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Prescribing Webster scalar curvature on CR manifolds of negative conformal invariants

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Abstract

In this paper, we are interested in solving the following partial differential equation

$$-\Delta_\theta u + Ru = f u^{1+2/n}$$

on a compact strictly pseudo-convex CR manifold (M, θ) of dimension $2n+1$ with $n \geqslant 1$. This problem naturally arises when solving the prescribing Webster scalar curvature problem on M with the prescribed function f. Using variational techniques, we prove several non-existence, existence, and multiplicity results when the function f is sign-changing.

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Contents

1. Introduction

The problem of finding a conformal metric on a manifold with certain prescribed curvature function has been extensively studied during the last few decades. A typical model is the prescribing scalar curvature problem on closed Riemannian manifolds (i.e. compact without boundary). More precisely, let (M, g) be an n-dimensional closed manifold with $n \geqslant 3$. A conformal change of metrics, say $\widetilde{g} = u^{4/(n-2)} g$, of the background metric g admits the following scalar curvature

$$\mathrm{Scal}_{\widetilde{g}} = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_g u + \mathrm{Scal}_g\, u \right)$$

where $\Delta_g = \mathrm{div}(\nabla)$ is the Laplace–Beltrami operator with respect to the metric g and Scal_g is the scalar curvature of the metric g. For a given smooth function f, it is immediately to see that the problem of solving $\mathrm{Scal}_{\widetilde{g}} = f$ is equivalent to solving the following partial differential equation

$$-\frac{4(n-1)}{n-2} \Delta_g u + \mathrm{Scal}_g\, u = f u^{\frac{n+2}{n-2}} \quad \text{on } M \tag{1.1}$$

for $u > 0$. Clearly, this problem includes the well-known Yamabe problem as a special case when the candidate function f is constant. While the Yamabe problem had already been settled down by a series of seminal works due to Yamabe, Trudinger, Aubin, and Schoen, Eq. (1.1) in its generic form remains open, see [1]. Since Eq. (1.1) is conformal invariant, when solving (1.1), one often uses the so-called Yamabe invariant to characterize the catalogue of possible metrics

g which eventually helps us to fix a sign for Scal_g which depends on the sign of the Yamabe invariant.

While the case of positive Yamabe invariants remains less understood especially when (M, g) is the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$, more or less the case of non-positive Yamabe invariants is well-understood by a series of works due to Kadzan–Warner, Ouyang, Rauzy, see [29–31] and the references therein. Intuitively, when the background metric g is of negative Yamabe invariant, i.e. $\mathrm{Scal}_g < 0$, the condition $\int_M f\,dv_g < 0$ is necessary. Clearly, the most interesting case in this catalogue is the case when f changes sign and $\int_M f\,dv_g < 0$. In literature, there are two different routes that have been used to solve (1.1). The first set of works is based on the geometric implementation of the problem where one fixes Scal_g and tries to find conditions for the candidate f, for example a work by Rauzy [31]. In [31], it was proved by variational techniques that when the set $\{x \in M : f(x) \geqslant 0\}$ has positive measure, Scal_g cannot be too negative. In fact, $|\mathrm{Scal}_g|$ is bounded from above by some number λ_f, depending only on the set $\{x \in M : f(x) \geqslant 0\}$, which can be characterized by the following variational problem

$$\lambda_f = \begin{cases} \inf\limits_{u\in\mathscr{A}} \dfrac{\int_M |\nabla u|_g^2\,d\mu_g}{\int_M u^2\,d\mu_g}, & \text{if } \mathscr{A} \neq \emptyset, \\ +\infty, & \text{if } \mathscr{A} = \emptyset, \end{cases} \tag{1.2}$$

here $f^{\pm} = \max\{\pm f, 0\}$ and

$$\mathscr{A} = \left\{u \in H^1(M) : u \geqslant 0, u \not\equiv 0, \int_M |f^-| u\,d\mu_g = 0\right\}.$$

In addition, it was proved in [31] that if $\sup_M f^+$ is small enough compared with f^-, Eq. (1.1) admits at least one positive smooth solution. In the second route, one can free (1.1) from geometry and fix f instead of Scal_g, for example two works by Ouyang [29,30]. In these works, using bifurcation method, Ouyang proved, among other things, that depending on how small $|\mathrm{Scal}_g|$ is Eq. (1.1) always admits either one or two positive smooth solutions. As far as we know, this is the first multiplicity result for (1.1) when $\mathrm{Scal}_g < 0$.

As a natural analogue of the prescribed scalar curvature problem for the CR geometry, one can consider the prescribed Webster (pseudo-hermitian) scalar curvature problem on compact CR manifolds which can be formulated as follows. Let (M, θ) be a compact strictly pseudo-convex CR manifold without boundary of real dimension $2n + 1$ with $n \geqslant 1$. Given any smooth function h on M, it is natural to ask: *Does there exist a contact form $\widehat{\theta}$ conformally related to θ in the sense that $\widehat{\theta} = u^{2/n}\theta$ for some smooth function $u > 0$ such that h is the Webster scalar curvature of the Webster metric $g_{\widehat{\theta}}$ associated with the contact form $\widehat{\theta}$?* Following the same way as in the Riemannian case, the Webster metric $g_{\widehat{\theta}}$ associated with $\widehat{\theta}$ obeys its scalar curvature which is given by

$$\mathrm{Scal}_{\widehat{\theta}} = u^{-\frac{n+2}{n}} \left(-\frac{2(n+1)}{n} \Delta_\theta u + \mathrm{Scal}_\theta\, u\right),$$

where Δ_θ is the sub-Laplacian with respect to the contact form θ, and Scal_θ is the Webster scalar curvature of the Webster metric g_θ associated with the contact form θ. Clearly, the problem of solving $\mathrm{Scal}_{\widehat{\theta}} = h$ is equivalent to finding positive solutions u to the following PDE

$$-\Delta_\theta u + \frac{n}{2(n+1)} \operatorname{Scal}_\theta u = \frac{n}{2(n+1)} h u^{1+2/n} \quad \text{on } M. \tag{1.3}$$

When h is constant, Eq. (1.3) is known as the CR Yamabe problem. In a series of seminal papers [16–18], Jerison and Lee extensively studied the Yamabe problem on CR manifolds. As always, the works by Jerison and Lee also depend on the sign of following invariant

$$\mu(M,\theta) = \inf_{u \in \mathbf{S}_1^2(M), u \not\equiv 0} \frac{\int_M \left[(2+2/n)|\nabla_\theta u|_\theta^2 + \operatorname{Scal}_\theta u^2\right] \theta \wedge (d\theta)^n}{\left(\int_M |u|^{2+2/n} \theta \wedge (d\theta)^n\right)^{n/(n+1)}},$$

where $\mathbf{S}_1^2(M)$ is the Folland–Stein space, see Section 2 below. Later on, Gamara and Yacoub [13,14] treated the cases left open by Jerison and Lee. On the contrary, to the best of authors' knowledge, only very few results have been established on the prescribed Webster scalar curvature problem, see [9,11,15,23,32], in spit of the vary existing results on its Riemannian analogue, see [2–8,20–22,28] and the references therein. Among them, Refs. [23] and [32] considered the prescribing Webster scalar problem on CR spheres; Ho [15] showed, via a flow method, that Eq. (1.3) has a smooth positive solution if both the Webster scalar curvature $\operatorname{Scal}_\theta$ and the candidate function h are strictly negative.

The primary aim of the paper is to carry the Rauzy and Ouyang results from the context of Riemannian geometry to CR geometry. As such, in this article, we investigate the prescribing Webster scalar curvature problem (1.3) on compact CR manifolds with negative conformal invariants, that is to say $\mu(M,\theta) < 0$. To study (1.3), we mainly follow the Rauzy variational method in [31] plus some modification taken from a recent paper by the first author together with Xu in [25], see also [24,26,27]. Loosely speaking, in [25], they proved some existence and multiplicity results of the Einstein-scalar field Lichnerowicz equations on closed Riemannian manifolds which includes (1.1) as a special case.

Before stating our main results and for the sake of simplicity, let us denote $R = n \operatorname{Scal}_\theta / (2n+2)$ and $f = nh/(2n+2)$. Then we can rewrite (1.3) as follows

$$-\Delta_\theta u + Ru = f u^{1+2/n} \quad \text{on } M. \tag{1.4}$$

Our main results are included in the three theorems below. First, we obtain the following existence result when f changes sign.

Theorem 1.1. *Let (M,θ) be a compact strictly pseudo-convex CR manifold with a negative conformal invariant of dimension $2n+1$ with $n \geqslant 1$. Suppose that f is smooth function on M satisfying $\int_M f \;\theta \wedge (d\theta)^n < 0$, $\sup_M f > 0$, and $|R| < \lambda_f$, where λ_f is given in (2.1) below. Then:*

(a) *There exists a constant $\mathcal{C}_1 > 0$ depending only on f^- which is given by (4.1) below such that if*

$$(\sup_M f^+) \left(\int_M |f^-| \; \theta \wedge (d\theta)^n \right)^{-1} < \mathcal{C}_1, \tag{1.5}$$

then Eq. (1.4) possesses at least one smooth positive solution; and

(b) *if we suppose further that*

$$(\sup_M f^+)\left(\int_M f\Phi^N \theta \wedge (d\theta)^n\right)^{-1} < \mathcal{C}_2, \tag{1.6}$$

and that

$$\int_M f\Phi^N \theta \wedge (d\theta)^n > 0 \tag{1.7}$$

for some smooth positive function Φ *in* M *and some positive constant* $\mathcal{C}_2$*, given in* (4.11) *below, depending on* Φ*, then Eq.* (1.4) *possesses at least two smooth positive solutions. In addition, if the function* Φ *satisfies*

$$\|\nabla_\theta \Phi\|_2^2 \, \|\Phi\|_N^{-2} \leqslant \frac{n}{n+1}, \tag{1.8}$$

and $\|\Phi\|_N \leqslant 1$ *then the constant* $\mathcal{C}_2$ *is independent of* Φ*.*

(c) *However, for given* f*, the condition* $|R| < \lambda_f$ *is not sufficient for the solvability of* (1.4) *in the following sense: Given any smooth function* f *and constant* $R < 0$ *with* $\sup_M f > 0$*,* $\int_M f\,\theta \wedge (d\theta)^n < 0$*, and* $|R| < \lambda_f$*, there exists a new continuous function* h *such that* $\sup_M h > 0$*,* $\int_M h\,\theta \wedge (d\theta)^n < 0$*, and* $\lambda_h > \lambda_f$ *but Eq.* (1.4) *with* f *replaced by* h *has no solution.*

Then, in the next result, we focus our attention on the case when $f \leqslant 0$. Although the condition $|R| < \lambda_f$ is not sufficient in the case $\sup_M f > 0$, nevertheless, in the case $\sup_M f = 0$, we are able to show that $|R| < \lambda_f$ is sufficient, thus obtaining necessary and sufficient conditions for the solvability of (1.4). To be exact, we shall prove the following theorem.

Theorem 1.2. *Let* (M, θ) *be a compact strictly pseudo-convex CR manifold with a negative conformal invariant of dimension* $2n+1$ *with* $n \geqslant 1$*. Suppose that* f *is a smooth non-positive function on* M *such that the set* $\{x \in M : f(x) = 0\}$ *has positive measure. Then Eq.* (1.4) *has a unique smooth positive solution if and only if* $|R| < \lambda_f$*.*

Finally, we show that once the function f having $\sup_M f > 0$ and $\int_M f\,\theta \wedge (d\theta)^n < 0$ is fixed and if $|R|$ is sufficiently small, then Eq. (1.4) always has positive smooth solutions. The following theorem is the content of this conclusion.

Theorem 1.3. *Let* (M, θ) *be a compact strictly pseudo-convex CR manifold with a negative conformal invariant of dimension* $2n+1$ *with* $n \geqslant 1$*. Suppose that* f *is smooth function on* M *satisfying* $\int_M f \;\theta \wedge (d\theta)^n < 0$*,* $\sup_M f > 0$*. Then, there exists a positive constant* $\mathcal{C}_3$ *given in* (5.4) *below such that if* $|R| < \mathcal{C}_3$*, there Eq.* (1.4) *admits at least one smooth positive solution.*

Let us now briefly mention the organization of the paper. Section 2 consists of preliminaries and notation. Also in this section, two necessary conditions for the solvability of Eq. (1.4) are also derived. In Section 3, we perform a careful analysis for the energy functional associated to (1.4).

Having all these preparation, we prove Theorem 1.1(a)–(b) in Section 4 while Theorems 1.2, 1.3, and 1.1(c) will be proved in Section 5. Finally, we put some basic and useful results in Appendices A, B, and C.

2. Notations and necessary conditions

To start this section, we first collect some well-known facts from CR geometry, for interested reader, we refer to [10].

As mentioned earlier, by M we mean an orientable CR manifold without boundary of CR dimension n. This is also equivalent to saying that M is an orientable differentiable manifold of real dimension $2n+1$ endowed with a pair $(H(M), J)$ where $H(M)$ is a subbundle of the tangent bundle $T(M)$ of real rank $2n$ and J is an integrable complex structure on $H(M)$. Since M is orientable, there exists a 1-form θ called pseudo-Hermitian structure on M. Then, we can associate each structure θ to a bilinear form G_θ, called Levi form, which is defined only on $H(M)$ by

$$G_\theta(X,Y) = -(d\theta)(JX,Y) \quad \forall X,Y \in H(M).$$

Since G_θ is symmetric and J-invariant, we then call (M,θ) strictly pseudo-convex CR manifold if the Levi form G_θ associated with the structure θ is positive definite. The structure θ is then a contact form which immediately induces on M the volume form $\theta \wedge (d\theta)^n$.

