angle BIC$, and since P and I are on the same side of BC, the points B, C, I and P are concyclic. In other words, P lies on the circumcircle ω of triangle BCI.



Let Ω be the circumcircle of triangle ABC. It is a well-known fact that the centre of ω is the midpoint M of the arc BC of Ω. This is also the point where the angle bisector AI intersects Ω.

From triangle APM we have

$$AP + PM \geq AM = AI + IM = AI + PM.$$

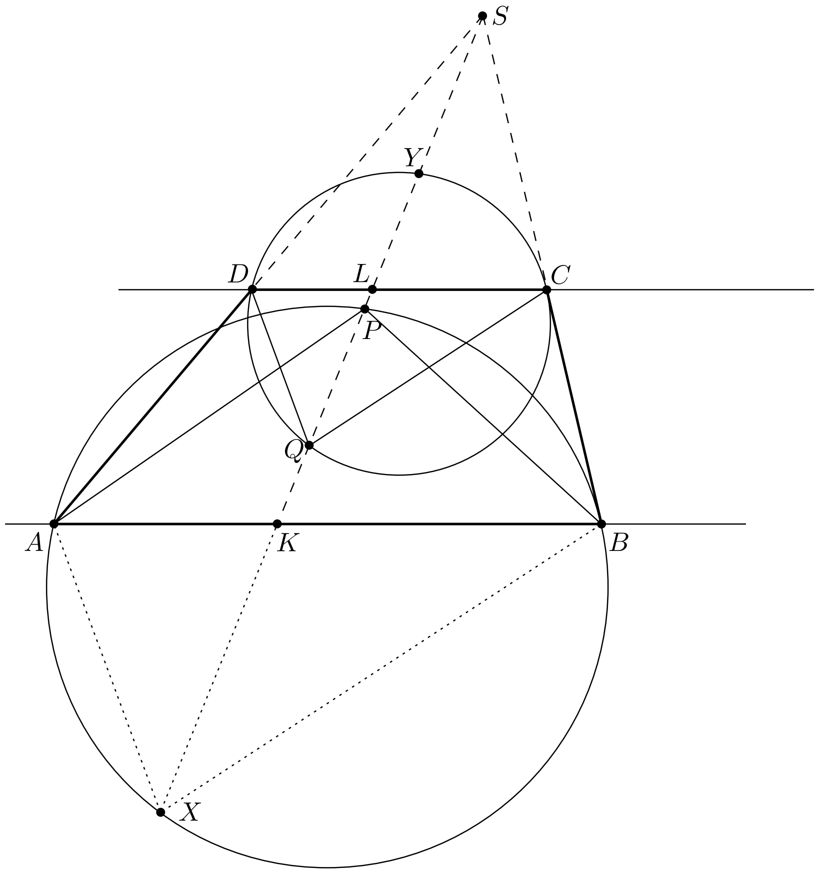
Therefore $AP \geq AI$. Equality holds if and only if P lies on the line segment AI, which occurs if and only if $P = I$.

G2. Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

Solution 1. Because $AB \parallel CD$, the relation $AK/KB = DL/LC$ readily implies that the lines AD, BC and KL have a common point S.



Consider the second intersection points X and Y of the line SK with the circles (ABP) and (CDQ), respectively. Since $APBX$ is a cyclic quadrilateral and $AB \parallel CD$, one has

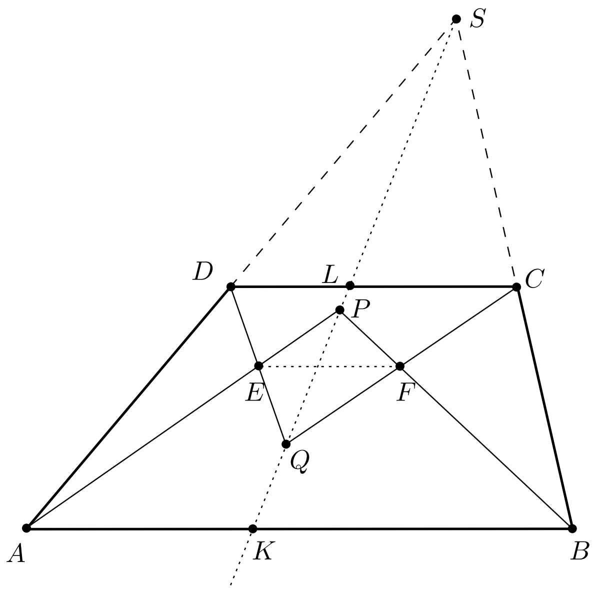
$$\angle AXB = 180^\circ - \angle APB = 180^\circ - \angle BCD = \angle ABC.$$

This shows that BC is tangent to the circle (ABP) at B. Likewise, BC is tangent to the circle (CDQ) at C. Therefore $SP \cdot SX = SB^2$ and $SQ \cdot SY = SC^2$.

Let h be the homothety with centre S and ratio SC/SB. Since $h(B) = C$, the above conclusion about tangency implies that h takes circle (ABP) to circle (CDQ). Also, h takes AB to CD, and it easily follows that $h(P) = Y$, $h(X) = Q$, yielding $SP/SY = SB/SC = SX/SQ$.

Equalities $SP \cdot SX = SB^2$ and $SQ/SX = SC/SB$ imply $SP \cdot SQ = SB \cdot SC$, which is equivalent to P, Q, B and C being concyclic.

Solution 2. The case where $P = Q$ is trivial. Thus assume that P and Q are two distinct points. As in the first solution, notice that the lines AD, BC and KL concur at a point S.



Let the lines AP and DQ meet at E, and let BP and CQ meet at F. Then $\angle EPF = \angle BCD$ and $\angle FQE = \angle ABC$ by the condition of the problem. Since the angles BCD and ABC add up to $180°$, it follows that $PEQF$ is a cyclic quadrilateral.

Applying Menelaus' theorem, first to triangle ASP and line DQ and then to triangle BSP and line CQ, we have

$$\frac{AD}{DS} \cdot \frac{SQ}{QP} \cdot \frac{PE}{EA} = 1 \quad \text{and} \quad \frac{BC}{CS} \cdot \frac{SQ}{QP} \cdot \frac{PF}{FB} = 1.$$

The first factors in these equations are equal, as $AB \| CD$. Thus the last factors are also equal, which implies that EF is parallel to AB and CD. Using this and the cyclicity of $PEQF$, we obtain

$$\angle BCD = \angle BCF + \angle FCD = \angle BCQ + \angle EFQ = \angle BCQ + \angle EPQ.$$

On the other hand,

$$\angle BCD = \angle APB = \angle EPF = \angle EPQ + \angle QPF,$$

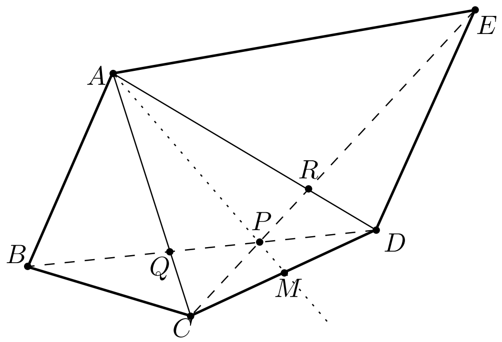
and consequently $\angle BCQ = \angle QPF$. The latter angle either coincides with $\angle QPB$ or is supplementary to $\angle QPB$, depending on whether Q lies between K and P or not. In either case it follows that P, Q, B and C are concyclic.

G3. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Solution. Let the diagonals AC and BD meet at Q, the diagonals AD and CE meet at R, and let the ray AP meet the side CD at M. We want to prove that $CM = MD$ holds.



The idea is to show that Q and R divide AC and AD in the same ratio, or more precisely

$$\frac{AQ}{QC} = \frac{AR}{RD} \tag{1}$$

(which is equivalent to $QR \| CD$). The given angle equalities imply that the triangles ABC, ACD and ADE are similar. We therefore have

$$\frac{AB}{AC} = \frac{AC}{AD} = \frac{AD}{AE}.$$

Since $\angle BAD = \angle BAC + \angle CAD = \angle CAD + \angle DAE = \angle CAE$, it follows from $AB/AC = AD/AE$ that the triangles ABD and ACE are also similar. Their angle bisectors in A are AQ and AR, respectively, so that

$$\frac{AB}{AC} = \frac{AQ}{AR}.$$

Because $AB/AC = AC/AD$, we obtain $AQ/AR = AC/AD$, which is equivalent to (1). Now Ceva's theorem for the triangle ACD yields

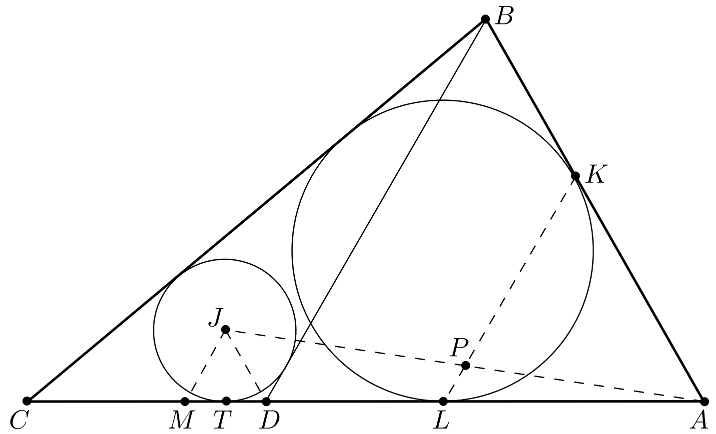
$$\frac{AQ}{QC} \cdot \frac{CM}{MD} \cdot \frac{DR}{RA} = 1.$$

In view of (1), this reduces to $CM = MD$, which completes the proof.

Comment. Relation (1) immediately follows from the fact that quadrilaterals $ABCD$ and $ACDE$ are similar.

G4. A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

Solution. Denote by P be the common point of AJ and KL. Let the parallel to KL through J meet AC at M. Then P is the midpoint of AJ if and only if $AM = 2 \cdot AL$, which we are about to show.



Denoting $\angle BAC = 2\alpha$, the equalities $BA = BD$ and $AK = AL$ imply $\angle ADB = 2\alpha$ and $\angle ALK = 90^\circ - \alpha$. Since DJ bisects $\angle BDC$, we obtain $\angle CDJ = \frac{1}{2} \cdot (180^\circ - \angle ADB) = 90^\circ - \alpha$. Also $\angle DMJ = \angle ALK = 90^\circ - \alpha$ since $JM \| KL$. It follows that $JD = JM$.

Let the incircle of triangle BCD touch its side CD at T. Then $JT \perp CD$, meaning that JT is the altitude to the base DM of the isosceles triangle DMJ. It now follows that $DT = MT$, and we have

$$DM = 2 \cdot DT = BD + CD - BC.$$

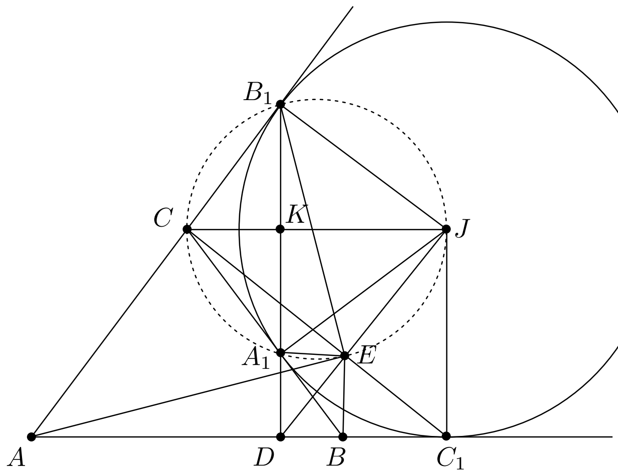
Therefore

$$\begin{aligned} AM &= AD + (BD + CD - BC) \\ &= AD + AB + DC - BC \\ &= AC + AB - BC \\ &= 2 \cdot AL, \end{aligned}$$

which completes the proof.

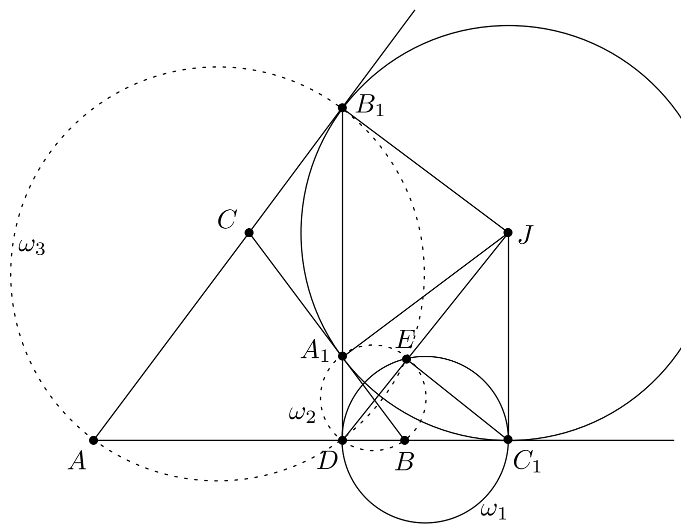
G5. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

Solution 1. Let K be the intersection point of lines JC and A_1B_1. Obviously $JC \perp A_1B_1$ and since $A_1B_1 \perp AB$, the lines JK and C_1D are parallel and equal. From the right triangle B_1CJ we obtain $JC_1^2 = JB_1^2 = JC \cdot JK = JC \cdot C_1D$ from which we infer that $DC_1/C_1J = C_1J/JC$ and the right triangles DC_1J and C_1JC are similar. Hence $\angle C_1DJ = \angle JC_1C$, which implies that the lines DJ and C_1C are perpendicular, i.e. the points C_1, E, C are collinear.



Since $\angle CA_1J = \angle CB_1J = \angle CEJ = 90°$, points A_1, B_1 and E lie on the circle of diameter CJ. Then $\angle DBA_1 = \angle A_1CJ = \angle DEA_1$, which implies that quadrilateral BEA_1D is cyclic; therefore $\angle A_1EB = 90°$.

Quadrilateral $ADEB_1$ is also cyclic because $\angle EB_1A = \angle EJC = \angle EDC_1$, therefore we obtain $\angle AEB_1 = \angle ADB = 90°$.



Solution 2. Consider the circles ω_1, ω_2 and ω_3 of diameters C_1D, A_1B and AB_1, respectively. Line segments JC_1, JB_1 and JA_1 are tangents to those circles and, due to the right angle at D, ω_2 and ω_3 pass through point D. Since $\angle C_1ED$ is a right angle, point E lies on circle ω_1, therefore

$$JC_1^2 = JD \cdot JE.$$

Since $JA_1 = JB_1 = JC_1$ are all radii of the excircle, we also have

$$JA_1^2 = JD \cdot JE \qquad \text{and} \qquad JB_1^2 = JD \cdot JE.$$

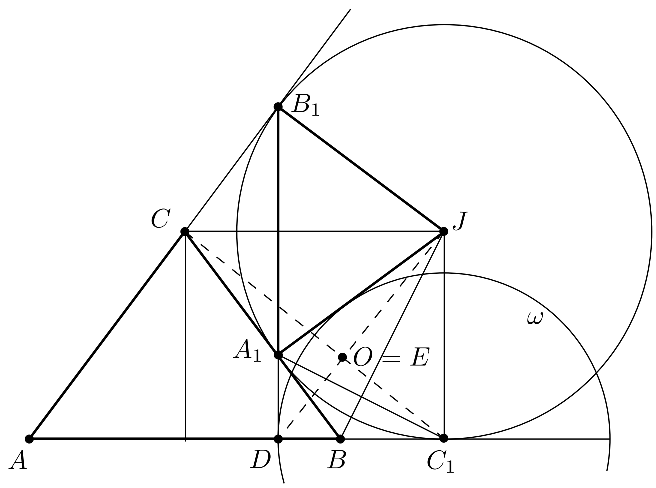
These equalities show that E lies on circles ω_2 and ω_3 as well, so $\angle BEA_1 = \angle AEB_1 = 90°$.

Solution 3. First note that A_1B_1 is perpendicular to the external angle bisector CJ of $\angle BCA$ and parallel to the internal angle bisector of that angle. Therefore, A_1B_1 is perpendicular to AB if and only if triangle ABC is isosceles, $AC = BC$. In that case the external bisector CJ is parallel to AB.

Triangles ABC and B_1A_1J are similar, as their corresponding sides are perpendicular. In particular, we have $\angle DA_1J = \angle C_1BA_1$; moreover, from cyclic deltoid JA_1BC_1,

$$\angle C_1A_1J = \angle C_1BJ = \frac{1}{2}\angle C_1BA_1 = \frac{1}{2}\angle DA_1J.$$

Therefore, A_1C_1 bisects angle $\angle DA_1J$.



In triangle B_1A_1J, line JC_1 is the external bisector at vertex J. The point C_1 is the intersection of two external angle bisectors (at A_1 and J) so C_1 is the centre of the excircle ω, tangent to side A_1J, and to the extension of B_1A_1 at point D.

Now consider the similarity transform φ which moves B_1 to A, A_1 to B and J to C. This similarity can be decomposed into a rotation by $90°$ around a certain point O and a homothety from the same centre. This similarity moves point C_1 (the centre of excircle ω) to J and moves D (the point of tangency) to C_1.

Since the rotation angle is $90°$, we have $\angle XO\varphi(X) = 90°$ for an arbitrary point $X \neq O$. For $X = D$ and $X = C_1$ we obtain $\angle DOC_1 = \angle C_1OJ = 90°$. Therefore O lies on line segment DJ and C_1O is perpendicular to DJ. This means that $O = E$.

For $X = A_1$ and $X = B_1$ we obtain $\angle A_1OB = \angle B_1OA = 90°$, i.e.

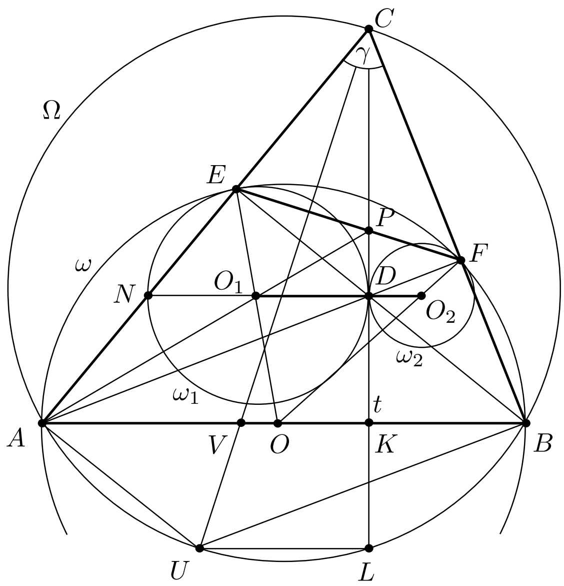
$$\angle BEA_1 = \angle AEB_1 = 90°.$$

Comment. Choosing $X = J$, it also follows that $\angle JEC = 90°$ which proves that lines DJ and CC_1 intersect at point E. However, this is true more generally, without the assumption that A_1B_1 and AB are perpendicular, because points C and D are conjugates with respect to the excircle. The last observation could replace the first paragraph of Solution 1.

G6. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E and O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Solution 1. Point E is the centre of a homothety h which takes circle ω_1 to circle ω. The radii O_1D and OB of these circles are parallel as both are perpendicular to line t. Also, O_1D and OB are on the same side of line EO, hence h takes O_1D to OB. Consequently, points E, D and B are collinear. Likewise, points F, D and A are collinear as well.

Let lines AE and BF intersect at C. Since AF and BE are altitudes in triangle ABC, their common point D is the orthocentre of this triangle. So CD is perpendicular to AB, implying that C lies on line t. Note that triangle ABC is acute-angled. We mention the well-known fact that triangles FEC and ABC are similar in ratio $\cos\gamma$, where $\gamma = \angle ACB$. In addition, points C, E, D and F lie on the circle with diameter CD.