Moreover, θ on a strictly pseudo-convex CR manifold (M,θ) also determines a "normal" vector field T on M, called the Reeb vector field of θ. Via the Reeb vector field T, one can extend the Levi form G_θ on $H(M)$ to a semi-Riemannian metric g_θ on $T(M)$, called the Webster metric of (M,θ). Let

$$\pi_H : T(M) \to H(M)$$

be the projection associated to the direct sum $T(M) = H(M) \oplus \mathbb{R}T$. Now, with the structure θ, we can construct a unique affine connection ∇, called the Tanaka–Webster connection on $T(M)$. Using ∇ and π_H, we can define the "horizontal" gradient ∇_θ by

$$\nabla_\theta u = \pi_H \nabla u.$$

Again, using the connection ∇ and the projection π_H, one can define the sub-Laplacian Δ_θ acting on a C^2-function u via

$$\Delta_\theta u = \operatorname{div}(\pi_H \nabla u).$$

Here ∇u is the ordinary gradient of u with respect to g_θ which can be written as $g_\theta(\nabla u, X) = X(u)$ for any X. Then integration by parts gives

$$\int_M (\Delta_\theta u) f \,\theta \wedge (d\theta)^n = -\int_M \langle \nabla_\theta u, \nabla_\theta f\rangle_\theta \,\theta \wedge (d\theta)^n$$

for any smooth function f. In the preceding formula, $\langle , \rangle_\theta$ denotes the inner product via the Levi form G_θ (or the Webster metric g_θ since both $\nabla_\theta u$ and $\nabla_\theta v$ are horizontal). When $u \equiv v$, we sometimes simply write $|\nabla_\theta u|^2$ instead of $\langle \nabla_\theta u, \nabla_\theta u\rangle_\theta$.

Having ∇ and g_θ in hand, one can talk about the curvature theory such as the curvature tensor fields, the pseudo-Hermitian Ricci and scalar curvature. Having all these, we denote by Scal_θ the pseudo-Hermitian scalar curvature associated with the Webster metric g_θ and the connection ∇, called the Webster scalar curvature, see [10, Proposition 2.9]. At the very beginning, since we assume $\mu(M, \theta) < 0$, we may further assume without loss of generality that Scal_θ is a negative constant and that

$$\mathrm{vol}(M, \theta) = \int_M \theta \wedge (d\theta)^n = 1$$

since there always exists such a metric in the conformal class of θ. In particular, $R < 0$ is constant.

In the context of CR manifolds, instead of using the standard Sobolev space $H^1(M)$, we find solutions of (1.4) in the so-called Folland–Stein space $\mathbf{S}_1^2(M)$ which is the completion of $C^\infty(M)$ with respect to the norm

$$\|u\| = \left(\int_M |\nabla_\theta u|^2 \ \theta \wedge (d\theta)^n + \int_M |u|^2 \ \theta \wedge (d\theta)^n\right)^{1/2}.$$

For notational simplicity, we simply denote by $\|\cdot\|_p$ and $\|\cdot\|_{\mathbf{S}_1^2(M)}$ the norms in $L^p(M)$ and $\mathbf{S}_1^2(M)$ respectively. Besides, the following dimensional constants

$$N = 2 + \frac{2}{n}, \quad 2^\flat = 2 + \frac{1}{n}$$

will also be used in the rest of the paper. Suppose that f is a smooth function on M and as before by $f^\pm$ we mean $f^- = \inf(f, 0)$ and $f^+ = \sup(f, 0)$. Similar to (1.2), we also define

$$\lambda_f = \begin{cases} \inf\limits_{u \in \mathscr{A}} \dfrac{\int_M |\nabla_\theta u|^2 \theta \wedge (d\theta)^n}{\int_M u^2 \ \theta \wedge (d\theta)^n}, & \text{if } \mathscr{A} \neq \emptyset, \\ +\infty, & \text{if } \mathscr{A} = \emptyset, \end{cases} \tag{2.1}$$

where the set $\mathscr{A}$ is now given as follows

$$\mathscr{A} = \left\{u \in \mathbf{S}_1^2(M) : u \geqslant 0, u \not\equiv 0, \int_M |f^-| u \ \theta \wedge (d\theta)^n = 0\right\}.$$

Since we are interested in the critical case, throughout this paper, we always assume $q \in (2^\flat, N)$. Moreover, we will use the following Sobolev inequality

$$\|u\|_N^2 \leqslant \mathcal{K}_1 \|\nabla u\|_2^2 + \mathcal{A}_1 \|u\|_2^2. \tag{2.2}$$

If we denote $\mathfrak{C} = \mathcal{K}_1 + \mathcal{A}_1$, then we obtain from (2.2) the following simpler Sobolev inequality

$$\|u\|_N^2 \leqslant \mathfrak{C}\|u\|_{\mathbf{S}_1^2(M)}^2. \tag{2.3}$$

Notice that $\mathcal{K}_1$ may not be the best Sobolev constant for the embedding $\mathbf{S}_1^2(M) \hookrightarrow L^N(M)$. If the manifold is the Heisenberg group or CR spheres, then the best constant has been found by Jerison and Lee in [18] (see also [12]). However, for generic CR manifolds, we have not seen any proof of the best Sobolev constant. Hence, in the present paper, it is safe to use the inequality (2.3).

2.1. A necessary condition for f

The aim of this subsection is to derive a necessary condition for f so that Eq. (1.4) admits a positive smooth solution.

Proposition 2.1. *Suppose that Eq. (1.4) has a positive smooth solution then* $\int_M f\ \theta \wedge (d\theta)^n < 0$.

Proof. Assume that $u > 0$ is a smooth solution of (1.4). By multiplying both sides of (1.4) by u^{1-N}, integrating over M and the fact that $R < 0$, we obtain

$$\int\limits_M (-\Delta_\theta u) u^{1-N}\ \theta \wedge (d\theta)^n > \int\limits_M f\ \theta \wedge (d\theta)^n.$$

It follows from the divergence theorem that

$$\int\limits_M (-\Delta_\theta u) u^{1-N}\ \theta \wedge (d\theta)^n = (1-N) \int\limits_M u^{-N} |\nabla_\theta u|^2\ \theta \wedge (d\theta)^n.$$

This equality and the fact that $N > 2$ imply that $\int_M f\ \theta \wedge (d\theta)^n < 0$ as claimed. □

2.2. A necessary condition for R

In this subsection, we show that the condition $|R| < \lambda_f$ is necessary if $\lambda_f < +\infty$ in order for Eq. (1.4) to have positive smooth solutions. As in [25], our proof makes use of a Picone type identity as follows

Lemma 2.2. *Assume that* $v \in \mathbf{S}_1^2(M)$ *with* $v \geqslant 0$ *and* $v \not\equiv 0$. *Let* $u > 0$ *be a smooth function. Then we have*

$$\int\limits_M |\nabla_\theta v|^2\ \theta \wedge (d\theta)^n = -\int\limits_M \frac{\Delta_\theta u}{u} v^2\ \theta \wedge (d\theta)^n + \int\limits_M u^2 \left|\nabla_\theta \left(\frac{v}{u}\right)\right|^2\ \theta \wedge (d\theta)^n.$$

Proof. It follows from density, integration by parts, and a direct computation. We omit the detail and refer the reader to [24] for a detailed proof in the context of Riemannian manifolds. □

Proposition 2.3. *If Eq. (1.4) has a positive smooth solution, then it is necessary to have* $|R| < \lambda_f$.

Proof. We only need to consider the case $\lambda_f < +\infty$ since otherwise it is trivial. Choose an arbitrary $v \in \mathscr{A}$ and assume that u is a positive smooth solution to (1.4). Then it follows from Lemma 2.2 and (1.4) that

$$\begin{aligned}\int_M |\nabla_\theta v|^2\ \theta\wedge(d\theta)^n &= -\int_M \frac{\Delta_\theta u}{u} v^2\ \theta\wedge(d\theta)^n + \int_M u^2\Big|\nabla_\theta\Big(\frac{v}{u}\Big)\Big|^2\ \theta\wedge(d\theta)^n\\ &= |R|\int_M v^2\ \theta\wedge(d\theta)^n + \int_M f u^{N-2} v^2\ \theta\wedge(d\theta)^n\\ &\quad + \int_M u^2\Big|\nabla_\theta\Big(\frac{v}{u}\Big)\Big|^2\ \theta\wedge(d\theta)^n\\ &\geqslant |R|\int_M v^2\ \theta\wedge(d\theta)^n + \int_M u^2\Big|\nabla_\theta\Big(\frac{v}{u}\Big)\Big|^2\ \theta\wedge(d\theta)^n.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\Big(\int_M |\nabla_\theta v|^2\ \theta\wedge(d\theta)^n\Big)\Big(\int_M v^2\ \theta\wedge(d\theta)^n\Big)^{-1}\\ &\quad\geqslant |R| + \Big(\int_M u^2\Big|\nabla_\theta\big(\frac{v}{u}\big)\Big|^2\ \theta\wedge(d\theta)^n\Big)\Big(\int_M v^2\ \theta\wedge(d\theta)^n\Big)^{-1},\end{aligned}\tag{2.4}$$

which implies by the definition of λ_f that $\lambda_f \geqslant |R| > 0$. Observe that $v/u \in \mathscr{A}$. Then we have

$$\begin{aligned}&\Big(\int_M u^2\Big|\nabla_\theta\big(\frac{v}{u}\big)\Big|^2\ \theta\wedge(d\theta)^n\Big)\Big(\int_M v^2\ \theta\wedge(d\theta)^n\Big)^{-1}\\ &= \Big(\int_M u^2\Big|\nabla_\theta\big(\frac{v}{u}\big)\Big|^2\ \theta\wedge(d\theta)^n\Big)\Big(\int_M u^2\big(\frac{v}{u}\big)^2\ \theta\wedge(d\theta)^n\Big)^{-1}\\ &\geqslant \Big(\frac{\inf u}{\sup u}\Big)^2\Big(\int_M \Big|\nabla_\theta\big(\frac{v}{u}\big)\Big|^2\ \theta\wedge(d\theta)^n\Big)\Big(\int_M \big(\frac{v}{u}\big)^2\ \theta\wedge(d\theta)^n\Big)^{-1}\\ &\geqslant \lambda_f\Big(\frac{\inf u}{\sup u}\Big)^2.\end{aligned}\tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\lambda_f \geqslant |R| + \lambda_f\Big(\frac{\inf u}{\sup u}\Big)^2.$$

The estimate above and the fact $\lambda_f > 0$ gives us the desired result. $\square$

3. The analysis of the energy functionals

As a fist step to tackle (1.4), we consider the following subcritical problem

$$-\Delta_\theta u + Ru = f u^{q-1}. \tag{3.1}$$

Our main purpose is to show the limit exists as $q \to N$ under some assumptions. It is well known that the energy functional associated with problem (3.1) is given by

$$F_q(u) = \frac{1}{2}\int\limits_M |\nabla_\theta u|^2 \theta \wedge (d\theta)^n + \frac{R}{2}\int\limits_M u^2 \theta \wedge (d\theta)^n - \frac{1}{q}\int\limits_M f u^q \ \theta \wedge (d\theta)^n,$$

where u is a function that belongs to the set

$$\mathscr{B}_{k,q} = \left\{u \in \mathbf{S}_1^2(M) : u \geqslant 0, \|u\|_q = k^{1/q}\right\}.$$

Note that $\mathscr{B}_{k,q}$ is not empty since $k^{1/q} \in \mathscr{B}_{k,q}$, hence we can set

$$\mu_{k,q} = \inf_{u\in\mathscr{B}_{k,q}} F_q(u).$$

It is not hard to see, by the Hölder inequality, that $F_q(u) \geqslant Rk^{2/q}/2 - (\sup_M f)k/q$ for any $u \in \mathscr{B}_{k,q}$. Hence

$$\mu_{k,q} \geqslant \frac{R}{2}k^{2/q} - \frac{k}{q}\sup_M f, \tag{3.2}$$

which implies that $\mu_{k,q} > -\infty$ so long as k is finite. On the other hand, using the test function $u = k^{1/q}$, we further obtain

$$\mu_{k,q} \leqslant \frac{R}{2}k^{\frac{2}{q}} - \frac{k}{q}\int\limits_M f \ \theta \wedge (d\theta)^n, \tag{3.3}$$

which implies that $\mu_{k,q} < +\infty$.

3.1. $\mu_{k,q}$ is achieved

In this subsection, we show that if k and q are fixed, then $\mu_{k,q}$ is achieved by some smooth function, say u_q. Indeed, let $(u_j)_j$ be a minimizing sequence for $\mu_{k,q}$ in $\mathscr{B}_{k,q}$. Then the Hölder inequality yields $\|u_j\|_2 \leqslant k^{1/q}$, and since $F_q(u_j) \leqslant \mu_{k,q} + 1$ for sufficiently large j, we arrive at

$$\frac{1}{2}\|\nabla u_j\|_2^2 \leqslant \mu_{k,q} + 1 + \frac{k}{q}\sup_M f - \frac{R}{2}k^{2/q}.$$

Hence, the sequence $(u_j)_j$ is bounded in $\mathbf{S}_1^2(M)$. By the Sobolev embedding theorem, up to a subsequence, there exists $u_q \in \mathbf{S}_1^2(M)$ such that

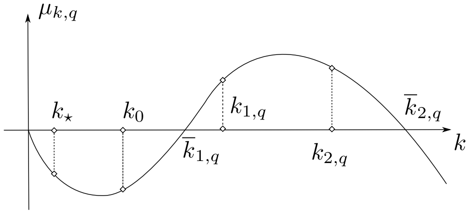

Fig. 1. The asymptotic behavior of $\mu_{k,q}$ when $\sup_M f > 0$.