Let P be the common point of lines EF and t. We are going to prove that P lies on line AO_1. Denote by N the second common point of circle ω_1 and AC; this is the point of ω_1 diametrically opposite to D. By Menelaus' theorem for triangle DCN, points A, O_1 and P are collinear if and only if

$$\frac{CA}{AN} \cdot \frac{NO_1}{O_1D} \cdot \frac{DP}{PC} = 1.$$

Because $NO_1 = O_1D$, this reduces to $CA/AN = CP/PD$. Let line t meet AB at K. Then $CA/AN = CK/KD$, so it suffices to show that

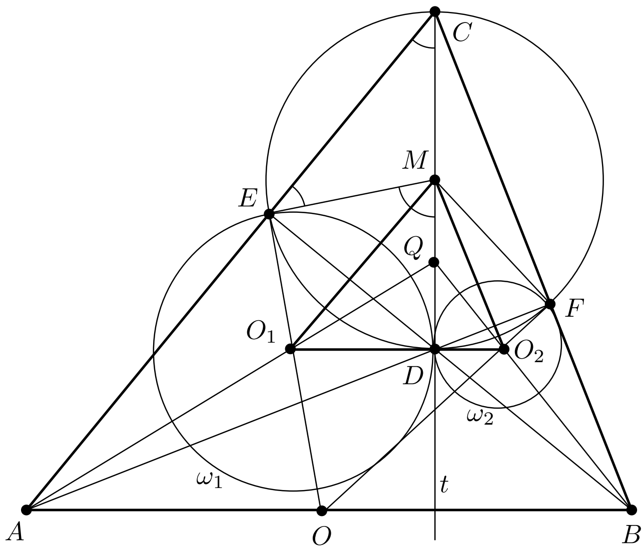
$$\frac{CP}{PD} = \frac{CK}{KD}. \tag{1}$$

To verify (1), consider the circumcircle Ω of triangle ABC. Draw its diameter CU through C, and let CU meet AB at V. Extend CK to meet Ω at L. Since AB is parallel to UL, we have $\angle ACU = \angle BCL$. On the other hand $\angle EFC = \angle BAC$, $\angle FEC = \angle ABC$ and $EF/AB = \cos\gamma$, as stated above. So reflection in the bisector of $\angle ACB$ followed by a homothety with centre C and ratio $1/\cos\gamma$ takes triangle FEC to triangle ABC. Consequently, this transformation takes CD to CU, which implies $CP/PD = CV/VU$. Next, we have $KL = KD$, because D

is the orthocentre of triangle ABC. Hence $CK/KD = CK/KL$. Finally, $CV/VU = CK/KL$ because AB is parallel to UL. Relation (1) follows, proving that P lies on line AO_1. By symmetry, P also lies on line AO_2 which completes the solution.

Solution 2. We proceed as in the first solution to define a triangle ABC with orthocentre D, in which AF and BE are altitudes.

Denote by M the midpoint of CD. The quadrilateral $CEDF$ is inscribed in a circle with centre M, hence $MC = ME = MD = MF$.



Consider triangles ABC and O_1O_2M. Lines O_1O_2 and AB are parallel, both of them being perpendicular to line t. Next, MO_1 is the line of centres of circles (CEF) and ω_1 whose common chord is DE. Hence MO_1 bisects $\angle DME$ which is the external angle at M in the isosceles triangle CEM. It follows that $\angle DMO_1 = \angle DCA$, so that MO_1 is parallel to AC. Likewise, MO_2 is parallel to BC.

Thus the respective sides of triangles ABC and O_1O_2M are parallel; in addition, these triangles are not congruent. Hence there is a homothety taking ABC to O_1O_2M. The lines AO_1, BO_2 and $CM = t$ are concurrent at the centre Q of this homothety.

Finally, apply Pappus' theorem to the triples of collinear points A, O, B and O_2, D, O_1. The theorem implies that the points $AD \cap OO_2 = F$, $AO_1 \cap BO_2 = Q$ and $OO_1 \cap BD = E$ are collinear. In other words, line EF passes through the common point Q of AO_1, BO_2 and t.

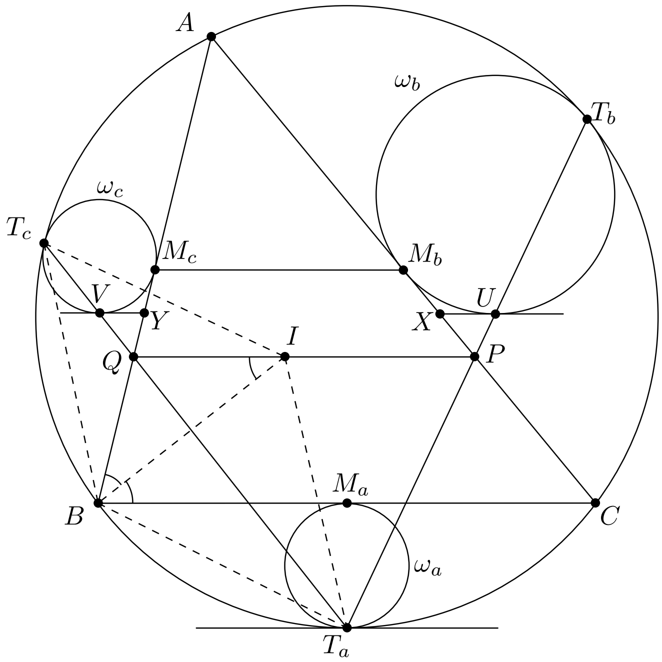
Comment. Relation (1) from Solution 1 expresses the well-known fact that points P and K are harmonic conjugates with respect to points C and D. It is also easy to justify it by direct computation. Denoting $\angle CAB = \alpha$, $\angle ABC = \beta$, it is straightforward to obtain $CP/PD = CK/KD = \tan\alpha\tan\beta$.

G7. In a triangle ABC, let M_a, M_b, M_c be respectively the midpoints of the sides BC, CA, AB and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the opposite vertices. For $i \in \{a, b, c\}$, let ω_i be the circle with M_iT_i as diameter. Let p_i be the common external tangent to ω_j, ω_k ($\{i, j, k\} = \{a, b, c\}$) such that ω_i lies on the opposite side of p_i than ω_j, ω_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

Solution. Let T_aT_b intersect circle ω_b at T_b and U, and let T_aT_c intersect circle ω_c at T_c and V. Further, let UX be the tangent to ω_b at U, with X on AC, and let VY be the tangent to ω_c at V, with Y on AB. The homothety with centre T_b and ratio T_bT_a/T_bU maps the circle ω_b onto the circumcircle of ABC and the line UX onto the line tangent to the circumcircle at T_a, which is parallel to BC; thus $UX \| BC$. The same is true of VY, so that $UX \| BC \| VY$.

Let T_aT_b cut AC at P and let T_aT_c cut AB at Q. The point X lies on the hypotenuse PM_b of the right triangle PUM_b and is equidistant from U and M_b. So X is the midpoint of M_bP. Similarly Y is the midpoint of M_cQ.

Denote the incentre of triangle ABC as usual by I. It is a known fact that $T_aI = T_aB$ and $T_cI = T_cB$. Therefore the points B and I are symmetric across T_aT_c, and consequently $\angle QIB = \angle QBI = \angle IBC$. This implies that BC is parallel to the line IQ, and likewise, to IP. In other words, PQ is the line parallel to BC passing through I.



Clearly $M_bM_c \| BC$. So PM_bM_cQ is a trapezoid and the segment XY connects the midpoints of its nonparallel sides; hence $XY \| BC$. This combined with the previously established relations $UX \| BC \| VY$ shows that all the four points U, X, Y, V lie on a line which is the common tangent to circles ω_b, ω_c. Since it leaves these two circles on one side and the circle ω_a on the other, this line is just the line p_a from the problem statement.

Line p_a runs midway between I and M_bM_c. Analogous conclusions hold for the lines p_b and p_c. So these three lines form a triangle homothetic from centre I to triangle $M_aM_bM_c$ in ratio $1/2$, hence similar to ABC in ratio $1/4$.

G8. Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \qquad \text{and} \qquad \angle PBA + \angle PCD \leq 90^\circ.$$

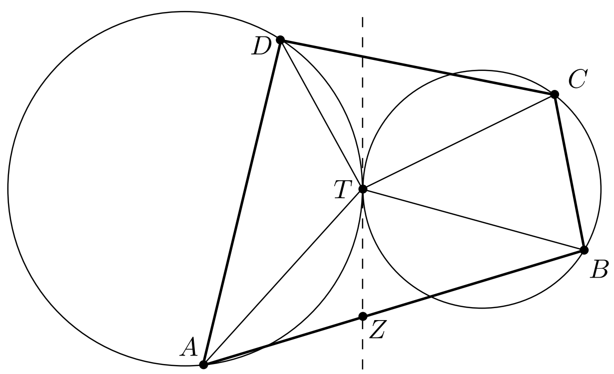
Prove that $AB + CD \geq BC + AD$.

Solution. We start with a preliminary observation. Let T be a point inside the quadrilateral $ABCD$. Then:

Circles (BCT) *and* (DAT) *are tangent at* T
if and only if $\quad \angle ADT + \angle BCT = \angle ATB.$ (1)

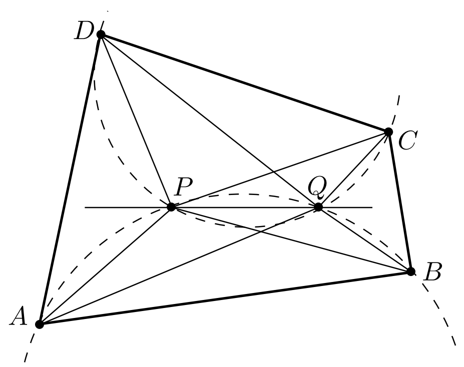
Indeed, if the two circles touch each other then their common tangent at T intersects the segment AB at a point Z, and so $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$, by the tangent-chord theorem. Thus $\angle ADT + \angle BCT = \angle ATZ + \angle BTZ = \angle ATB$.

And conversely, if $\angle ADT + \angle BCT = \angle ATB$ then one can draw from T a ray TZ with Z on AB so that $\angle ADT = \angle ATZ$, $\angle BCT = \angle BTZ$. The first of these equalities implies that TZ is tangent to the circle (DAT); by the second equality, TZ is tangent to the circle (BCT), so the two circles are tangent at T.



So the equivalence (1) is settled. It will be used later on. Now pass to the actual solution. Its key idea is to introduce the circumcircles of triangles ABP and CDP and to consider their second intersection Q (assume for the moment that they indeed meet at two distinct points P and Q).

Since the point A lies outside the circle (BCP), we have $\angle BCP + \angle BAP < 180^\circ$. Therefore the point C lies outside the circle (ABP). Analogously, D also lies outside that circle. It follows that P and Q lie on the same arc CD of the circle (BCP).



By symmetry, P and Q lie on the same arc AB of the circle (ABP). Thus the point Q lies either inside the angle BPC or inside the angle APD. Without loss of generality assume that Q lies inside the angle BPC. Then

$$\angle AQD = \angle PQA + \angle PQD = \angle PBA + \angle PCD \leq 90^\circ, \tag{2}$$

by the condition of the problem.

In the cyclic quadrilaterals $APQB$ and $DPQC$, the angles at vertices A and D are acute. So their angles at Q are obtuse. This implies that Q lies not only inside the angle BPC but in fact inside the triangle BPC, hence also inside the quadrilateral $ABCD$.

Now an argument similar to that used in deriving (2) shows that

$$\angle BQC = \angle PAB + \angle PDC \leq 90^\circ. \tag{3}$$

Moreover, since $\angle PCQ = \angle PDQ$, we get

$$\angle ADQ + \angle BCQ = \angle ADP + \angle PDQ + \angle BCP - \angle PCQ = \angle ADP + \angle BCP.$$

The last sum is equal to $\angle APB$, according to the observation (1) applied to $T = P$. And because $\angle APB = \angle AQB$, we obtain

$$\angle ADQ + \angle BCQ = \angle AQB.$$

Applying now (1) to $T = Q$ we conclude that the circles (BCQ) and (DAQ) are externally tangent at Q. (We have assumed $P \neq Q$; but if $P = Q$ then the last conclusion holds trivially.)

Finally consider the halfdiscs with diameters BC and DA constructed inwardly to the quadrilateral $ABCD$. They have centres at M and N, the midpoints of BC and DA respectively. In view of (2) and (3), these two halfdiscs lie entirely inside the circles (BQC) and (AQD); and since these circles are tangent, the two halfdiscs cannot overlap. Hence $MN \geq \frac{1}{2}BC + \frac{1}{2}DA$.

On the other hand, since $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{CD})$, we have $MN \leq \frac{1}{2}(AB + CD)$. Thus indeed $AB + CD \geq BC + DA$, as claimed.

G9. Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Solution. We will work with oriented angles between lines. For two straight lines ℓ, m in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transforms line ℓ into a line parallel to m (the choice of the rotation centre is irrelevant). This is a signed quantity; values differing by a multiple of π are identified, so that

$$\angle(\ell, m) = -\angle(m, \ell), \qquad \angle(\ell, m) + \angle(m, n) = \angle(\ell, n).$$

If ℓ is the line through points K, L and m is the line through M, N, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters K, L are freely interchangeable; and so are M, N.

The counterpart of the classical theorem about cyclic quadrilaterals is the following:
If K, L, M, N are four noncollinear points in the plane then

$$K, L, M, N \text{ are concyclic if and only if } \angle(KM, LM) = \angle(KN, LN). \qquad (1)$$

Passing to the solution proper, we first show that the three circles (AB_1C_1), (BC_1A_1), (CA_1B_1) have a common point. So, let (AB_1C_1) and (BC_1A_1) intersect at the points C_1 and P. Then by (1)

$$\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1)$$

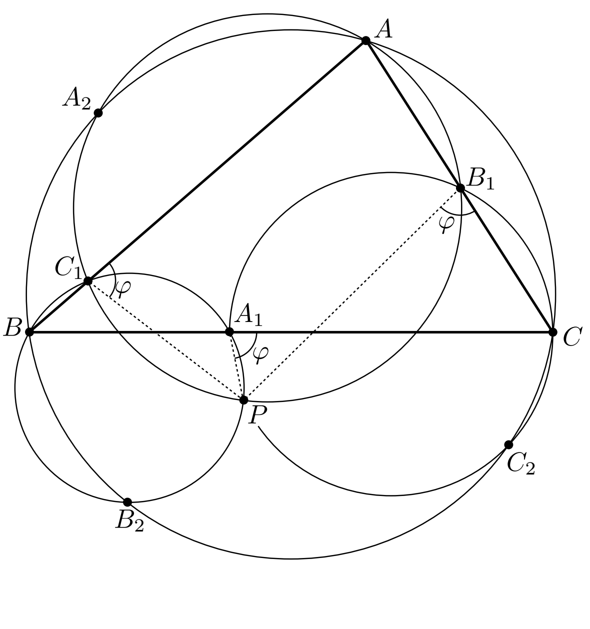
$$= \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).$$

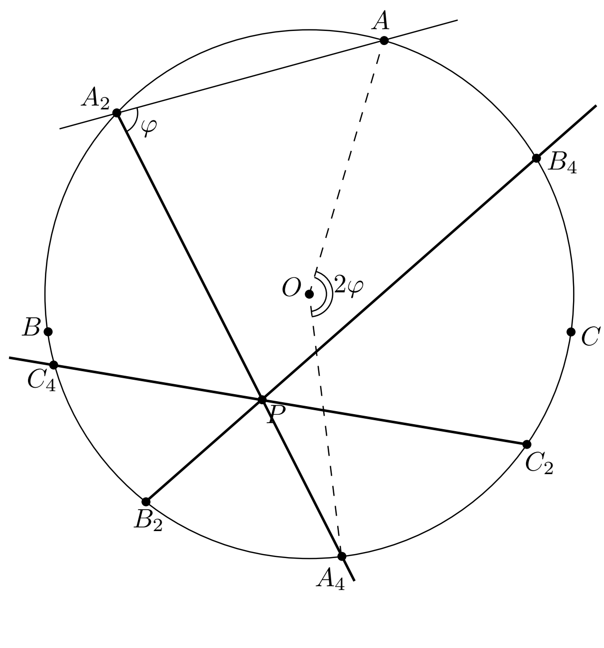
Denote this angle by φ.

The equality between the outer terms shows, again by (1), that the points A_1, B_1, P, C are concyclic. Thus P is the common point of the three mentioned circles.

From now on the basic property (1) will be used without explicit reference. We have

$$\varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB). \qquad (2)$$





Let lines A_2P, B_2P, C_2P meet the circle (ABC) again at A_4, B_4, C_4, respectively. As

$$\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,$$

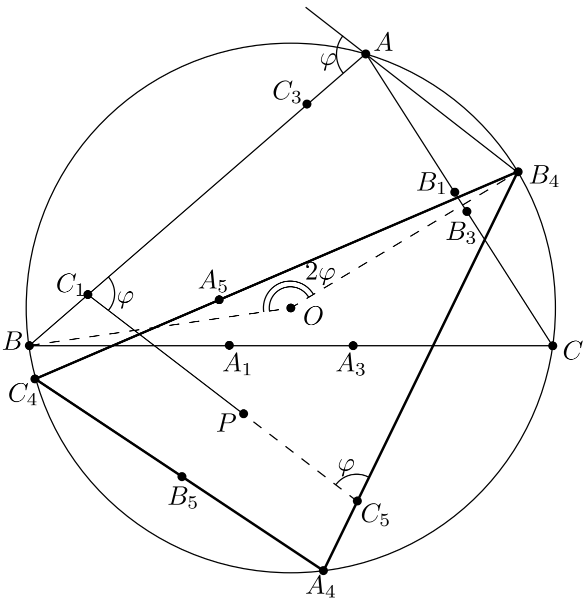
we see that line A_2A is the image of line A_2A_4 under rotation about A_2 by the angle φ. Hence the point A is the image of A_4 under rotation by 2φ about O, the centre of (ABC). The same rotation sends B_4 to B and C_4 to C. Triangle ABC is the image of $A_4B_4C_4$ in this map. Thus

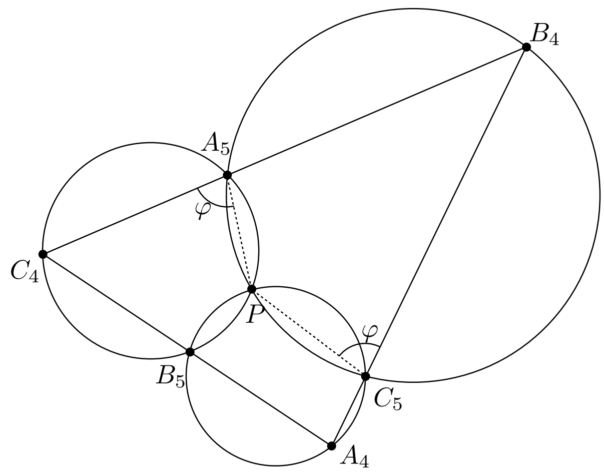
$$\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi. \tag{3}$$

Since the rotation by 2φ about O takes B_4 to B, we have $\angle(AB_4, AB) = \varphi$. Hence by (2)

$$\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,$$

which means that $AB_4 \| PC_1$.





Let C_5 be the intersection of lines PC_1 and A_4B_4; define A_5, B_5 analogously. So $AB_4 \| C_1C_5$ and, by (3) and (2),

$$\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi; \tag{4}$$

i.e., $\angle(B_4C_5, C_5C_1) = \varphi$. This combined with $\angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi$ (see (2)) proves that the quadrilateral $AB_4C_5C_1$ is an isosceles trapezoid with $AC_1 = B_4C_5$.

Interchanging the roles of A and B we infer that also $BC_1 = A_4C_5$. And since $AC_1 + BC_1 = AB = A_4B_4$, it follows that the point C_5 lies on the line segment A_4B_4 and partitions it into segments A_4C_5, B_4C_5 of lengths BC_1 $(= AC_3)$ and AC_1 $(= BC_3)$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto ABC carries C_5 onto C_3. Likewise, it sends A_5 to A_3 and B_5 to B_3. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent. It now suffices to show that the latter is similar to $A_2B_2C_2$.