- $u_j \rightharpoonup u_q$ weakly in $\mathbf{S}_1^2(M)$, and
- $u_j \to u_q$ strongly in $L^q(M)$.

This shows that $u_q \geqslant 0$ and $\|u_q\|_q^q = k$. In particular, we have just shown that $u_q \in \mathscr{B}_{k,q}$. Since F_q is weakly lower semi-continuous, we also get $\mu_{k,q} = \lim_{j\to+\infty} F_q(u_j) \geqslant F_q(u_q)$. This and the fact that $\mu_{k,q} \in \mathscr{B}_{k,q}$ thus showing that $\mu_{k,q} = F_q(u_q)$. We are only left to show the smoothness and positivity of u_q. The standard regularity theorem and maximum principle show that $u_q \in C^\infty(M)$ and $u_q > 0$, see for example [16, Theorem 5.15].

3.2. Asymptotic behavior of $\mu_{k,q}$

In this subsection, we will describe the asymptotic behavior of $\mu_{k,q}$ as k varies which can be illustrated in Fig. 1.

First, we study $\mu_{k,q}$ when k is small. Obviously, when $k = 0$, we easily see that $\mu_{0,q} = 0$. When $k > 0$ and small, we obtain the following result.

Lemma 3.1. *If $\sup_M f \geqslant 0$, then there exist k_0 such that $\mu_{k,q} < 0$, for all $0 < k \leqslant k_0$. Moreover, there is a positive number $k_\star < 1$ independent of q with $k_\star < k_0$ such that $\mu_{k_0,q} < \mu_{k_\star,q} < 0$.*

Proof. First, we solve the following equation

$$\frac{1}{2} R k_0^{2/q} - \frac{k_0}{q} \int\limits_M f \ \theta \wedge (d\theta)^n = \frac{1}{4} R k_0^{2/q}$$

to obtain

$$k_0 = \left(\frac{q|R|}{4\left|\int_M f \ \theta \wedge (d\theta)^n\right|} \right)^{q/(q-2)}.$$

It is not hard to see that for such choice of k_0, we have $\mu_{k,q} < 0$ for all $0 < k \leqslant k_0$. Observe that

$$k_0 \geqslant \left(\frac{|R|}{2\left(\left|\int_M f \ \theta \wedge (d\theta)^n\right| + |R|\right)} \right)^{2^\flat/(2^\flat-2)}.$$

Then, it follows from (3.3) and $R < 0$ that

$$\mu_{k_0,q} < \frac{R}{8} k_0^{2/q} \leqslant \frac{R}{8} \left(\frac{|R|}{2\left(\left|\int_M f \ \theta \wedge (d\theta)^n\right| + |R|\right)} \right)^{2/(2^\flat-2)}. \tag{3.4}$$

Keep in mind that $2/(2^\flat - 2) = 2n$. Now, let $k < 1$ solve the following inequality

$$\frac{R}{2}k^{2/q} - \frac{k}{q}\sup_M f \geqslant \frac{R}{8}\left(\frac{|R|}{2\left(\left|\int_M f\ \theta\wedge(d\theta)^n\right| + |R|\right)}\right)^{2n}, \tag{3.5}$$

which is equivalent to solving

$$\begin{aligned}\frac{|R|}{8}\left(\frac{|R|}{2\left(\left|\int_M f\ \theta\wedge(d\theta)^n\right| + |R|\right)}\right)^{2n} &\geqslant \frac{|R|}{2}k^{2/q} + \frac{k}{q}\sup_M f\\ &\geqslant \frac{|R|}{N}k + \frac{k}{N}\sup_M f,\end{aligned}$$

where we have used the fact that $q > 2$ and $k < 1$. Hence, we have

$$k \leqslant \frac{N|R|}{8(|R| + \sup_M f)}\left(\frac{|R|}{2\left(\left|\int_M f\ \theta\wedge(d\theta)^n\right| + |R|\right)}\right)^{2n}. \tag{3.6}$$

We then set

$$k_\star = \frac{N|R|}{16(|R| + \sup_M f)}\left(\frac{|R|}{2\left(\left|\int_M f\ \theta\wedge(d\theta)^n\right| + |R|\right)}\right)^{2n}.$$

Then, thanks to $N \leqslant 4$, clearly $k_\star < 1$ and $k_\star$ is independent of q. In addition, thanks to $2^\flat/(2^\flat - 2) = 2n + 1$, a simple calculation shows that

$$k_\star < \left(\frac{|R|}{2\left(\left|\int_M f\ \theta\wedge(d\theta)^n\right| + |R|\right)}\right)^{2^\flat/(2^\flat-2)} \leqslant k_0,$$

Finally, since $k_\star$ satisfies (3.6), we conclude from (3.2), (3.4) and (3.5) that $\mu_{k_0,q} < \mu_{k_\star,q} < 0$ as claimed. □

Next, we will study the asymptotic behavior of $\mu_{k,q}$ when $k \to +\infty$. But before doing so, we want show that if $\sup_M f > 0$, then $\mu_{k,q}$ is bounded above by a constant which is independent of q. This fact will play some role in our later argument.

Lemma 3.2. *If $\sup_M f > 0$, then there is some $k_{\star\star} > 1$ sufficiently large and independent of q such that $\mu_{k,q} < 0$ for all $k \geqslant k_{\star\star}$.*

Proof. Choose $x_0 \in M$ such that $f(x_0) > 0$, for example, we can select x_0 in such a way that $f(x_0) = \sup_M f$. By the continuity of f, there exists some r_0 sufficiently small such that $f(x) > 0$, for any $x \in \overline{B}_{r_0}(x_0)$ and $f(x) \geqslant 0$ for any $x \in \overline{B}_{2r_0}(x_0)$. Let $\phi : [0, +\infty) \to [0, 1]$ be a smooth non-negative function such that

$$\phi(s) = \begin{cases} 1, & 0 \leqslant s \leqslant r_0^2,\\ 0, & s \geqslant 4r_0^2.\end{cases}$$

For small r_0, it is clear that the function $\mathrm{dist}(x, x_0)^2$ is smooth. We then define

$$w(x) = \phi(\mathrm{dist}(x, x_0)^2), \quad x \in M$$

and set

$$g(t) = \int_M f e^{tw}\ \theta \wedge (d\theta)^n, \quad t \in \mathbb{R}.$$

Obviously, g is continuous and $g(0) < 0$ by the assumption $\int_M f\ \theta \wedge (d\theta)^n < 0$. For any $t \in \mathbb{R}$, we have

$$\begin{aligned} g(t) &\geqslant \Big(\sup_{\overline{B}_{r_0}(x_0)} f^+\Big) \int_{B_{r_0}(x_0)} e^{tw}\ \theta \wedge (d\theta)^n + \int_M f^- e^{tw}\ \theta \wedge (d\theta)^n \\ &\geqslant \Big(\sup_{\overline{B}_{r_0}(x_0)} f^+\Big) \mathrm{vol}(B_{r_0}(x_0)) e^t - \int_{M \backslash B_{2r_0}(x_0)} |f^-|\ \theta \wedge (d\theta)^n. \end{aligned}$$

Hence, there exists some t_0 sufficiently large such that $g(t_0) \geqslant 1$ for all $t \geqslant t_0$. Moreover, we have

$$\begin{aligned} g'(t) &= \int_M f w e^{tw}\ \theta \wedge (d\theta)^n \\ &= \int_{B_{2r_0}(x_0)} f^+ w e^{tw}\ \theta \wedge (d\theta)^n > 0, \end{aligned}$$

which implies that $g(t)$ is monotone increasing and $g(t) \geqslant 1$ for any $t \geqslant t_0$. Now, let $v(x) = ce^{t_0 w(x)}$, $x \in M$, where c is a positive constant chosen in such a way that $\|v\|_q = 1$. By the construction above, the function $e^{t_0 w}$ is independent of q. Therefore,

$$\int_M f v^q\ \theta \wedge (d\theta)^n = c^q g(q t_0) \geqslant \Big(\int_M e^{N t_0 w}\ \theta \wedge (d\theta)^n\Big)^{-1}. \tag{3.7}$$

Starting with $k \geqslant 1$, since $k^{1/q} v \in \mathscr{B}_{k,q}$, $R < 0$ and $c \leqslant 1$, we can estimate by (3.7)

$$\begin{aligned} F_q(k^{\frac{1}{q}} v) &\leqslant \frac{1}{2} k^{2/q} \big(\|\nabla_\theta v\|_2^2 + R\|v\|_2^2\big) - \frac{k}{q} \int_M f v^q\ \theta \wedge (d\theta)^n \\ &\leqslant \frac{1}{2} k^{2/2^\flat} \|\nabla_\theta e^{t_0 w}\|_2^2 - \frac{k}{N} \Big(\int_M e^{N t_0 w}\ \theta \wedge (d\theta)^n\Big)^{-1}. \end{aligned} \tag{3.8}$$

Due to the fact that $2/2^\flat < 1$, it is clear that right hand side of (3.8), as a function of k, decreasing to $-\infty$ as $k \to +\infty$. Since it is independent of q, we obviously have the existence of some $k_{\star\star}$ as in the statement of the lemma. □

Remark 3.3.

(1) If $\sup_M f < 0$, i.e. f is strictly negative everywhere, then we have $F_q(u) \geqslant Rk^{2/q} + k|\sup_M f|$. Hence, if k is sufficiently large, then $\mu_{k,q} > 0$.
(2) The most delicate case is $\sup_M f = 0$. We will conclude that there exists k_0 such that $\mu_{k,q} > 0$ for all $k > k_0$ in the proof of Proposition below.

Before completing the subsection, we prove another interesting property of $\mu_{k,q}$ which says that $\mu_{k,q}$ is continuous with respect to k.

Proposition 3.4. *$\mu_{k,q}$ is continuous with respect to k.*

Proof. Since $\mu_{k,q}$ is well-defined at any point k, we have to verify that for each k fixed and for any sequence $k_j \to k$ there holds $\mu_{k_j,q} \to \mu_{k,q}$ as $j \to +\infty$. This is equivalent to showing that for any subsequence $(k_{j_l})_l$ of $(k_j)_j$, there exists a subsequence of $(k_{j_{l_m}})_m$ of $(k_{j_l})_l$ such that $\mu_{k_{j_{l_m}},q} \to \mu_{k,q}$ as $m \to +\infty$. For simplicity, we still denote $(k_{j_l})_l$ by $(k_j)_j$. From Subsection 3.1, we suppose that $\mu_{k,q}$ and $\mu_{k_j,q}$ are achieved by $u \in \mathscr{B}_{k,q}$ and $u_j \in \mathscr{B}_{k_j,q}$ respectively. Keep in mind that u and u_j are positive smooth functions on M.

Our aim is to prove the boundedness of $(u_j)_j$ in $\mathbf{S}_1^2(M)$. It then suffices to control $\|\nabla_\theta u_j\|_{L^2}$. As in Subsection 3.1, we have

$$\int_M |\nabla_\theta u_j|^2 \theta \wedge (d\theta)^n < 2\left(\mu_{k_j,q} + 1 - \frac{R}{2} k_j^{2/q} + \frac{k_j}{q}\sup_M f\right). \tag{3.9}$$

Thus, it suffices to control $\mu_{k_j,q}$. By the homogeneity we can find a sequence of positive numbers $(t_j)_j$ such that $t_j u \in \mathscr{B}_{k_j,q}$. Since $k_j \to k$ as $j \to +\infty$ and $k_j^{2/q} = \|t_j u\|_q = t_j k^{2/q}$, we immediately see that $t_j \to 1$ as $j \to +\infty$. Now we can use $t_j u$ to control $\mu_{k_j,q}$. Indeed, using the function $t_j u$ we know that

$$\mu_{k_j,q} \leqslant t_j^2 \left(\frac{1}{2}\int_M |\nabla_\theta u|^2 \theta \wedge (d\theta)^n + \frac{R}{2}\int_M u^2 \theta \wedge (d\theta)^n\right) - \frac{1}{q} t_j^q \int_M f u^q \theta \wedge (d\theta)^n. \tag{3.10}$$

Notice that u is fixed and t_j belongs to a neighborhood of 1 for large j. Thus, $(\mu_{k_j,q})_j$ is bounded which also implies by (3.10) that $(\|\nabla_\theta u_j\|_2)_j$ is bounded. Hence $(u_j)_j$ is bounded in $\mathbf{S}_1^2(M)$.

Being bounded, there exists $\bar{u} \in \mathbf{S}_1^2(M)$ such that, up to a subsequence, $u_j \to \bar{u}$ strongly in $L^p(M)$ for any $p \in [1, N)$. Consequently, $\lim_{j\to+\infty} \|u_j\|_q = \|\bar{u}\|_q = k^{2/q}$, that is, $\bar{u} \in \mathscr{B}_{k,q}$. In particular, $F_q(u) \leqslant F_q(\bar{u})$. We now use weak lower semi-continuity property of F_q to deduce that

$$F_q(u) \leqslant F_q(\bar{u}) \leqslant \liminf_{j\to+\infty} F_q(u_j).$$

We now use our estimate for $\mu_{k_j,q}$ above to see that $\limsup_{j\to+\infty} \mu_{k_j,q} \leqslant F_q(u)$. This is due to the Lebesgue Dominated Convergence Theorem and the fact that $t_j \to 1$ as $j \to +\infty$. Therefore, $\lim_{j\to+\infty} \mu_{k_j,q} = \mu_{k,q}$ which proves the continuity of $\mu_{k,q}$. □

The next subsection is originally due to Rauzy [31, Subsection IV.3] in the context of Riemannian geometry. However, this result still holds in the context of compact CR manifolds and for the sake of clarity and in order to make the paper self-contained, we borrow the argument in [31] to reprove [31, Subsection IV.3] in this new setting.

3.3. *The study of* $\lambda_{f,\eta,q}$

As in Rauzy [31], for given $\eta > 0$, we define

$$\lambda_{f,\eta,q} = \inf_{u \in \mathscr{A}(\eta,q)} \frac{\|\nabla_\theta u\|_2^2}{\|u\|_2^2},$$

where the set $\mathscr{A}(\eta, q)$ is defined as follows

$$\mathscr{A}(\eta, q) = \Big\{ u \in \mathbf{S}_1^2(M) : u \geqslant 0, \|u\|_q = 1, \int_M |f^-| u^q \; \theta \wedge (d\theta)^n = \eta \int_M |f^-| \; \theta \wedge (d\theta)^n \Big\}.$$

Also define

$$\lambda'_{f,\eta,q} = \inf_{u \in \mathscr{A}'(\eta,q)} \frac{\|\nabla_\theta u\|_2^2}{\|u\|_2^2},$$

with

$$\mathscr{A}'(\eta, q) = \Big\{ u \in \mathbf{S}_1^2(M) : u \geqslant 0, \|u\|_q = 1, \int_M |f^-| u^q \; \theta \wedge (d\theta)^n \leqslant \eta \int_M |f^-| \; \theta \wedge (d\theta)^n \Big\}.$$

Notice that $\mathscr{A}(\eta, q)$ is not empty, since there always exists a C^∞ function u such that $\|u\|_q = 1$ whose support set is in the set

$$\Big\{ x \in M : |f^-|(x) < \eta \int_M |f^-| \; \theta \wedge (d\theta)^n \Big\}.$$

According to the curvature candidate f, we will split our argument into two cases.

Case I. Suppose the set $\{x \in M : f(x) \geqslant 0\}$ is not small, that is equivalent to saying

$$\int_{\{f \geqslant 0\}} \mathbf{1} \; \theta \wedge (d\theta)^n > 0.$$

If the preceding inequality hold, then it is not hard to see that $\mathscr{A}$ is not empty, hence, $\lambda_f < +\infty$. We are going to show that $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$. But before doing so, we want to explore some properties of $\lambda_{f,\eta,q}$.

Lemma 3.5. *For any* $\eta > 0$ *fixed, there holds* $\lambda_{f,\eta,q} = \lambda'_{f,\eta,q}$.

Proof. Since $\mathscr{A}(\eta, q) \subset \mathscr{A}'(\eta, q)$, we have $\lambda_{f,\eta,q} \geqslant \lambda'_{f,\eta,q}$. Now, we claim that $\lambda_{f,\eta,q} \leqslant \lambda'_{f,\eta,q}$. Let $(v_j)_j \subset \mathscr{A}'(\eta, q)$ be a minimizing sequence for $\lambda'_{f,\eta,q}$, then it follows from $\|\nabla_\theta v_j\|_2^2 \|v_j\|_2^{-2} \to \lambda'_{f,\eta,q}$ that v_i form a bounded sequence in $\mathbf{S}_1^2(M)$. By a common procedure that has already used several times, up to a subsequence, there exists $v \in \mathbf{S}_1^2(M)$ such that

- $v_j \rightharpoonup v$ weakly in $\mathbf{S}_1^2(M)$ and
- $v_j \to v$ strongly in $L^2(M)$ and $L^q(M)$.

Consequently, $\|v\|_q = 1$, $\|v\|_2 = \lim_{j\to\infty} \|v_i\|_2$, and the following holds

$$\int\limits_M |f^-| v^q \;\theta \wedge (d\theta)^n \leqslant \eta \int\limits_M |f^-|.$$

This particularly implies $v \in \mathscr{A}'(\eta, q)$. Since $\|\nabla_\theta v\|_2 \leqslant \lim_{i\to\infty} \|\nabla_\theta v_i\|_2$ also holds, we conclude further that $\|\nabla_\theta v\|_2^2 \|v\|_2^{-2} \leqslant \lambda'_{f,\eta,q}$. Thus, $\lambda'_{f,\eta,q}$ is achieved by the function v. To rule out the possibility of a strict inequality, i.e. the following

$$\int\limits_M |f^-| v^q \;\theta \wedge (d\theta)^n = \eta \int\limits_M |f^-| \;\theta \wedge (d\theta)^n \tag{3.11}$$

should occur, we assume by contradiction that

$$\int\limits_M |f^-| v^q \;\theta \wedge (d\theta)^n < \eta \int\limits_M |f^-| \;\theta \wedge (d\theta)^n$$

holds. Hence, there exists a positive constant α such that

$$\int\limits_M |f^-| (v+\alpha)^q \;\theta \wedge (d\theta)^n = \eta \int\limits_M |f^-| \;\theta \wedge (d\theta)^n.$$

Then $\|v+\alpha\|_q \geqslant 1$ and thus

$$\|\nabla_\theta (v+\alpha)\|_2^2 \|v+\alpha\|_2^{-2} < \|\nabla_\theta v\|_2^2 \|v\|_2^{-2} = \lambda'_{f,\eta,q}.$$

Keep in mind that $(v+a)\|v+a\|_2^{-2} \in \mathscr{A}'(\eta, q)$, we then obtain a contradiction to the definition of $\lambda'_{f,\eta,q}$, which implies that (3.11) holds. Hence, $v \in \mathscr{A}(\eta, q)$. Then, we get $\lambda_{f,\eta,q} \leqslant \|\nabla_\theta v\|_2^2 \|v\|_2^{-2} = \lambda'_{f,\eta,q}$. The proof now follows. □

Lemma 3.6. *As a function of η, $\lambda_{f,\eta,q}$ is monotone decreasing and bounded by λ_f.*

Proof. From Lemma 3.5, it suffices to show that $\lambda'_{f,\eta,q}$ is monotone decreasing. Indeed, if $\eta_1 \leqslant \eta_2$, then $\mathscr{A}'(\eta_1, q) \subset \mathscr{A}'(\eta_2, q)$. Hence, $\lambda'_{f,\eta_1,q} \geqslant \lambda'_{f,\eta_2,q}$. Moreover, if $\eta = 0$, then $\lambda_{f,\eta,q} = \lambda_f$. From the fact that $\lambda_{f,\eta,q}$ is decreasing with respect to η, it implies that $\lambda_{f,\eta,q} \leqslant \lambda_f$. □

At this point, we will show the convergence of $\lambda_{f,\eta,q}$ as $\eta \to 0$, which is the following lemma.

Lemma 3.7. *For each $q \in (2^\flat, N)$ fixed, there holds $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$.*

Proof. Suppose that $\lambda_{f,\eta,q}$ is achieved by some function $v_{\eta,q} \in \mathscr{A}(\eta, q)$. Then $v_{\eta,q}$ will form a bounded sequence in $\mathbf{S}_1^2(M)$ when η varies. It follows from the Hölder inequality and Lemma 3.6 that $\|v_{\eta,q}\|_2^2 \leqslant 1$ and $\|\nabla_\theta v_{\eta,q}\|_2^2 \leqslant \lambda_{f,\eta,q} \leqslant \lambda_f$. Therefore, up to a subsequence, we immediately obtain

- $v_{\eta,q} \rightharpoonup v$ weakly in $\mathbf{S}_1^2(M)$ and
- $v_{\eta,q} \to v_q$ strongly in $L^2(M)$ and $L^q(M)$.

Then, $v_q \geqslant 0$, $\|v_q\|_q = 1$, and $\int_M |f^-| v_q^q \ \theta \wedge (d\theta)^n = 0$. This implies that $v_q \in \mathscr{A}$ and that $\|v_q\|_2 = \lim_{\eta\to 0} \|v_{\eta,q}\|_2$. Furthermore,

$$\begin{aligned}\|\nabla_\theta v_q\|_2^2 &\leqslant \lim_{\eta\to 0} \left(\lambda_{f,\eta,q}\|v_{\eta,q}\|_2^2\right)\\ &\leqslant \lambda_f \lim_{\eta\to 0} \|v_{\eta,q}\|_2^2 \leqslant \lambda_f \|v_q\|_2^2,\end{aligned}$$

which implies that $\|\nabla_\theta v_q\|_2^2 \|v_q\|_2^{-2} = \lambda_f$. Hence, $\lambda_{f,\eta,q} \to \lambda_f$ as $\eta \to 0$. □

Lemma 3.8. *For each fixed $\varepsilon > 0$, there exists $\eta_0 > 0$ such that for any $\eta < \eta_0$, there exists $q_\eta \in (2^\flat, N)$ such that $\lambda_{f,\eta,q} \geqslant \lambda_f - \varepsilon$ for all $q \in (q_\eta, N)$.*

Proof. By contradiction, we suppose that there is $\varepsilon_0 > 0$ such that for any η_0, there exists $\eta < \eta_0$ and for any corresponding q_η, there exists $q > q_\eta$ such that $\lambda_{f,\eta,q} < \lambda_f - \varepsilon$.

Let $v_{\eta,q}$ be a function which realizes $\lambda_{f,\eta,q}$ and $\|v_{\eta,q}\|_q = 1$. Then $\|v_{\eta,q}\|_2 \leqslant 1$ and $\|\nabla_\theta v_{\eta,q}\|_2^2 \|v_{\eta,q}\|_2^{-2} = \lambda_{f,\eta,q}$. For η chosen above, there exists a sequence $q \to N$ such that

$$\|\nabla_\theta v_{\eta,q}\|_2^2 \|v_{\eta,q}\|_2^{-2} = \lambda_{f,\eta,q} < \lambda_f - \varepsilon.$$

These $v_{\eta,q}$ form a bounded sequence in $\mathbf{S}_1^2(M)$. A standard argument the implies that there exists a function v_η such that $v_{\eta,q}$ converges to v_η weakly in $\mathbf{S}_1^2(M)$ and strongly in $L^2(M)$. We then have $\|\nabla_\theta v_\eta\|_2^2 \leqslant \liminf_{q\to N} \|\nabla_\theta v_{\eta,q}\|_2^2$. This fact and the strong convergence in $L^2(M)$ imply that

$$\|\nabla_\theta v_\eta\|_2^2 \leqslant (\lambda_f - \varepsilon)\|v_{\eta,q}\|_2^2.$$

By the Sobolev and Hölder inequalities together with Lemma 3.6, we know that

$$\begin{aligned}1 &\leqslant \|v_{\eta,q}\|_N^2\\ &\leqslant \mathfrak{C}\left(\frac{\|\nabla_\theta v_{\eta,q}\|_2^2}{\|v_{\eta,q}\|_2^2} + 1\right)\|v_{\eta,q}\|_2^2\\ &\leqslant \mathfrak{C}(\lambda_f + 1)\|v_{\eta,q}\|_2^2.\end{aligned}$$

Hence, $\|v_{\eta,q}\|_2^2 \geqslant [\mathfrak{C}(\lambda_f + 1)]^{-1}$. Passing to the limit as $q \to N$, we obtain

$$\|v_\eta\|_2^2 \geqslant \frac{1}{\mathfrak{C}(\lambda_f + 1)}. \tag{3.12}$$

Next, let $1 < k < N$, then for any $q \geqslant k$, we have, by the Hölder inequality, $\int_M v_{\eta,q}^k \leqslant 1$ and

$$\begin{aligned}\int\limits_M |f^-| v_{\eta,q}^k \; \theta \wedge (d\theta)^n &\leqslant \Big(\int\limits_M |f^-| v_{\eta,q}^q \; \theta \wedge (d\theta)^n\Big)^{k/q} \Big(\int\limits_M |f^-| \; \theta \wedge (d\theta)^n\Big)^{1-k/q} \\ &\leqslant \eta^{k/q} \int\limits_M |f^-| \; \theta \wedge (d\theta)^n.\end{aligned}$$

Letting $q \to N$ yields

$$\int\limits_M v_\eta^k \leqslant 1$$

and

$$\int\limits_M |f^-| v_\eta^k \; \theta \wedge (d\theta)^n \leqslant \eta^{k/N} \int\limits_M |f^-| \; \theta \wedge (d\theta)^n.$$

Now, we let $\eta_0 \to 0$, then clearly $\eta \to 0$. The boundedness of (v_η) in $\mathbf{S}_1^2(M)$ implies that there exists $v \in \mathbf{S}_1^2(M)$ which, by (3.12), is nonnegative and not identically zero such that up to a subsequence

- $v_\eta \rightharpoonup v$ weakly in $\mathbf{S}_1^2(M)$ and
- $v_\eta \to v$ strongly in $L^2(M)$.

Then v satisfies

$$\|\nabla_\theta v\|_2^2 \leqslant (\lambda_f - \varepsilon)\|v\|_2^2.$$

By the Fatou lemma, we have

$$\begin{aligned}0 \leqslant \int\limits_M |f^-| v^k \; \theta \wedge (d\theta)^n &\leqslant \liminf_{\eta \to 0} \int\limits_M |f^-| v_\eta^k \; \theta \wedge (d\theta)^n \\ &\leqslant \liminf_{\eta \to 0} \eta^{k/N} \int\limits_M |f^-| \; \theta \wedge (d\theta)^n = 0.\end{aligned}$$

In particular, we conclude that $\int_M |f^-| v \;\; \theta \wedge (d\theta)^n = 0$. Consequently, $v \in \mathscr{A}$ and thus $\|\nabla_\theta v\|_2^2 \|v\|_2^{-2} \geqslant \lambda_f$, which is a contradiction. □

Case II. Otherwise, we assume that

$$\int_{\{f \geqslant 0\}} \mathbf{1}\ \theta \wedge (d\theta)^n = 0, \quad \sup_M f = 0.$$

In this case, it is easy to see that $\mathscr{A} = \emptyset$. Hence, $\lambda_f = +\infty$. However, we will show that $\lambda_{f,\eta,q}$ approaches to infinity as η goes to zero, which is the following lemma.