Lines B_4C_5 and PC_5 coincide respectively with A_4B_4 and PC_1. Thus by (4)

$$\angle(B_4C_5, PC_5) = \varphi.$$

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$

These relations imply that the points P, B_4, C_5, A_5 are concyclic. Analogously, P, C_4, A_5, B_5 and P, A_4, B_5, C_5 are concyclic quadruples. Therefore

$$\angle(A_5B_5, C_5B_5) = \angle(A_5B_5, PB_5) + \angle(PB_5, C_5B_5) = \angle(A_5C_4, PC_4) + \angle(PA_4, C_5A_4). \qquad (5)$$

On the other hand, since the points $A_2, B_2, C_2, A_4, B_4, C_4$ all lie on the circle (ABC), we have

$$\angle(A_2B_2, C_2B_2) = \angle(A_2B_2, B_4B_2) + \angle(B_4B_2, C_2B_2) = \angle(A_2A_4, B_4A_4) + \angle(B_4C_4, C_2C_4). \quad (6)$$

But the lines A_2A_4, B_4A_4, B_4C_4, C_2C_4 coincide respectively with PA_4, C_5A_4, A_5C_4, PC_4. So the sums on the right-hand sides of (5) and (6) are equal, leading to equality between their left-hand sides: $\angle(A_5B_5, C_5B_5) = \angle(A_2B_2, C_2B_2)$. Hence (by cyclic shift, once more) also $\angle(B_5C_5, A_5C_5) = \angle(B_2C_2, A_2C_2)$ and $\angle(C_5A_5, B_5A_5) = \angle(C_2A_2, B_2A_2)$. This means that the triangles $A_5B_5C_5$ and $A_2B_2C_2$ have their corresponding angles equal, and consequently they are similar.

Comment 1. This is the way in which the proof has been presented by the proposer. Trying to work it out in the language of classical geometry, so as to avoid oriented angles, one is led to difficulties due to the fact that the reasoning becomes heavily case-dependent. Disposition of relevant points can vary in many respects. Angles which are equal in one case become supplementary in another. Although it seems not hard to translate all formulas from the shapes they have in one situation to the one they have in another, the real trouble is to identify all cases possible and rigorously verify that the key conclusions retain validity in each case.

The use of oriented angles is a very efficient method to omit this trouble. It seems to be the most appropriate environment in which the solution can be elaborated.

Comment 2. Actually, the fact that the circles (AB_1C_1), (BC_1A_1) and (CA_1B_1) have a common point does not require a proof; it is known as Miquel's theorem.

G10. To each side a of a convex polygon we assign the maximum area of a triangle contained in the polygon and having a as one of its sides. Show that the sum of the areas assigned to all sides of the polygon is not less than twice the area of the polygon.

Solution 1.

Lemma. Every convex $(2n)$-gon, of area S, has a side and a vertex that jointly span a triangle of area not less than S/n.

Proof. By *main diagonals* of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by Δ_b the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles Δ_b, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', $\ldots$, where $A, B, C, \ldots$ are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence $A, B, C, \ldots$ before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\Delta_{\ell'}$, $\ell' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles Δ_b jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that $[\Delta_b] + [\Delta_{b'}] \geq S/n$, where $[\cdots]$ stands for the area of a region. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\Delta_b] + [\Delta_{b'}] \geq S/n,$$

proving the lemma. □

Now, let $\mathcal{P}$ be any convex polygon, of area S, with m sides $a_1, \ldots, a_m$. Let S_i be the area of the greatest triangle in $\mathcal{P}$ with side a_i. Suppose, contrary to the assertion, that

$$\sum_{i=1}^{m} \frac{S_i}{S} < 2.$$

Then there exist rational numbers $q_1, \ldots, q_m$ such that $\sum q_i = 2$ and $q_i > S_i/S$ for each i.

Let n be a common denominator of the m fractions $q_1, \ldots, q_m$. Write $q_i = k_i/n$; so $\sum k_i = 2n$. Partition each side a_i of $\mathcal{P}$ into k_i equal segments, creating a convex $(2n)$-gon of area S (with some angles of size 180°), to which we apply the lemma. Accordingly, this refined polygon has a side b and a vertex H spanning a triangle T of area $[T] \geq S/n$. If b is a piece of a side a_i of $\mathcal{P}$, then the triangle W with base a_i and summit H has area

$$[W] = k_i \cdot [T] \geq k_i \cdot S/n = q_i \cdot S > S_i,$$

in contradiction with the definition of S_i. This ends the proof.

Solution 2. As in the first solution, we allow again angles of size 180° at some vertices of the convex polygons considered.

To each convex n-gon $\mathcal{P} = A_1A_2\dots A_n$ we assign a centrally symmetric convex $(2n)$-gon $\mathcal{Q}$ with side vectors $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. The construction is as follows. Attach the $2n$ vectors $\pm\overrightarrow{A_iA_{i+1}}$ at a common origin and label them $\overrightarrow{\mathbf{b}_1}, \overrightarrow{\mathbf{b}_2}, \dots, \overrightarrow{\mathbf{b}_{2n}}$ in counterclockwise direction; the choice of the first vector $\overrightarrow{\mathbf{b}_1}$ is irrelevant. The order of labelling is well-defined if $\mathcal{P}$ has neither parallel sides nor angles equal to 180°. Otherwise several collinear vectors with the same direction are labelled consecutively $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+r}}$. One can assume that in such cases the respective opposite vectors occur in the order $-\overrightarrow{\mathbf{b}_j}, -\overrightarrow{\mathbf{b}_{j+1}}, \dots, -\overrightarrow{\mathbf{b}_{j+r}}$, ensuring that $\overrightarrow{\mathbf{b}_{j+n}} = -\overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. Indices are taken cyclically here and in similar situations below.

Choose points $B_1, B_2, \dots, B_{2n}$ satisfying $\overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$ for $j = 1, \dots, 2n$. The polygonal line $\mathcal{Q} = B_1B_2\dots B_{2n}$ is closed, since $\sum_{j=1}^{2n} \overrightarrow{\mathbf{b}_j} = \overrightarrow{0}$. Moreover, $\mathcal{Q}$ is a convex $(2n)$-gon due to the arrangement of the vectors $\overrightarrow{\mathbf{b}_j}$, possibly with 180°-angles. The side vectors of $\mathcal{Q}$ are $\pm\overrightarrow{A_iA_{i+1}}$, $1 \le i \le n$. So in particular $\mathcal{Q}$ is centrally symmetric, because it contains as side vectors $\overrightarrow{A_iA_{i+1}}$ and $-\overrightarrow{A_iA_{i+1}}$ for each $i = 1, \dots, n$. Note that B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are opposite sides of $\mathcal{Q}$, $1 \le j \le n$. We call $\mathcal{Q}$ the *associate* of $\mathcal{P}$.

Let S_i be the maximum area of a triangle with side A_iA_{i+1} in $\mathcal{P}$, $1 \le i \le n$. We prove that

$$[B_1B_2\dots B_{2n}] = 2\sum_{i=1}^{n} S_i \tag{1}$$

and

$$[B_1B_2\dots B_{2n}] \ge 4\,[A_1A_2\dots A_n]\,. \tag{2}$$

It is clear that (1) and (2) imply the conclusion of the original problem.

Lemma. For a side A_iA_{i+1} of $\mathcal{P}$, let h_i be the maximum distance from a point of $\mathcal{P}$ to line A_iA_{i+1}, $i = 1, \dots, n$. Denote by B_jB_{j+1} the side of $\mathcal{Q}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}}$. Then the distance between B_jB_{j+1} and its opposite side in $\mathcal{Q}$ is equal to $2h_i$.

Proof. Choose a vertex A_k of $\mathcal{P}$ at distance h_i from line A_iA_{i+1}. Let $\mathbf{u}$ be the unit vector perpendicular to A_iA_{i+1} and pointing inside $\mathcal{P}$. Denoting by $\mathbf{x}\cdot\mathbf{y}$ the dot product of vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$h = \mathbf{u}\cdot\overrightarrow{A_iA_k} = \mathbf{u}\cdot(\overrightarrow{A_iA_{i+1}} + \cdots + \overrightarrow{A_{k-1}A_k}) = \mathbf{u}\cdot(\overrightarrow{A_iA_{i-1}} + \cdots + \overrightarrow{A_{k+1}A_k}).$$

In $\mathcal{Q}$, the distance H_i between the opposite sides B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ is given by

$$H_i = \mathbf{u}\cdot(\overrightarrow{B_jB_{j+1}} + \cdots + \overrightarrow{B_{j+n-1}B_{j+n}}) = \mathbf{u}\cdot(\overrightarrow{\mathbf{b}_j} + \overrightarrow{\mathbf{b}_{j+1}} + \cdots + \overrightarrow{\mathbf{b}_{j+n-1}}).$$

The choice of vertex A_k implies that the n consecutive vectors $\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}, \dots, \overrightarrow{\mathbf{b}_{j+n-1}}$ are precisely $\overrightarrow{A_iA_{i+1}}, \dots, \overrightarrow{A_{k-1}A_k}$ and $\overrightarrow{A_iA_{i-1}}, \dots, \overrightarrow{A_{k+1}A_k}$, taken in some order. This implies $H_i = 2h_i$. □

For a proof of (1), apply the lemma to each side of $\mathcal{P}$. If O the centre of $\mathcal{Q}$ then, using the notation of the lemma,

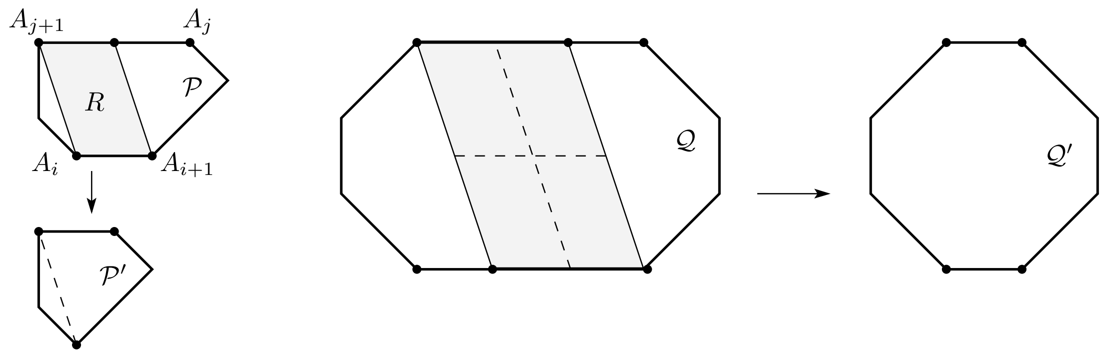
$$[B_jB_{j+1}O] = [B_{j+n}B_{j+n+1}O] = [A_iA_{i+1}A_k] = S_i\,.$$

Summation over all sides of $\mathcal{P}$ yields (1).