Lemma 3.9. *Fix $q \in (2^\flat, N)$, then $\lambda_{f,\eta,q} \to +\infty$ as $\eta \to 0$.*

Proof. $\lambda_{f,\eta,q}$ is achieved by a function $v_{\eta,q} \in \mathscr{A}(\eta, q)$. Assume that $v_{\eta,q}$ can form a bounded sequence in $\mathbf{S}_1^2(M)$ as η varies. Then, the standard argument shows that a subsequence of $v_{\eta,q}$ converges weakly in $\mathbf{S}_1^2(M)$ and strongly in $L^2(M)$ and $L^q(M)$ to v_q with $\|v_q\|_q = 1$ and $\int_M |f^-| v_q^q\ \theta \wedge (d\theta)^n = 0$. Hence, $v_q = 0$ a.e., which is clearly a contradiction. □

Lemma 3.10. *There exists $\eta_0 > 0$ such that for any $\eta < \eta_0$, there is q_η such that $\lambda_{f,\eta,q} > |R|$ for all $q > q_\eta$.*

Proof. We prove it again by contradiction. Let $(\eta_j)_j$ be a sequence of η that tends to zero such that there exists $q_j \in (2^\flat, N)$ such that $\lambda_{f,\eta_j,q_j} \leqslant |R|$. Notice that λ_{f,η_j,q_j} is achieved by a function v_j with $\|v_j\|_q = 1$ and $\|\nabla_\theta v_j\|_2^2 \leqslant \lambda_{f,\eta_j,q_j}$. The sequence is then bounded in $\mathbf{S}_1^2(M)$. Hence, a subsequence of v_j converges weakly in $\mathbf{S}_1^2(M)$ and strongly in $L^2(M)$ to a function v. Moreover, there is a subsequence of q_j converges to q with $q \in [2, N]$. By the Fatou lemma, we have

$$\begin{aligned} 0 \leqslant \int_M |f^-| v^q\ \theta \wedge (d\theta)^n &\leqslant \liminf_{j\to\infty} \int_M |f^-| v_j^{q_j}\ \theta \wedge (d\theta)^n \\ &\leqslant \liminf_{j\to\infty} \eta_j \int_M |f^-|\ \theta \wedge (d\theta)^n = 0. \end{aligned}$$

Hence, $\int_M |f^-| v^q\ \theta \wedge (d\theta)^n = 0$, which implies that $v = 0$ a.e. Thus $\|v\|_2 = 0$. By the fact that $\lim_{j\to\infty} \|v_j\|_2^2 = 0$ and the Sobolev inequality $\mathfrak{C}\|\nabla_\theta v_j\|_2^2 \geqslant 1 - \mathfrak{C}\|v_j\|_2^2$, we obtain a contradiction with the boundedness of λ_{f,η_j,q_j}. □

3.4. $\mu_{k,q} > 0$ *for some* k

With the properties of $\lambda_{f,\eta,q}$ studied in the previous lemmas, we will prove that for some k, $\mu_{k,q} > 0$.

Proposition 3.11. *Let $q \in (2^\flat, N)$.*

(i) *Assume that* $\sup_M f > 0$ *and $\lambda_f > |R|$, then there is some number $\eta_0 > 0$ such that*

$$\varepsilon = \frac{\lambda_{f,\eta_0,q} + R}{2} > \frac{3}{8}(\lambda_f + R). \tag{3.13}$$

Moreover, if we set

$$C_q = \frac{\eta_0}{4|R|} \inf\left(\frac{\varepsilon}{\mathfrak{C}[1+|R|+2\varepsilon]}, \frac{|R|}{2}\right), \tag{3.14}$$

and if we suppose that

$$(\sup_M f)\left(\int_M |f^-| \ \theta \wedge (d\theta)^n\right)^{-1} < C_q, \tag{3.15}$$

then there exists an interval $I_q = [k_{1,q}, k_{1,q}]$ *such that* $\mu_{k,q} > 0$ *for any* $k \in I_q$.

(ii) *In the case* $\sup_M f = 0$, *if*
- *either* $\int_{\{f \geqslant 0\}} \mathbf{1} \ \theta \wedge (d\theta)^n = 0$,
- *or* $\int_{\{f \geqslant 0\}} \mathbf{1} \ \theta \wedge (d\theta)^n \neq 0$ *plus* $\lambda_f > |R|$,

then there exists an interval $I_q = [k_{1,q}, +\infty)$ *such that* $\mu_{k,q} > 0$ *for any* $k \in I_q$.

Proof. (i) If $\sup_M f > 0$ and $\lambda_f > |R|$, then by Lemma 3.8, there exists $0 < \eta_0 < 2$ and its corresponding $q_{\eta_0} \in (2^\flat, N)$ such that

$$0 \leqslant \lambda_f - \lambda_{f,\eta_0,q} \leqslant \frac{1}{4}(\lambda_f - |R|)$$

for any $q \in (q_{\eta_0}, N)$. This immediately implies (3.13). Now, let $u \geqslant 0$ be a non-identical zero function in $\mathbf{S}_1^2(M)$ such that $\|u\|_q^q = k$ with

$$k^{(q-2)/q} > \frac{q|R|}{\eta_0} \int_M |f^-| \ \theta \wedge (d\theta)^n.$$

Set

$$k_{1,q} = \left(\frac{q|R|}{\eta_0} \int_M |f^-| \ \theta \wedge (d\theta)^n\right)^{q/(q-2)}.$$

We then consider the following two cases:

Case 1. Suppose

$$\int_M |f^-| u^q \ \theta \wedge (d\theta)^n \geqslant \eta_0 k \int_M |f^-| \ \theta \wedge (d\theta)^n.$$

Let

$$G_q(u) = \frac{1}{2}\|\nabla_\theta u\|_2^2 + \frac{R}{2}\|u\|_2^2 + \frac{1}{q}\int_M |f^-| u^q \ \theta \wedge (d\theta)^n.$$

Then by the choice of k, we have

$$
\begin{aligned}
G_q(u) &\geqslant \frac{R}{2}\|u\|_2^2 + \frac{\eta_0 k}{q}\int\limits_M |f^-| \ \theta \wedge (d\theta)^n \\
&\geqslant \frac{|R|}{2}k^{2/q}\left(\frac{2\eta_0 \int_M |f^-| \ \theta \wedge (d\theta)^n}{|qR|}k^{1-2/q} - 1\right) \\
&\geqslant \frac{|R|}{2}k^{2/q}.
\end{aligned}
$$

Case 2. Suppose

$$
\int\limits_M |f^-|u^q \ \theta \wedge (d\theta)^n < \eta_0 k \int\limits_M |f^-| \ \theta \wedge (d\theta)^n.
$$

In this case, we have $k^{1/q}u \in \mathscr{A}'(\eta, q)$. It follows from Lemma 3.5 that $\|\nabla_\theta u\|_2^2 \|u\|_2^{-2} \geqslant \lambda_{f,\eta_0,q}$. Hence,

$$
\begin{aligned}
G_q(u) &\geqslant \frac{1}{2}\big(\lambda_{f,\eta_0,q} + R\big)\|u\|_2^2 + \frac{1}{q}\int\limits_M |f^-|u^q \ \theta \wedge (d\theta)^n \\
&= \varepsilon\|u\|_2^2 + \frac{1}{q}\int\limits_M |f^-|u^q \ \theta \wedge (d\theta)^n.
\end{aligned}
$$

Set $\alpha + \beta = \varepsilon, \alpha = \beta/|R|$. Then

$$
\begin{aligned}
G_q(u) \geqslant{}& \frac{2\beta}{|R|}\left(\frac{1}{2}\|\nabla_\theta u\|_2^2 + \frac{1}{q}\int\limits_M |f^-|u^q \ \theta \wedge (d\theta)^n - G_q(u)\right) \\
&+ \frac{1}{q}\int\limits_M |f^-|u^q \ \theta \wedge (d\theta)^n + \alpha\|u\|_2^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left(1 + \frac{2\beta}{|R|}\right)G_q(u) &\geqslant \alpha\|u\|_2^2 + \frac{\beta}{|R|}\|\nabla_\theta u\|_2^2 \\
&= \frac{\beta}{|R|}\left(\|\nabla_\theta u\|_2^2 + \frac{\alpha|R|}{\beta}\|u\|_2^2\right).
\end{aligned}
$$

By the definition of α and β and the Sobolev inequality $\mathfrak{C}(\|\nabla_\theta u\|_2^2 + \|u\|_2^2) \geqslant k^{2/q}$, we have

$$
G_q(u) \geqslant \frac{\varepsilon k^{2/q}}{\mathfrak{C}(1 + |R| + 2\varepsilon)}. \tag{3.16}
$$

Let

$$m = \min\left\{\frac{\varepsilon}{\mathfrak{C}(1+|R|+2\varepsilon)}, \frac{|R|}{2}\right\}.$$

Then, from (3.16) and our assumption $\sup_M f \leqslant C_q \int_M |f^-| \; \theta\wedge(d\theta)^n$ with $C_q = m\eta_0/4|R|$, it follows that

$$\begin{aligned}
F_q(u) &= G_q(u) - \frac{1}{q}\int\limits_M f^+ u^q \;\; \theta\wedge(d\theta)^n \\
&\geqslant G_q(u) - \frac{k}{q}\sup_M f \\
&\geqslant mk^{2/q} - \frac{k}{q}C_q \int\limits_M |f^-| \;\theta\wedge(d\theta)^n,
\end{aligned}$$

for any $k > k_{1,q}$. Now, if we suppose

$$\begin{aligned}
k &< \left(\frac{qm}{2C_q \int_M |f^-| \;\theta\wedge(d\theta)^n}\right)^{q/(q-2)} \\
&= \left(\frac{2q|R|}{\eta_0 \int_M |f^-| \;\theta\wedge(d\theta)^n}\right)^{q/(q-2)} = 2^{q/(q-2)} k_{1,q},
\end{aligned}$$

then we can verify $F_q(u) > \frac{1}{2}mk^{2/q} > 0$. By setting $k_{2,q} = 2^{q/(q-2)}k_{1,q}$, we thus complete the proof of this part.

(ii) If $\sup_M f = 0$, then $F_q(u) = G_q(u)$. From the proof of (i), it easily follows that $\mu_{k,q} > 0$ for any $k > k_{1,q}$. $\square$

3.5. The Palais–Smale condition

For later use and self-contained, we will prove the Palais–Smale compact condition.

Proposition 3.12. *Suppose that the conditions (3.13)–(3.15) hold. Then for each $\varepsilon > 0$ fixed, the function $F_q(u)$ satisfies the Palais–Smale condition.*

Proof. Assume that $(v_j)_j \subset \mathbf{S}_1^2(M)$ is a Palais–Smale sequence for $F_q(u)$, that is, there exists a constant $\mathcal{C}$ such that

$$F_q(v_j) \to \mathcal{C}, \quad dF_q(v_j) \to 0, \;\text{ as }\; j\to\infty.$$

As the first step, we show that, up to a subsequence, $(v_j)_j$ is bounded in $\mathbf{S}_1^2(M)$. By means of the Palais–Smale sequence, we can derive

$$\frac{1}{2}\|\nabla_\theta v_j\|_2^2 + \frac{R}{2}\|v_j\|_2^2 - \frac{1}{q}\int\limits_M f|v_j|^q \;\;\theta\wedge(d\theta)^n = \mathcal{C} + o(1) \tag{3.17}$$

and

$$\int\limits_M \langle \nabla_\theta v_j, \nabla_\theta \xi \rangle_\theta \, \theta \wedge (d\theta)^n + R \int\limits_M v_j \xi \;\, \theta \wedge (d\theta)^n - \int\limits_M f v_j^{q-1} \xi \;\, \theta \wedge (d\theta)^n = o(1) \|\xi\|_{\mathbf{S}_1^2(M)} \tag{3.18}$$

for any $\xi \in \mathbf{S}_1^2(M)$. By setting $\xi = v_j$ in (3.18), we obtain

$$\|\nabla_\theta v_j\|_2^2 + R\|v_j\|_2^2 - \int\limits_M f|v_j|^q \;\, \theta \wedge (d\theta)^n = o(1)\|v_j\|_{\mathbf{S}_1^2(M)} \tag{3.19}$$

For simplicity, let us set $k_j = \|v_j\|_p^p$. We then consider the following two cases as in the proof of Proposition 3.11

Case 1. Suppose that, up to a subsequence, $(v_j)_j$ satisfies

$$\int\limits_M |f^-| |v_j|^q \;\, \theta \wedge (d\theta)^n \geqslant \eta_0 k_j \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n.$$

Using (3.14) and (3.15), we obtain

$$\begin{aligned} F_q(v_j) &\geqslant \frac{R}{2} k_j^{2/q} + \frac{\eta_0 k_j}{q} \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n - \frac{1}{q} \int\limits_M f^+ |v_j|^q \;\, \theta \wedge (d\theta)^n \\ &\geqslant \frac{R}{2} k_j^{2/q} + \frac{\eta_0 k_j}{q} \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n - \frac{k_j}{q} \sup_M f \\ &\geqslant \frac{R}{2} k_j^{2/q} + \frac{\eta_0 k_j}{q} \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n - \frac{k_j}{q} \frac{\eta_0}{8} \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n \\ &= \left(\frac{7\eta_0}{8} \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n \right) \frac{k_j}{q} - \frac{|R|}{2} k_j^{2/q}. \end{aligned}$$

The estimate above and the fact that $F_q(v_j) \to \mathcal{C}$ imply that $(k_j)_j$ is bounded, which, in other words, means that $(v_j)_j$ is bounded in $L^p(M)$. Then from the Hölder inequality and (3.17), it follows that $(v_j)_j$ is also bounded in $\mathbf{S}_1^2(M)$.