Set $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ for a convex polygon $\mathcal{P}$ with associate $\mathcal{Q}$. Inequality (2) means that $d(\mathcal{P}) \ge 0$ for each convex polygon $\mathcal{P}$. The last inequality will be proved by induction on the number ℓ of side directions of $\mathcal{P}$, i. e. the number of pairwise nonparallel lines each containing a side of $\mathcal{P}$.

We choose to start the induction with $\ell = 1$ as a base case, meaning that certain degenerate polygons are allowed. More exactly, we regard as *degenerate* convex polygons all closed polygonal lines of the form $X_1X_2 \ldots X_kY_1Y_2 \ldots Y_mX_1$, where $X_1, X_2, \ldots, X_k$ are points in this order on a line segment X_1Y_1, and so are $Y_m, Y_{m-1}, \ldots, Y_1$. The initial construction applies to degenerate polygons; their associates are also degenerate, and the value of d is zero. For the inductive step, consider a convex polygon $\mathcal{P}$ which determines ℓ side directions, assuming that $d(\mathcal{P}) \geq 0$ for polygons with smaller values of ℓ.

Suppose first that $\mathcal{P}$ *has a pair of parallel sides*, i. e. sides on distinct parallel lines. Let A_iA_{i+1} and A_jA_{j+1} be such a pair, and let $A_iA_{i+1} \leq A_jA_{j+1}$. Remove from $\mathcal{P}$ the parallelogram R determined by vectors $\overrightarrow{A_iA_{i+1}}$ and $\overrightarrow{A_iA_{j+1}}$. Two polygons are obtained in this way. Translating one of them by vector $\overrightarrow{A_iA_{i+1}}$ yields a new convex polygon $\mathcal{P}'$, of area $[\mathcal{P}] - [R]$ and with value of ℓ not exceeding the one of $\mathcal{P}$. The construction just described will be called operation $\mathbf{A}$.



The associate of $\mathcal{P}'$ is obtained from $\mathcal{Q}$ upon decreasing the lengths of two opposite sides by an amount of $2A_iA_{i+1}$. By the lemma, the distance between these opposite sides is twice the distance between A_iA_{i+1} and A_jA_{j+1}. Thus operation $\mathbf{A}$ decreases $[\mathcal{Q}]$ by the area of a parallelogram with base and respective altitude twice the ones of R, i. e. by $4[R]$. Hence $\mathbf{A}$ leaves the difference $d(\mathcal{P}) = [\mathcal{Q}] - 4[\mathcal{P}]$ unchanged.

Now, if $\mathcal{P}'$ also has a pair of parallel sides, apply operation $\mathbf{A}$ to it. Keep doing so with the subsequent polygons obtained for as long as possible. Now, $\mathbf{A}$ decreases the number p of pairs of parallel sides in $\mathcal{P}$. Hence its repeated applications gradually reduce p to 0, and further applications of $\mathbf{A}$ will be impossible after several steps. For clarity, let us denote by $\mathcal{P}$ again the polygon obtained at that stage.

The inductive step is complete if $\mathcal{P}$ is degenerate. Otherwise $\ell > 1$ and $p = 0$, i. e. there are no parallel sides in $\mathcal{P}$. Observe that then $\ell \geq 3$. Indeed, $\ell = 2$ means that the vertices of $\mathcal{P}$ all lie on the boundary of a parallelogram, implying $p > 0$.

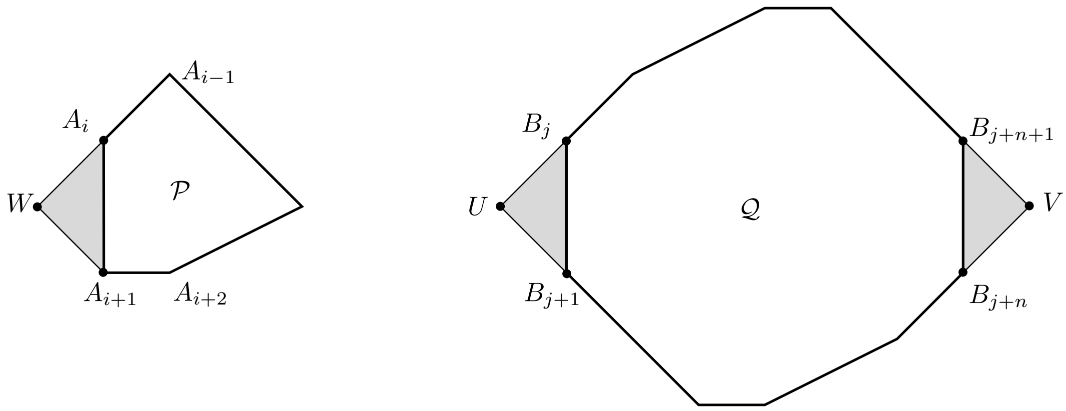
Furthermore, since $\mathcal{P}$ has no parallel sides, consecutive collinear vectors in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ (if any) correspond to consecutive 180°-angles in $\mathcal{P}$. Removing the vertices of such angles, we obtain a convex polygon with the same value of $d(\mathcal{P})$.

In summary, if operation $\mathbf{A}$ is impossible for a nondegenerate polygon $\mathcal{P}$, then $\ell \geq 3$. In addition, one may assume that $\mathcal{P}$ has no angles of size 180°.

The last two conditions then also hold for the associate $\mathcal{Q}$ of $\mathcal{P}$, and we perform the following construction. Since $\ell \geq 3$, there is a side B_jB_{j+1} of $\mathcal{Q}$ such that the sum of the angles at B_j and B_{j+1} is greater than 180°. (Such a side exists in each convex k-gon for $k > 4$.) Naturally, $B_{j+n}B_{j+n+1}$ is a side with the same property. Extend the pairs of sides $B_{j-1}B_j, B_{j+1}B_{j+2}$

and $B_{j+n-1}B_{j+n}, B_{j+n+1}B_{j+n+2}$ to meet at U and V, respectively. Let $\mathcal{Q}'$ be the centrally symmetric convex $2(n+1)$-gon obtained from $\mathcal{Q}$ by inserting U and V into the sequence $B_1, \ldots, B_{2n}$ as new vertices between B_j, B_{j+1} and B_{j+n}, B_{j+n+1}, respectively. Informally, we adjoin to $\mathcal{Q}$ the congruent triangles $B_jB_{j+1}U$ and $B_{j+n}B_{j+n+1}V$. Note that B_j, B_{j+1}, B_{j+n} and B_{j+n+1} are kept as vertices of $\mathcal{Q}'$, although B_jB_{j+1} and $B_{j+n}B_{j+n+1}$ are no longer its sides.

Let A_iA_{i+1} be the side of $\mathcal{P}$ such that $\overrightarrow{A_iA_{i+1}} = \overrightarrow{B_jB_{j+1}} = \overrightarrow{\mathbf{b}_j}$. Consider the point W such that triangle $A_iA_{i+1}W$ is congruent to triangle $B_jB_{j+1}U$ and exterior to $\mathcal{P}$. Insert W into the sequence $A_1, A_2, \ldots, A_n$ as a new vertex between A_i and A_{i+1} to obtain an $(n+1)$-gon $\mathcal{P}'$. We claim that $\mathcal{P}'$ is convex and its associate is $\mathcal{Q}'$.



Vectors $\overrightarrow{A_iW}$ and $\overrightarrow{\mathbf{b}_{j-1}}$ are collinear and have the same direction, as well as vectors $\overrightarrow{WA_{i+1}}$ and $\overrightarrow{\mathbf{b}_{j+1}}$. Since $\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}$ are consecutive terms in the sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$, the angle inequalities $\angle(\overrightarrow{\mathbf{b}_{j-1}}, \overrightarrow{\mathbf{b}_j}) \leq \angle(\overrightarrow{A_{i-1}A_i}, \overrightarrow{\mathbf{b}_j})$ and $\angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{\mathbf{b}_{j+1}}) \leq \angle(\overrightarrow{\mathbf{b}_j}, \overrightarrow{A_{i+1}A_{i+2}})$ hold true. They show that $\mathcal{P}'$ is a convex polygon. To construct its associate, vectors $\pm\overrightarrow{A_iA_{i+1}} = \pm\overrightarrow{\mathbf{b}_j}$ must be deleted from the defining sequence $\left(\overrightarrow{\mathbf{b}_k}\right)$ of $\mathcal{Q}$, and the vectors $\pm\overrightarrow{A_iW}$, $\pm\overrightarrow{WA_{i+1}}$ must be inserted appropriately into it. The latter can be done as follows:

$$\ldots,\ \overrightarrow{\mathbf{b}_{j-1}},\ \overrightarrow{A_iW},\ \overrightarrow{WA_{i+1}},\ \overrightarrow{\mathbf{b}_{j+1}},\ \ldots,\ -\overrightarrow{\mathbf{b}_{j-1}},\ -\overrightarrow{A_iW},\ -\overrightarrow{WA_{i+1}},\ -\overrightarrow{\mathbf{b}_{j+1}},\ \ldots .$$

This updated sequence produces $\mathcal{Q}'$ as the associate of $\mathcal{P}'$.

It follows from the construction that $[\mathcal{P}'] = [\mathcal{P}] + [A_iA_{i+1}W]$ and $[\mathcal{Q}'] = [\mathcal{Q}] + 2[A_iA_{i+1}W]$. Therefore $d(\mathcal{P}') = d(\mathcal{P}) - 2[A_iA_{i+1}W] < d(\mathcal{P})$.

To finish the induction, it remains to notice that the value of ℓ for $\mathcal{P}'$ is less than the one for $\mathcal{P}$. This is because side A_iA_{i+1} was removed. The newly added sides A_iW and WA_{i+1} do not introduce new side directions. Each one of them is either parallel to a side of $\mathcal{P}$ or lies on the line determined by such a side. The proof is complete.

Number Theory

N1. Determine all pairs (x, y) of integers satisfying the equation

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. If (x, y) is a solution then obviously $x \geq 0$ and $(x, -y)$ is a solution too. For $x = 0$ we get the two solutions $(0, 2)$ and $(0, -2)$.

Now let (x, y) be a solution with $x > 0$; without loss of generality confine attention to $y > 0$. The equation rewritten as

$$2^x(1 + 2^{x+1}) = (y - 1)(y + 1)$$

shows that the factors $y - 1$ and $y + 1$ are even, exactly one of them divisible by 4. Hence $x \geq 3$ and one of these factors is divisible by 2^{x-1} but not by 2^x. So

$$y = 2^{x-1}m + \epsilon, \qquad m \text{ odd}, \qquad \epsilon = \pm 1. \tag{1}$$

Plugging this into the original equation we obtain

$$2^x\left(1 + 2^{x+1}\right) = \left(2^{x-1}m + \epsilon\right)^2 - 1 = 2^{2x-2}m^2 + 2^x m\epsilon,$$

or, equivalently

$$1 + 2^{x+1} = 2^{x-2}m^2 + m\epsilon.$$

Therefore

$$1 - \epsilon m = 2^{x-2}(m^2 - 8). \tag{2}$$

For $\epsilon = 1$ this yields $m^2 - 8 \leq 0$, i.e., $m = 1$, which fails to satisfy (2).

For $\epsilon = -1$ equation (2) gives us

$$1 + m = 2^{x-2}(m^2 - 8) \geq 2(m^2 - 8),$$

implying $2m^2 - m - 17 \leq 0$. Hence $m \leq 3$; on the other hand m cannot be 1 by (2). Because m is odd, we obtain $m = 3$, leading to $x = 4$. From (1) we get $y = 23$. These values indeed satisfy the given equation. Recall that then $y = -23$ is also good. Thus we have the complete list of solutions (x, y): $(0, 2)$, $(0, -2)$, $(4, 23)$, $(4, -23)$.

N2. For $x \in (0,1)$ let $y \in (0,1)$ be the number whose nth digit after the decimal point is the (2^n)th digit after the decimal point of x. Show that if x is rational then so is y.

Solution. Since x is rational, its digits repeat periodically starting at some point. We wish to show that this is also true for the digits of y, implying that y is rational.

Let d be the length of the period of x and let $d = 2^u \cdot v$, where v is odd. There is a positive integer w such that

$$2^w \equiv 1 \pmod{v}.$$

(For instance, one can choose w to be $\varphi(v)$, the value of Euler's function at v.) Therefore

$$2^{n+w} = 2^n \cdot 2^w \equiv 2^n \pmod{v}$$

for each n. Also, for $n \geq u$ we have

$$2^{n+w} \equiv 2^n \equiv 0 \pmod{2^u}.$$

It follows that, for all $n \geq u$, the relation

$$2^{n+w} \equiv 2^n \pmod{d}$$

holds. Thus, for n sufficiently large, the 2^{n+w}th digit of x is in the same spot in the cycle of x as its 2^nth digit, and so these digits are equal. Hence the $(n+w)$th digit of y is equal to its nth digit. This means that the digits of y repeat periodically with period w from some point on, as required.

N3. The sequence $f(1), f(2), f(3), \ldots$ is defined by

$$f(n) = \frac{1}{n}\left(\left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n}\right\rfloor\right),$$

where $\lfloor x\rfloor$ denotes the integer part of x.

(a) Prove that $f(n+1) > f(n)$ infinitely often.

(b) Prove that $f(n+1) < f(n)$ infinitely often.

Solution. Let $g(n) = nf(n)$ for $n \geq 1$ and $g(0) = 0$. We note that, for $k = 1, \ldots, n$,

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 0$$

if k is not a divisor of n and

$$\left\lfloor\frac{n}{k}\right\rfloor - \left\lfloor\frac{n-1}{k}\right\rfloor = 1$$

if k divides n. It therefore follows that if $d(n)$ is the number of positive divisors of $n \geq 1$ then

$$\begin{aligned} g(n) &= \left\lfloor\frac{n}{1}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + \cdots + \left\lfloor\frac{n}{n-1}\right\rfloor + \left\lfloor\frac{n}{n}\right\rfloor \\ &= \left\lfloor\frac{n-1}{1}\right\rfloor + \left\lfloor\frac{n-1}{2}\right\rfloor + \cdots + \left\lfloor\frac{n-1}{n-1}\right\rfloor + \left\lfloor\frac{n-1}{n}\right\rfloor + d(n) \\ &= g(n-1) + d(n). \end{aligned}$$

Hence

$$g(n) = g(n-1) + d(n) = g(n-2) + d(n-1) + d(n) = \cdots = d(1) + d(2) + \cdots + d(n),$$

meaning that

$$f(n) = \frac{d(1) + d(2) + \cdots + d(n)}{n}.$$

In other words, $f(n)$ is equal to the arithmetic mean of $d(1), d(2), \ldots, d(n)$. In order to prove the claims, it is therefore sufficient to show that $d(n+1) > f(n)$ and $d(n+1) < f(n)$ both hold infinitely often.

We note that $d(1) = 1$. For $n > 1$, $d(n) \geq 2$ holds, with equality if and only if n is prime. Since $f(6) = 7/3 > 2$, it follows that $f(n) > 2$ holds for all $n \geq 6$.

Since there are infinitely many primes, $d(n+1) = 2$ holds for infinitely many values of n, and for each such $n \geq 6$ we have $d(n+1) = 2 < f(n)$. This proves claim (b).

To prove (a), notice that the sequence $d(1), d(2), d(3), \ldots$ is unbounded (e. g. $d(2^k) = k+1$ for all k). Hence $d(n+1) > \max\{d(1), d(2), \ldots, d(n)\}$ for infinitely many n. For all such n, we have $d(n+1) > f(n)$. This completes the solution.

N4. Let P be a polynomial of degree $n > 1$ with integer coefficients and let k be any positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, with k pairs of parentheses. Prove that Q has no more than n integer fixed points, i.e. integers satisfying the equation $Q(x) = x$.

Solution. The claim is obvious if every integer fixed point of Q is a fixed point of P itself. For the sequel assume that this is not the case. Take any integer x_0 such that $Q(x_0) = x_0$, $P(x_0) \neq x_0$ and define inductively $x_{i+1} = P(x_i)$ for $i = 0, 1, 2, \ldots$; then $x_k = x_0$.

It is evident that

$$P(u) - P(v) \text{ is divisible by } u - v \text{ for distinct integers } u, v. \tag{1}$$

(Indeed, if $P(x) = \sum a_i x^i$ then each $a_i(u^i - v^i)$ is divisible by $u - v$.) Therefore each term in the chain of (nonzero) differences

$$x_0 - x_1, \quad x_1 - x_2, \quad \ldots, \quad x_{k-1} - x_k, \quad x_k - x_{k+1} \tag{2}$$

is a divisor of the next one; and since $x_k - x_{k+1} = x_0 - x_1$, all these differences have equal absolute values. For $x_m = \min(x_1, \ldots, x_k)$ this means that $x_{m-1} - x_m = -(x_m - x_{m+1})$. Thus $x_{m-1} = x_{m+1} (\neq x_m)$. It follows that consecutive differences in the sequence (2) have opposite signs. Consequently, $x_0, x_1, x_2, \ldots$ is an alternating sequence of two distinct values. In other words, every integer fixed point of Q is a fixed point of the polynomial $P(P(x))$. Our task is to prove that there are at most n such points.

Let a be one of them so that $b = P(a) \neq a$ (we have assumed that such an a exists); then $a = P(b)$. Take any other integer fixed point α of $P(P(x))$ and let $P(\alpha) = \beta$, so that $P(\beta) = \alpha$; the numbers α and β need not be distinct (α can be a fixed point of P), but each of α, β is different from each of a, b. Applying property (1) to the four pairs of integers (α, a), (β, b), (α, b), (β, a) we get that the numbers $\alpha - a$ and $\beta - b$ divide each other, and also $\alpha - b$ and $\beta - a$ divide each other. Consequently

$$\alpha - b = \pm(\beta - a), \qquad \alpha - a = \pm(\beta - b). \tag{3}$$

Suppose we have a plus in both instances: $\alpha - b = \beta - a$ and $\alpha - a = \beta - b$. Subtraction yields $a - b = b - a$, a contradiction, as $a \neq b$. Therefore at least one equality in (3) holds with a minus sign. For each of them this means that $\alpha + \beta = a + b$; equivalently $a + b - \alpha - P(\alpha) = 0$.

Denote $a + b$ by C. We have shown that every integer fixed point of Q other that a and b is a root of the polynomial $F(x) = C - x - P(x)$. This is of course true for a and b as well. And since P has degree $n > 1$, the polynomial F has the same degree, so it cannot have more than n roots. Hence the result.

Comment. The first part of the solution, showing that integer fixed points of any iterate of P are in fact fixed points of the second iterate $P \circ P$ is standard; moreover, this fact has already appeared in contests. We however do not consider this as a major drawback to the problem because the only tricky moment comes up only at the next stage of the reasoning—to apply the divisibility property (1) to points from distinct 2-orbits of P. Yet maybe it would be more appropriate to state the problem in a version involving $k = 2$ only.

N5. Find all integer solutions of the equation

$$\frac{x^7-1}{x-1}=y^5-1.$$

Solution. The equation has no integer solutions. To show this, we first prove a lemma.

Lemma. If x is an integer and p is a prime divisor of $\frac{x^7-1}{x-1}$ then either $p\equiv 1 \pmod 7$ or $p=7$.

Proof. Both x^7-1 and $x^{p-1}-1$ are divisible by p, by hypothesis and by Fermat's little theorem, respectively. Suppose that 7 does not divide $p-1$. Then $\gcd(p-1,7)=1$, so there exist integers k and m such that $7k+(p-1)m=1$. We therefore have

$$x\equiv x^{7k+(p-1)m}\equiv (x^7)^k\cdot(x^{p-1})^m\equiv 1 \pmod p,$$

and so

$$\frac{x^7-1}{x-1}=1+x+\cdots+x^6\equiv 7 \pmod p.$$

It follows that p divides 7, hence $p=7$ must hold if $p\equiv 1 \pmod 7$ does not, as stated. □

The lemma shows that each positive divisor d of $\frac{x^7-1}{x-1}$ satisfies either $d\equiv 0 \pmod 7$ or $d\equiv 1 \pmod 7$.