Case 2. Otherwise, for all j sufficiently large, $(v_j)_j$ satisfies

$$\int\limits_M |f^-| |v_j|^q \;\, \theta \wedge (d\theta)^n < \eta_0 k_j \int\limits_M |f^-| \;\, \theta \wedge (d\theta)^n.$$

Using (3.17) and (3.19), we have

$$-\frac{1}{q} \int\limits_M f |v_j|^q \;\, \theta \wedge (d\theta)^n = -\frac{2}{q-2} \mathcal{C} + o(1) \|v_j\|_{\mathbf{S}_1^2(M)} + o(1).$$

Observe that, from the definition of $\lambda_{f,\eta_0,q}$, there holds $\|\nabla_\theta v_j\|_2^2 \geqslant \lambda_{f,\eta_0,q}\|v_j\|_2^2$. This fact together with the equality above imply that

$$\begin{aligned} F_q(v_j) &\geqslant \frac{1}{2}\|\nabla_\theta v_j\|_2^2 + \frac{R}{2}\|v_j\|_2^2 - \frac{2}{q-2}\mathcal{C} + o(1)\|v_j\|_{\mathbf{S}_1^2(M)} + o(1) \\ &\geqslant \left(\frac{\lambda_{f,\eta_0,q}+R}{2} + o(1)\right)\|v_j\|_2^2 - \frac{2}{q-2}\mathcal{C}. \end{aligned} \tag{3.20}$$

Now, if $\|v_j\|_2 \to +\infty$, as $j \to \infty$, then we clearly reach a contradiction by taking the limit in the previous inequality since $\lambda_{f,\eta_0,q} + R > 0$ and $F_q(v_j) \to \mathcal{C}$. Therefore, $(v_j)_j$ is bounded in $L^2(M)$, which in turn implies that $\|\nabla_\theta v_j\|_2$ is also bounded in view of (3.20). Consequently, $(v_j)_j$ is bounded in $\mathbf{S}_1^2(M)$. Combining cases 1 and 2, we complete the first step. Then, there exists $v \in \mathbf{S}_1^2(M)$ such that up to a subsequence

- $v_j \rightharpoonup v$ in $\mathbf{S}_1^2(M)$,
- $v_j \to v$ strongly in $L^2(M)$, and
- $v_j \to v$ a.e. in M.

Using (3.18) with ξ replaced by $v_j - v$, we obtain

$$\begin{aligned} &\int_M \langle \nabla_\theta v_j, \nabla_\theta(v_j - v)\rangle_\theta\, \theta \wedge (d\theta)^n + R\int_M v_j(v_j - v)\ \theta \wedge (d\theta)^n \\ &\qquad - \int_M f v_j^{q-1}(v_j - v)\ \theta \wedge (d\theta)^n \to 0 \end{aligned}$$

as $j \to \infty$. It is not hard to see that the second term will go to zero as $j \to \infty$. By using the Hölder inequality and the fact that $v_j \to v$ strongly in $L^{N/(N-(q-1))}(M)$, we can conclude that the third term goes to zero either. Hence, we obtain

$$\int_M \langle \nabla_\theta, \nabla_\theta(v_j - v)\rangle_\theta\, \theta \wedge (d\theta)^n \to 0$$

as $j \to \infty$. In view of the following identity

$$\begin{aligned} \int_M |\nabla_\theta v_j - \nabla_\theta v|^2\ \theta \wedge (d\theta)^n &= \int_M \langle \nabla_\theta v_j, \nabla_\theta v_j - \nabla_\theta v\rangle_\theta\, \theta \wedge (d\theta)^n \\ &\quad - \int_M \langle \nabla_\theta v, \nabla_\theta v_j - \nabla_\theta v\rangle_\theta\, \theta \wedge (d\theta)^n, \end{aligned}$$

the fact that $v_j \to v$ strongly in $\mathbf{S}_1^2(M)$ and $\nabla_\theta v_j \rightharpoonup \nabla_\theta v$ weakly in $L^2(M)$, we obtain that $v_j \to v$ strongly in $\mathbf{S}_1^2(M)$. This completes the proof of the Palais–Smale condition. □

4. Proof of Theorem 1.1(a)–(b)

The main purpose of this section is to prove Theorem 1.1(a) and (b). The key idea is to find two solutions u_1 and u_2. One of them is a local minimum of the energy functional with critical exponent; the other is a saddle point. Then under a suitable assumption on f, we are able to show that the energy level of u_1 and u_2 are different. Hence, u_1 and u_2 are distinct which means that Eq. (1.4) has at least two positive solutions.

4.1. The existence of the first solution

Proposition 4.1. *Let f be a smooth function on M with $\int_M f\ \theta \wedge (d\theta)^n < 0$, $\sup_M f > 0$ and $|R| < \lambda_f$. If there exists a positive number $\mathcal{C}_1$ given by (4.1) below such that*

$$(\sup_M f)\left(\int_M |f^-|\ \theta \wedge (d\theta)^n\right)^{-1} \leqslant \mathcal{C}_1,$$

then Eq. (1.4) admits at least one smooth positive solution.

Proof. From Proposition 3.11(i), it follows that there exist η_0 and its corresponding $q_{\eta_0} \in (2^\flat, N)$ such that $\varepsilon = (\lambda_{f,\eta_0,q} + R)/2 > 3(\lambda_f + R)/8$ for any $q \in (q_{\eta_0}, N)$. By Lemma 3.6, we have $3(\lambda_f + R)/8 \leqslant \varepsilon \leqslant (\lambda_f + R)/2$, which implies that $C_q \geqslant \mathcal{C}_1$ where

$$\mathcal{C}_1 = \frac{\eta_0}{4|R|} \min\left\{\frac{3}{8}\frac{\lambda_f + R}{\mathfrak{C}(1 + \lambda_f)}, \frac{|R|}{2}\right\}. \tag{4.1}$$

Note that $\mathcal{C}_1$ is independent of q and hence never vanishing for $q \in (q_{\eta_0}, N)$. Observe that

$$\lim_{q \to N} k_{1,q} = \left(\frac{N|R|}{\eta_0 \int_M |f^-|\ \theta \wedge (d\theta)^n}\right)^{n+1} = \ell$$

and

$$\lim_{q \to N} k_{2,q} = 2^{n+1}\ell.$$

By Proposition 3.11(i) again, there exists an interval $I_q = [k_{1,q}, k_{2,q}]$ such that $\mu_{k,q} > 0$ for any $k \in I_q$. In view of Lemma 3.1, we can conclude that $k_\star < k_0 < k_{1,q}$ where $k_\star$ and k_0 are given in Lemma 3.1.

With the information above in hand, we now divide the proof into three claims for clarity.

Claim 1. Eq. (3.1) has a positive solution with strictly negative energy $\mu_{k_1,q}$.

Proof of Claim 1. We define

$$\mu_{k_1,q} = \inf_{u \in \mathscr{D}_{k,q}} F_q(u),$$

where

$$\mathscr{D}_{k,q} = \{u \in \mathbf{S}_1^2(M) : k_\star \leqslant \|u\|_q^q \leqslant k_{1,q}\}.$$

Since $k_{1,q}$ is monotone function of q, we have $\|u\|_q^q \leqslant l$ for any $u \in \mathscr{D}_{k,q}$. It follows from Section 3.1 and Lemma 3.1 that $\mu_{k_1,q}$ is finite and strictly negative. Similar arguments to those in Section 3.1 show that $\mu_{k_1,q}$ is achieved by some positive smooth function $u_{1,q}$. In particular, $\mu_{k_1,q}$ is the energy of $u_{1,q}$. Obviously, $u_{1,q}$ is a solution of (3.1). If we set $\|u_{1,q}\|_q^q = k_1$, we then immediately have $k_1 \in (k_\star, k_{\star\star})$.

Claim 2. Eq. (3.1) has a positive smooth solution $u_{2,q}$ with strictly positive energy $\mu_{k_2,q}$.

Proof of Claim 2. Let $k^\star$ be a real number such that

$$\mu_{k^\star,q} = \max\{\mu_{k,q} : k_{1,q} \leqslant k \leqslant k_{2,q}\}.$$

Obviously, $\mu_{k^\star,q} > 0$. By Proposition 3.4, we can choose $\bar{k}_1 \in (k_0, k_{1,q})$ and $\bar{k}_2 \in (k_{2,q}, k_{\star\star})$ such that $\mu_{\bar{k}_1,q} = \mu_{\bar{k}_2,q} = 0$. By the argument in Section 3.1, we see that $\mu_{\bar{k}_1,q}$ and $\mu_{\bar{k}_2,q}$ can be achieved, say by $u_{\bar{k}_1,q}$ and $u_{\bar{k}_2,q}$ respectively. We now set

$$\Gamma = \big\{\gamma \in C\big([0,1]; \mathbf{S}_1^2(M)\big) : \gamma(0) = u_{\bar{k}_1,q}, \gamma(1) = \mu_{\bar{k}_2,q}\big\}.$$

Consider the functional $E(v) = F_q(u_{\bar{k}_1,q} + v)$ for any non-negative real valued function v with

$$\|v\| = \left(\int_M |u_{\bar{k}_1,q} + v|^q\right)^{1/q}.$$

Notice that $E(0) = 0$. Let $\rho = (k^\star)^{1/q}$. If $\|v\| = \rho$, by setting $u = u_{\bar{k}_1,q} + v$, then $\|u\|_q^q = k^\star$. Hence

$$E(v) = F_q(u) \geqslant \mu_{k^\star,q} > 0.$$

Next we set $v_1 = u_{\bar{k}_2,q} - u_{\bar{k}_1,q}$, then $E(v_1) = 0$ and $\|v_1\| = (\bar{k}_2)^{1/q} > \rho$. Note that our functional E satisfies the Palais–Smale condition as we have shown for F_q. Hence, by the standard Mountain-Pass theorem, we can conclude that

$$\mu_{k_2,q} = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} E(\gamma(t) - u_{\bar{k}_1}, q)$$

is a critical value of the functional E. Clearly, $\mu_{k_2,q} > 0$. Hence, there exists a non-negative Palais–Smale sequence $(u_j)_j \subset \mathbf{S}_1^2(M)$ for the functional F_q at the level $\mu_{k_2,q}$. Consequently, Proposition 3.12 implies that, up to a subsequence, $u_j \to u_{2,q}$ strongly in $\mathbf{S}_1^2(M)$ for some $u_{2,q} \in \mathbf{S}_1^2(M)$ as $j \to +\infty$. Therefore, the function $u_{2,q}$ with positive energy $\mu_{k_2,q}$ satisfies Eq. (3.1) in the weak sense where we denote $\|u_{2,q}\|_q^q = k_2$. The regularity theorem and the maximum principle as in Section 3.1 imply that $u_{2,q} \in C^\infty(M)$ and $u_{2,q} > 0$. Finally, in view of Lemma 3.2, we know that $0 < k_2 < k_{\star\star}$.

Claim 3. Eq. (1.4) has at least one positive solution.

Proof of Claim 3. By Claims 1 and 2 above, for each $q \in (2^\flat, N)$, we know that there exist two positive smooth functions $u_{1,q}$ and $u_{2,q}$ which solve (3.1). In addition, theses solutions are critical points of the functional F_q. Therefore, on one hand we have

$$\mu_{k_i,q} = \frac{1}{2}\int\limits_M |\nabla_\theta u_{i,q}|^2 \ \theta \wedge (d\theta)^n + \frac{R}{2}\int\limits_M (u_{i,q})^2 \ \theta \wedge (d\theta)^n - \frac{1}{q}\int\limits_M f(u_{i,q})^q \ \theta \wedge (d\theta)^n$$

while on the other hand we have

$$\int\limits_M |\nabla_\theta u_{i,q}|^2 \ \theta \wedge (d\theta)^n + R\int\limits_M (u_{i,q})^2 \theta \wedge (d\theta)^n = \int\limits_M f(u_{i,q})^q \ \theta \wedge (d\theta)^n. \tag{4.2}$$

Hence, we obtain the following relation

$$\mu_{k_i,q} = \left(\frac{1}{2} - \frac{1}{q}\right)\int\limits_M f(u_{i,q})^q \ \theta \wedge (d\theta)^n.$$

Recall that $\|u_{i,q}\|_q^q = k_i$. Therefore, we can estimate $\mu_{k_i,q}$ as follows

$$\mu_{k_i,q} \leqslant \frac{\sup_M f}{2(n+1)} k_i$$

Now, from (4.2) we obtain

$$\int\limits_M |\nabla_\theta u_{i,q}|^2 \ \theta \wedge (d\theta)^n \leqslant (\sup_M f) k_i + |R| k_i^{2/q}.$$

Hence

$$\|u_{i,q}\|^2_{\mathbf{S}_1^2(M)} \leqslant (\sup_M f) k_i + (|R|+1) k_i^{2/q}, \tag{4.3}$$

which implies that the sequence $(u_{i,q})_q$ is bounded $\mathbf{S}_1^2(M)$ since $k_i \leqslant k_{\star\star}$ and $q \in (2, N)$. Thus, up to a subsequence, there exists $u_i \in \mathbf{S}_1^2(M)$ such that

- $u_{i,q} \rightharpoonup u_i$ in $\mathbf{S}_1^2(M)$,
- $u_{i,q} \to u_i$ strongly in $L^2(M)$, and
- $u_{i,q} \to u_i$ a.e. in M,

as $q \to N$. Notice that $u_{i,q}$ satisfies

$$\int_M \langle \nabla_\theta u_{i,q}, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + R\int_M u_{i,q} v \ \theta \wedge (d\theta)^n - \int_M f(u_{i,q})^{q-1} v \ \theta \wedge (d\theta)^n = 0 \tag{4.4}$$

for any $v \in \mathbf{S}_1^2(M)$. Since $u_{i,q} \rightharpoonup u_i$ weakly in $\mathbf{S}_1^2(M)$ and $u_{i,q} \to u_i$ strongly in $L^2(M)$, we obtain

$$\int_M \langle \nabla_\theta u_{i,q} - \nabla_\theta u_i, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n \to 0$$

and

$$\int_M (u_{i,q} - u_i) v \ \theta \wedge (d\theta)^n \to 0$$

as $q \to N$. Since $u_{i,q} \to u_i$ a.e. in M as $q \to N$, $(u_{i,q})^{q-1} \to (u_i)^{N-1}$ a.e. in M as well. Now, by the Hölder and Sobolev inequalities, we obtain

$$\|(u_{i,q})^{q-1}\|_{N/(N-1)} \leqslant \|u_{i,q}\|_N^{q-1} \leqslant \mathfrak{C}^{q-1} \|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-1},$$

which implies that the sequence $\{(u_{i,q})^{q-1}\}_i$ is bounded in $L^{N/(N-1)}(M)$. Hence, we know that $(u_{i,q})^{q-1} \rightharpoonup (u_i)^{N-1}$ weakly in $L^{N/(N-1)}(M)$. Notice that if $v \in L^N(M)$, then $fv \in L^N(M)$ since f is smooth. Hence, by the definition of the weak convergence, we have

$$\int_M f(u_{i,q})^{q-1} v \ \theta \wedge (d\theta)^n \to \int_M f(u_i)^{N-1} v \ \theta \wedge (d\theta)^n,$$

as $q \to N$. We are now in a position to send $q \to N$ in (4.4) thus proving that u_i are weak solutions to (1.4). Finally, the regularity theorem and the maximum principle as in Section 3.1 with q replaced by N show that $u_i \in C^\infty(M)$ and $u_i > 0$ in M. □

Until now, we only show that u_i are solutions of (1.4). However, we don't have enough information to guarantee that these solutions are distinct. Hence, we only complete the existence part. In the next subsection we show that u_i are in fact different provided $\sup_M f$ is sufficiently small.

4.2. The existence of the second solution

In the previous subsection, we showed that there are two functions u_1, u_2 solving (1.4), however, it is not clear that whether or not these functions are distinct. Recall that the energies of u_i are given as follows

$$F_N(u_i) = \frac{1}{2}\int_M |\nabla_\theta u_i|^2 \ \theta \wedge (d\theta)^n + \frac{R}{2}\int_M |u_i|^2 \ \theta \wedge (d\theta)^n - \frac{1}{N}\int_M f u_i^N \ \theta \wedge (d\theta)^n.$$

The problem comes from the fact that even the two sequences of solutions for subcritical problems $(u_{i,q})_q$ converge and

$$\lim_{q\to N} F_q(u_{1,q}) \neq \lim_{q\to N} F_q(u_{2,q}),$$

it may happen that

$$F_N(u_1) = F_N(u_2).$$

This is basically due to the fact that we lack strong convergence to ensure that $F_N(u_i) = \lim_{q\to N} F_q(u_{i,q})$ for $i = 1, 2$.

As always, we want to compare $F_N(u_1)$ and $F_N(u_2)$ and this can be done once we can show that $\lim_{q\to N} F_q(u_{i,q}) = F_N(u_i)$ for $i = 1, 2$. From the expression of $F_q(u_{i,q})$, we find that the only difficult part is to show that

$$\int_M f(u_{i,q})^q \;\theta\wedge(d\theta)^n \to \int_M f(u_i)^N \;\theta\wedge(d\theta)^n, \tag{4.5}$$

as $q \to N$. Here, if we make $\sup_M f$ sufficiently small, then such convergence can be guaranteed. It is worth noticing that this idea was used once in [25], however, it is not clear how small $\sup_M f$ is in order to guarantee such a convergence. In the present work, we slightly modify the argument used in [25] to get a more explicit result.

Before doing so, we first set $\Psi_q = \Phi/\|\Phi\|_q$. Then the following lemma is elementary.

Lemma 4.2. *We have*

$$\left(\int_M |\nabla_\theta \Psi_q|^2\,\theta\wedge(d\theta)^n\right)^{q/(q-2)} \to \left(\int_M |\nabla_\theta \Phi|^2\,\theta\wedge(d\theta)^n\right)^{n+1} \|\Phi\|_N^{-2(n+1)},$$

and

$$\left(\int_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{-1} \to \left(\int_M f\Phi^N\,\theta\wedge(d\theta)^n\right)^{-1} \|\Phi\|_N^N$$

as $q \to N$.

Proof. It suffices to prove the last limit since the first limit can be handled in a similar way by using the fact that $q/(2-q) \to -(n+1)$ as $q \to N$. However, since

$$\int_M f\Psi_q^q\,\theta\wedge(d\theta)^n = \left(\int_M f\Phi^q\,\theta\wedge(d\theta)^n\right)\|\Phi\|_q^{-q},$$

$$\int_M f\Phi^q\,\theta\wedge(d\theta)^n \to \int_M f\Phi^N\,\theta\wedge(d\theta)^n$$

and

$$\|\Phi\|_q^{-q} \to \|\Phi\|_N^{-N}$$

as $q \to N$, this conclusion easily deduces. □

In the following result, we aim to bound k_i from above using information from Φ. Note that the bound $k_{\star\star}$ appearing in Lemma 3.2 is not good enough since it involves the function f in such a way that it is difficult to handle later.

Lemma 4.3. *There exists a constant $C_{1\Phi}$ depending only on n and Φ such that*

$$k_i \leqslant C_{1\Phi}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{q/(2-q)}$$

for all q sufficiently close to N. Moreover, if the condition (1.8) holds, that is

$$\|\nabla_\theta\Phi\|_2^2\,\|\Phi\|_N^{-2} \leqslant \frac{n}{n+1},$$

then we can select $C_{1\Phi}=1$.

Proof. Let Φ be a positive smooth function on M such that $\int_M f\Phi^N\ \theta\wedge(d\theta)^n>0$. Let $\Psi_q=\Phi/\|\Phi\|_q$ as above. Hence $\|\Psi_q\|_q=1$ for any $q\in(2^\flat,N)$. Then by Lemma 4.2, there holds $\int_M f\Psi_q^q\ \theta\wedge(d\theta)^n>0$ for all q sufficiently close to N. Since $R<0$, we easily get

$$F_q(k^{1/q}\Psi_q) \leqslant \frac{1}{2}k^{2/q}\|\nabla_\theta\Psi_q\|_2^2 - \frac{k}{q}\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n. \tag{4.6}$$

By solving the following equation for k

$$\frac{1}{2}k^{2/q}\|\nabla_\theta\Psi_q\|_2^2 - \frac{k}{q}\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n = 0,$$

we obtain

$$k = \left(\frac{q}{2}\right)^{q/(q-2)}\left\|\nabla_\theta\Psi_q\right\|_2^{2q/(q-2)}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{q/(2-q)}.$$

Note that as $q\to N=2+2/n$ and in view of Lemma 4.2, there hold

$$\left\|\nabla_\theta\Psi_q\right\|_2^{2q/(q-2)} \to \|\nabla_\theta\Phi\|_2^{2(n+1)}\,\|\Phi\|_N^{-2(n+1)}$$

and $(q/2)^{q/(q-2)} \nearrow (1+1/n)^{n+1}$. Therefore, for q close to N, there holds

$$k \leqslant \left(1+\frac{1}{n}\right)^{n+1}\left(\frac{\|\nabla_\theta \Phi\|_2}{\|\Phi\|_N}\right)^{2(n+1)}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{\frac{q}{2-q}}$$

$$\leqslant C_{1\Phi}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{\frac{q}{2-q}},$$

where we set

$$C_{1\Phi}=\max\left\{\left(1+\frac{1}{n}\right)^{n+1}\|\nabla_\theta\Phi\|_2^{2(n+1)}\|\Phi\|_N^{-2(n+1)},1\right\}. \tag{4.7}$$

Then it is easy to see that $\mu_{k,q}\leqslant 0$ for any

$$k\geqslant C_{1\Phi}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{\frac{q}{2-q}}.$$

Clearly, if the condition (1.8) holds, we can select $C_{1\Phi}=1$. In view of the asymptotic behavior of $\mu_{k,q}$ in Section 3. The proof now follows. □

In the sequel, we aim to estimate the right hand side of (4.3) in terms of $k_i^{2/q}$. To do so, we make an ansatz assumption that the condition (1.6) holds for some $\mathcal{C}_2$ to be determined.

Lemma 4.4. *When q is sufficiently closed to N and assuming the condition (1.6) holds, there holds*

$$(\sup_M f)k_i^{1-2/q}\leqslant \mathcal{C}_2\left(C_{1\Phi}\right)^{1/(n+1)}\|\Phi\|_N^N.$$

Moreover, if the condition (1.8) holds, then we have $(\sup_M f)k_i^{1-2/q}\leqslant\mathcal{C}_2$.

Proof. Using Lemma 4.3, we obtain

$$k_i\leqslant C_{1\Phi}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{q/(2-q)}.$$

Therefore,

$$(\sup_M f)k_i^{1-2/q}\leqslant(\sup_M f)\left(C_{1\Phi}\right)^{1-2/q}\left(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\right)^{-1}$$

$$\leqslant\mathcal{C}_2\left(C_{1\Phi}\right)^{1/(n+1)}\|\Phi\|_N^N,$$

since $1-2/q \to 1/(n+1)$ as $q \to N$. When the condition (1.8) holds, the claim follows from the fact that we can select $C_{1\Phi}=1$. $\square$

Proposition 4.5. *Assume that all requirements in Proposition 4.1 are fulfilled and that there exists a smooth positive function Φ with $\int_M f\Phi^N \ \theta\wedge(d\theta)^n>0$ such that f satisfies the condition (1.6), that is*

$$(\sup_M f)\bigg(\int_M f\Phi^N \ \theta\wedge(d\theta)^n\bigg)^{-1} < \mathcal{C}_2,$$

where the number $\mathcal{C}_2>0$ is given in (4.11) below. Then

$$\int_M f(u_{i,q})^q \ \theta\wedge(d\theta)^n \to \int_M f(u_i)^N \ \theta\wedge(d\theta)^n$$

as $q\to N$. Moreover, if the condition (1.8) and $\|\Phi\|_N \leqslant 1$ hold, then the constant $\mathcal{C}_2$ is independent of Φ.

Proof. Set $v=(u_{i,q})^{1+2\delta}$ for some $\delta>0$ to be determined later. Then we have

$$\begin{aligned}\frac{1+2\delta}{(1+\delta)^2}\int_M |\nabla_\theta w_{i,q}|^2 \ \theta\wedge(d\theta)^n &= |R|\int_M (w_{i,q})^2 \ \theta\wedge(d\theta)^n \\ &\quad + \int_M f(w_{i,q})^2(u_{i,q})^{q-2} \ \theta\wedge(d\theta)^n,\end{aligned}$$

where $w_{i,q}=(u_{i,q})^{1+\delta}$. From this equality and the Sobolev inequality, it follows that

$$\begin{aligned}\|w_{i,q}\|_N^2 &\leqslant \bigg(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|R|+\mathcal{A}_1\bigg)\|w_{i,q}\|_2^2 \\ &\quad + \mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}\int_M f^+(w_{i,q})^2(u_{i,q})^{q-2} \ \theta\wedge(d\theta)^n. \end{aligned} \tag{4.8}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}&\int_M (w_{i,q})^2(u_{i,q})^{q-2}\theta\wedge(d\theta)^n \\ &\quad \leqslant \bigg(\int_M (w_{i,q})^N \ \theta\wedge(d\theta)^n\bigg)^{2/N}\bigg(\int_M (u_{i,q})^{\frac{N(q-2)}{N-2}} \ \theta\wedge(d\theta)^n\bigg)^{1-2/N}.\end{aligned}$$

Again, by the Hölder and Sobolev inequalities, we have

$$\int\limits_M (u_{i,q})^{\frac{N(q-2)}{N-2}} \;\theta\wedge(d\theta)^n \leqslant \Big(\int\limits_M (u_{i,q})^N \;\theta\wedge(d\theta)^n\Big)^{\frac{q-2}{N-2}}$$
$$\leqslant \big(\mathcal{K}_1+\mathcal{A}_1\big)^{\frac{N(q-2)}{2(N-2)}} \|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{N(q-2)/(N-2)}.$$

Therefore,

$$\int\limits_M (w_{i,q})^2(u_{i,q})^{q-2} \;\theta\wedge(d\theta)^n \leqslant \big(\mathcal{K}_1+\mathcal{A}_1\big)^{q/2-1}\|w_{i,q}\|_N^2\|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-2}.$$