Now assume that (x,y) is an integer solution of the original equation. Notice that $y-1>0$, because $\frac{x^7-1}{x-1}>0$ for all $x\neq 1$. Since $y-1$ divides $\frac{x^7-1}{x-1}=y^5-1$, we have $y\equiv 1 \pmod 7$ or $y\equiv 2 \pmod 7$ by the previous paragraph. In the first case, $1+y+y^2+y^3+y^4\equiv 5 \pmod 7$, and in the second $1+y+y^2+y^3+y^4\equiv 3 \pmod 7$. Both possibilities contradict the fact that the positive divisor $1+y+y^2+y^3+y^4$ of $\frac{x^7-1}{x-1}$ is congruent to 0 or 1 modulo 7. So the given equation has no integer solutions.

N6. Let $a > b > 1$ be relatively prime positive integers. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimal possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by = c.$$

An integer c is called a *local champion* if $w(c) \geq w(c \pm a)$ and $w(c) \geq w(c \pm b)$.

Find all local champions and determine their number.

Solution. Call the pair of integers (x, y) a *representation* of c if $ax + by = c$ and $|x| + |y|$ has the smallest possible value, i.e. $|x| + |y| = w(c)$.

We characterise the local champions by the following three observations.

Lemma 1. If (x, y) a representation of a local champion c then $xy < 0$.

Proof. Suppose indirectly that $x \geq 0$ and $y \geq 0$ and consider the values $w(c)$ and $w(c + a)$. All representations of the numbers c and $c + a$ in the form $au + bv$ can be written as

$$c = a(x - kb) + b(y + ka), \qquad c + a = a(x + 1 - kb) + b(y + ka)$$

where k is an arbitrary integer.

Since $|x| + |y|$ is minimal, we have

$$x + y = |x| + |y| \leq |x - kb| + |y + ka|$$

for all k. On the other hand, $w(c + a) \leq w(c)$, so there exists a k for which

$$|x + 1 - kb| + |y + ka| \leq |x| + |y| = x + y.$$

Then

$$(x + 1 - kb) + (y + ka) \leq |x + 1 - kb| + |y + ka| \leq x + y \leq |x - kb| + |y + ka|.$$

Comparing the first and the third expressions, we find $k(a - b) + 1 \leq 0$ implying $k < 0$. Comparing the second and fourth expressions, we get $|x + 1 - kb| \leq |x - kb|$, therefore $kb > x$; this is a contradiction.

If $x, y \leq 0$ then we can switch to $-c$, $-x$ and $-y$. □

From this point, write $c = ax - by$ instead of $c = ax + by$ and consider only those cases where x and y are nonzero and have the same sign. By Lemma 1, there is no loss of generality in doing so.

Lemma 2. Let $c = ax - by$ where $|x| + |y|$ is minimal and x, y have the same sign. The number c is a local champion if and only if $|x| < b$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$.

Proof. Without loss of generality we may assume $x, y > 0$.

The numbers $c - a$ and $c + b$ can be written as

$$c - a = a(x - 1) - by \quad \text{and} \quad c + b = ax - b(y - 1)$$

and trivially $w(c - a) \leq (x - 1) + y < w(c)$ and $w(c + b) \leq x + (y - 1) < w(c)$ in all cases.

Now assume that c is a local champion and consider $w(c + a)$. Since $w(c + a) \leq w(c)$, there exists an integer k such that

$$c + a = a(x + 1 - kb) - b(y - ka) \quad \text{and} \quad |x + 1 - kb| + |y - ka| \leq x + y.$$

This inequality cannot hold if $k \leq 0$, therefore $k > 0$. We prove that we can choose $k = 1$.

Consider the function $f(t) = |x+1-bt| + |y-at| - (x+y)$. This is a convex function and we have $f(0) = 1$ and $f(k) \le 0$. By Jensen's inequality, $f(1) \le \left(1-\frac{1}{k}\right)f(0) + \frac{1}{k}f(k) < 1$. But $f(1)$ is an integer. Therefore $f(1) \le 0$ and

$$|x+1-b| + |y-a| \le x+y.$$

Knowing $c = a(x-b) - b(y-a)$, we also have

$$x+y \le |x-b| + |y-a|.$$

Combining the two inequalities yields $|x+1-b| \le |x-b|$ which is equivalent to $x < b$.

Considering $w(c-b)$, we obtain similarly that $y < a$.

Now $|x-b| = b-x$, $|x+1-b| = b-x-1$ and $|y-a| = a-y$, therefore we have

$$(b-x-1) + (a-y) \le x+y \le (b-x) + (a-y),$$
$$\frac{a+b-1}{2} \le x+y \le \frac{a+b}{2}.$$

Hence $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$.

To prove the opposite direction, assume $0 < x < b$ and $x+y = \left\lfloor \frac{a+b}{2} \right\rfloor$. Since $a > b$, we also have $0 < y < a$. Then

$$w(c+a) \le |x+1-b| + |y-a| = a+b-1-(x+y) \le x+y = w(c)$$

and

$$w(c-b) \le |x-b| + |y+1-a| = a+b-1-(x+y) \le x+y = w(c)$$

therefore c is a local champion indeed. □

Lemma 3. Let $c = ax - by$ and assume that x and y have the same sign, $|x| < b$, $|y| < a$ and $|x| + |y| = \left\lfloor \frac{a+b}{2} \right\rfloor$. Then $w(c) = x+y$.

Proof. By definition $w(c) = \min\{|x-kb| + |y-ka| : \ k \in \mathbb{Z}\}$. If $k \le 0$ then obviously $|x-kb| + |y-ka| \ge x+y$. If $k \ge 1$ then

$$|x-kb| + |y-ka| = (kb-x) + (ka-y) = k(a+b) - (x+y) \ge (2k-1)(x+y) \ge x+y.$$

Therefore $w(c) = x+y$ indeed. □

Lemmas 1, 2 and 3 together yield that the set of local champions is

$$C = \left\{ \pm(ax-by) : \ 0 < x < b, \ x+y = \left\lfloor \frac{a+b}{2} \right\rfloor \right\}.$$

Denote by C^+ and C^- the two sets generated by the expressions $+(ax-by)$ and $-(ax-by)$, respectively. It is easy to see that both sets are arithmetic progressions of length $b-1$, with difference $a+b$.

If a and b are odd, then $C^+ = C^-$, because $a(-x) - b(-y) = a(b-x) - b(a-y)$ and $x+y = \frac{a+b}{2}$ is equivalent to $(b-x) + (a-y) = \frac{a+b}{2}$. In this case there exist $b-1$ local champions.

If a and b have opposite parities then the answer is different. For any $c_1 \in C^+$ and $c_2 \in C^-$,

$$2c_1 \equiv -2c_2 \equiv 2\left(a\frac{a+b-1}{2} - b \cdot 0\right) \equiv -a \pmod{a+b}$$

and

$$2c_1 - 2c_2 \equiv -2a \pmod{a+b}.$$

The number $a+b$ is odd and relatively prime to a, therefore the elements of C^+ and C^- belong to two different residue classes modulo $a+b$. Hence, the set C is the union of two disjoint arithmetic progressions and the number of all local champions is $2(b-1)$.

So the number of local champions is $b-1$ if both a and b are odd and $2(b-1)$ otherwise.

Comment. The original question, as stated by the proposer, was:

(a) Show that there exists only finitely many local champions;

(b) Show that there exists at least one local champion.

N7. Prove that, for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.

Solution. We will prove by induction on d that, for every positive integer N, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that, for each $i = 0, 1, 2, \ldots, d-1$, we have $b_i > N$ and

$$2^{b_i} + b_i \equiv i \pmod{d}.$$

This yields the claim for $m = b_0$.

The base case $d = 1$ is trivial. Take an $a > 1$ and assume that the statement holds for all $d < a$. Note that the remainders of 2^i modulo a repeat periodically starting with some exponent M. Let k be the length of the period; this means that $2^{M+k'} \equiv 2^M \pmod{a}$ holds only for those k' which are multiples of k. Note further that the period cannot contain all the a remainders, since 0 either is missing or is the only number in the period. Thus $k < a$.

Let $d = \gcd(a, k)$ and let $a' = a/d$, $k' = k/d$. Since $0 < k < a$, we also have $0 < d < a$. By the induction hypothesis, there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ such that $b_i > \max(2^M, N)$ and

$$2^{b_i} + b_i \equiv i \pmod{d} \quad \text{for} \quad i = 0, 1, 2, \ldots, d-1. \tag{1}$$

For each $i = 0, 1, \ldots, d-1$ consider the sequence

$$2^{b_i} + b_i, \;\; 2^{b_i+k} + (b_i + k), \;\; \ldots, \;\; 2^{b_i+(a'-1)k} + (b_i + (a'-1)k). \tag{2}$$

Modulo a, these numbers are congruent to

$$2^{b_i} + b_i, \;\; 2^{b_i} + (b_i + k), \;\; \ldots, \;\; 2^{b_i} + (b_i + (a'-1)k),$$

respectively. The d sequences contain $a'd = a$ numbers altogether. We shall now prove that no two of these numbers are congruent modulo a.

Suppose that

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{a} \tag{3}$$

for some values of $i, j \in \{0, 1, \ldots, d-1\}$ and $m, n \in \{0, 1, \ldots, a'-1\}$. Since d is a divisor of a, we also have

$$2^{b_i} + (b_i + mk) \equiv 2^{b_j} + (b_j + nk) \pmod{d}.$$

Because d is a divisor of k and in view of (1), we obtain $i \equiv j \pmod{d}$. As $i, j \in \{0, 1, \ldots, d-1\}$, this just means that $i = j$. Substituting this into (3) yields $mk \equiv nk \pmod{a}$. Therefore $mk' \equiv nk' \pmod{a'}$; and since a' and k' are coprime, we get $m \equiv n \pmod{a'}$. Hence also $m = n$.

It follows that the a numbers that make up the d sequences (2) satisfy all the requirements; they are certainly all greater than N because we chose each $b_i > \max(2^M, N)$. So the statement holds for a, completing the induction.