From (4.8) and the estimate above, it follows that

$$\begin{aligned}\|w_{i,q}\|_N^2 \leqslant{}& \Big(\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}|R|+\mathcal{A}_1\Big)\|w_{i,q}\|_2^2\\ &+\mathcal{K}_1\frac{(1+\delta)^2}{1+2\delta}(\sup_M f)\big(\mathcal{K}_1+\mathcal{A}_1\big)^{q/2-1}\|w_{i,q}\|_N^2\|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-2}.\end{aligned} \tag{4.9}$$

Making use of (4.3) and Lemma 4.4, we can estimate $\|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-2}$ in terms of Φ as follows: first we have

$$\begin{aligned}\|u_{i,q}\|_{\mathbf{S}_1^2(M)} &\leqslant \Big(\mathcal{C}_2\big(C_{1\Phi}\big)^{1/(n+1)}\|\Phi\|_N^N+|R|+1\Big)^{1/2}k_i^{1/q}\\ &= C_{2\Phi}k_i^{1/q}.\end{aligned} \tag{4.10}$$

In view of Lemma 4.3, we further obtain

$$\|u_{i,q}\|_{\mathbf{S}_1^2(M)} \leqslant C_{1\Phi}^{1/q}C_{2\Phi}\Big(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\Big)^{1/(2-q)}.$$

Hence

$$\|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-2} \leqslant C_{1\Phi}^{1-2/q}C_{2\Phi}^{q-2}\Big(\int\limits_M f\Psi_q^q\,\theta\wedge(d\theta)^n\Big)^{-1}.$$

Keep in mind that $C_{i\Phi}\geqslant 1$, for $i=1,2$, then the following convergences are obvious $C_{1\Phi}^{1-2/q}\nearrow C_{1\Phi}^{1/(n+1)}$, $C_{2\Phi}^{q-2}\nearrow C_{2\Phi}^{2/n}$. Thus, using Lemma 4.2, we have just shown that

$$\|u_{i,q}\|_{\mathbf{S}_1^2(M)}^{q-2} \leqslant C_{1\Phi}^{1/(n+1)}C_{2\Phi}^{2/n}\Big(\int\limits_M f\Phi^N\,\theta\wedge(d\theta)^n\Big)^{-1}\|\Phi\|_N^N$$

for any q close to N. By using the constant functions and the fact that $\mathrm{vol}(M,\theta)=1$, we obtain $\mathcal{A}_1 \geqslant 1$. Hence the convergence $(\mathcal{K}_1+\mathcal{A}_1)^{q/2-1} \nearrow (\mathcal{K}_1+\mathcal{A}_1)^{N/2-1}$ is obvious. This helps us to conclude the following key estimate

$$\mathcal{K}_1 (\sup_M f)(\mathcal{K}_1+\mathcal{A}_1)^{q/2-1} \|u_{i,q}\|^{q-2}_{\mathbf{S}^2_1(M)} \leqslant \mathcal{K}_1(\mathcal{K}_1+\mathcal{A}_1)^{1/n} C^{1/(n+1)}_{1\Phi} C^{2/n}_{2\Phi} \|\Phi\|^N_N (\sup_M f)\left(\int_M f\Phi^N \theta\wedge(d\theta)^n\right)^{-1}.$$

We then impose some condition on f such that the right hand side of the preceding inequality is less than $1/2$. To do so, let $\mathcal{C}_2$ be the unique positive solution of the following algebraic equation

$$\mathcal{C}_2 = \frac{1}{2\mathcal{K}_1}(\mathcal{K}_1+\mathcal{A}_1)^{-1/n} C^{-\frac{1}{n+1}}_{1\Phi}\left(\mathcal{C}_2 C^{\frac{1}{n+1}}_{1\Phi}\|\Phi\|^N_N + |R| + 1\right)^{-1/n} \|\Phi\|^{-N}_N. \tag{4.11}$$

Then, in view of (1.6) and the previous inequality, we conclude that

$$\mathcal{K}_1 (\sup_M f)(\mathcal{K}_1+\mathcal{A}_1)^{q/2-1} \|u_{i,q}\|^{q-2}_{\mathbf{S}^2_1(M)} < \frac{1}{2}$$

for q close to N. Now, we are able to choose δ such that $1+\delta < \frac{N}{2}$ and

$$\frac{(1+\delta)^2}{1+2\delta}\mathcal{K}_1 (\sup_M f)(\mathcal{K}_1+\mathcal{A}_1)^{q/2-1} \|u_{i,q}\|^{q-2}_{\mathbf{S}^2_1(M)} < \frac{1}{2} \tag{4.12}$$

holds for q close to N. Hence, from (4.9) and (4.12), it follows that

$$\|w_{i,q}\|^2_N \leqslant 2\left(\mathcal{K}_1 \frac{(1+\delta)^2}{1+2\delta}|R| + \mathcal{A}_1\right)\|w_{i,q}\|^2_2. \tag{4.13}$$

By our choice of δ, we have

$$\|w_{i,q}\|^2_2 = \|(u_{i,q})^{1+\delta}\|_2 = \|u_{i,q}\|^{1+\delta}_{2(1+\delta)} \leqslant \|u_{i,q}\|^{1+\delta}_N.$$

This estimate and the Sobolev inequality imply that $\|w_{i,q}\|_2$ can be bounded by some constant depending on Φ. It then follows from (4.13) that $(\|w_{i,q}\|_N)_q$ is bounded, that is, $(\|u_{i,q}\|_{N(1+\delta)})_q$ is bounded. By the Hölder inequality, we obtain

$$\|(u_{i,q})^q\|_{1+\delta} \leqslant \|u_{i,q}\|^q_{N(1+\delta)},$$

which implies that $((u_{i,q})^q)_q$ is bounded in $L^{1+\delta}(M)$. This and the fact that $(u_{i,q})^q \to (u_i)^N$ a.e. in M yield $(u_{i,q})^q \rightharpoonup (u_i)^N$ weakly in $L^{1+\delta}(M)$. Hence, from the definition of weak convergence and the fact that $L^{1+1/\delta}(M)$ is the dual space of $L^{1+\delta}(M)$ there holds

$$\int\limits_M f(u_{i,q})^q \ \theta \wedge (d\theta)^n \to \int\limits_M f(u_i)^N \ \theta \wedge (d\theta)^n,$$

as $q \to N$, since it is clear that $f \in L^{1+1/\delta}(M)$.

It remains to consider whenever the constant $\mathcal{C}_2$ is independent of Φ. First, suppose that the condition (1.8) holds, then we can select $C_{1\Phi} = 1$, and if we assume further that $\|\Phi\|_N \leqslant 1$ hold, then the constant $C_{2\Phi}$ simply becomes $(\mathcal{C} + |R| + 1)^{1/2}$. Now instead of solving (4.11), we solve the following algebraic equation

$$\mathcal{C}_2 = \frac{1}{2\mathcal{K}_1}\big(\mathcal{K}_1 + \mathcal{A}_1\big)^{-1/n}\big(\mathcal{C}_2 + |R| + 1\big)^{-1/n} \tag{4.14}$$

to find $\mathcal{C}_2$. Having such a constant $\mathcal{C}_2$ we then immediately obtain

$$\mathcal{K}_1\big(\mathcal{K}_1 + \mathcal{A}_1\big)^{1/n}\big(\mathcal{C}_2 + |R| + 1\big)^{1/n} \leqslant \frac{1}{2\mathcal{C}_2},$$

which is enough to conclude that

$$\frac{(1+\delta)^2}{1+2\delta}\mathcal{K}_1\big(\mathcal{K}_1 + \mathcal{A}_1\big)^{1/n} C_{2\Phi}^{2/n}\big(\sup_M f\big)\Bigg(\int\limits_M f\Phi^N \theta \wedge (d\theta)^n\Bigg)^{-1} < \frac{1}{2}$$

holds. Hence, the rest of the proof goes as before. □

It is worth noticing that the novelty of the condition $\|\Phi\|_N \leqslant 1$ is that it is easy to construct those functions Φ satisfying $\|\Phi\|_N \leqslant 1$ by a simple scaling with a sufficiently small constant. However, by imposing the condition $\|\Phi\|_N \leqslant 1$, one can easily check that the left hand side of (1.6) is always greater than 1, therefore, to fulfill (1.6), one needs $\mathcal{C}_2 > 1$. In the context of compact CR manifolds, since the optimal constants $\mathcal{K}_1, \mathcal{A}_1$ are unknown, we do not know whether or not the condition $\mathcal{C}_2 > 1$ actually holds. Nevertheless, in the context of compact Riemannian manifolds, the known constant $\mathcal{K}_1$ is relatively small. In addition, it is worth emphasizing that the technique introduced here works well in the context of bounded domains in $\mathbb{R}^n$, especially when we have zero Dirichlet boundary condition.

With the help of the proposition above, we can easily get that $\nabla_\theta u_{i,q} \to \nabla_\theta u_i$ strongly in $L^2(M)$.

Proposition 4.6. *Assume that the requirements in Proposition 4.5 are fulfilled, then there holds* $\|\nabla_\theta u_{i,q}\|_2 \to \|\nabla_\theta u_i\|_2$ *as* $q \to N$.

Proof. It suffices to prove that $\nabla_\theta u_{i,q} \to \nabla_\theta u_i$ strongly in $L^2(M)$. By replacing v with $u_{i,q} - u_i$ in (4.4), we arrive at

$$\int\limits_M \langle \nabla_\theta u_{i,q}, \nabla_\theta(u_{i,q} - u_i)\rangle_\theta \,\theta \wedge (d\theta)^n + R\int\limits_M u_{i,q}(u_{i,q} - u_i) \ \theta \wedge (d\theta)^n$$
$$- \int\limits_M f(u_{i,q})^{q-1}(u_{i,q} - u_i) \ \theta \wedge (d\theta)^n = 0 \tag{4.15}$$

Since $u_{i,q} \to u_i$ strongly in $L^2(M)$, the second term in (4.15) approaches to zero as $q \to N$. Notice that the third term can be rewritten as

$$\int\limits_M f(u_{i,q})^{q-1}(u_{i,q} - u_i)\ \theta \wedge (d\theta)^n = \int\limits_M f(u_{i,q})^q - f(u_{i,q})^N\ \theta \wedge (d\theta)^n$$
$$- \left(\int\limits_M f\big((u_{i,q})^{q-1} - u_i^{N-1}\big)u_i\ \theta \wedge (d\theta)^n \right)$$

Using Proposition 4.5, we obtain the fist term in the above equality goes to zero as $q \to N$. The fact $u_{i,q}^{q-1} \rightharpoonup u_i^{N-1}$ weakly in $L^{N/(N-2)}(M)$ and $fu_i \in L^N(M)$ imply that the second term goes to zero either. Hence the third term in (4.15) converges to zero as $q \to N$. Therefore, by (4.15), we have

$$\int\limits_M \langle \nabla_\theta u_{i,q}, \nabla_\theta (u_{i,q} - u_i)\rangle_\theta\, \theta \wedge (d\theta)^n \to 0$$

as $q \to N$. Using this and the fact that $\nabla_\theta u_{i,q} \rightharpoonup \nabla_\theta u_i$ weakly in $L^2(M)$, we obtain

$$\int\limits_M |\nabla_\theta (u_{i,q} - u_i)|^2\ \theta \wedge (d\theta)^n \to 0$$

as $q \to N$. In other words, $\nabla_\theta u_{i,q} \to \nabla_\theta u_i$ strongly in $L^2(M)$. □

At this point, we can easily conclude that Eq. (1.4) has at least two positive solutions. This is the content of the following result whose proof is straightforward.

Proposition 4.7. *Assume that all requirements in Proposition 4.5 are fulfilled. Then Eq. (1.4) has at least two smooth positive solutions, in which, one has strictly negative energy and the other has positive energy.*

5. Proof of Theorems 1.2, 1.3, and 1.1(c)

5.1. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 which provides a necessary and sufficient condition for the solvability of (1.4). Notice that we have already proved that the condition $|R| < \lambda_f$ is necessary, we will show that this condition is also sufficient. To do so, we again study the asymptotic behavior of $\mu_{k,q}$. However, unlike the case $\sup_M f > 0$, the curve $k \mapsto \mu_{k,q}$ takes a different shape as shown in Fig. 2.

Now, we prove the following proposition.

Proposition 5.1. *If* $\sup_M f \leqslant 0$ *and* $|R| < \lambda_f$, *then Eq. (1.4) admits a positive solution.*

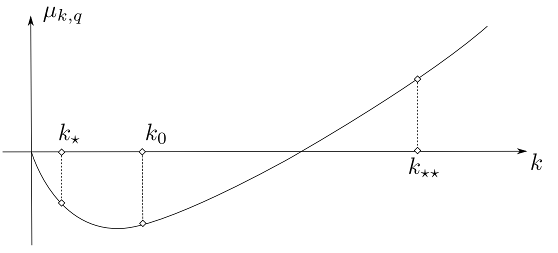

Fig. 2. The asymptotic behavior of $\mu_{k,q}$ when $\sup_M f = 0$.

Proof. Let $q \in (q_{\eta_0}, N)$. Then by solving the following equation

$$\frac{R}{2}k^{2/q} - \frac{k}{q}\int\limits_M f\ \theta \wedge (d\theta)^n = 0,$$

we can easily get $\mu_{k_0,q} \leqslant 0$ where

$$k_0 = \left(\frac{q}{2}\frac{R}{\int_M f\ \theta \wedge (d\theta)^n}\right)^{q/(q-2)}.$$

It is not hard to find k_1 and k_2 independent of q such that $k_1 < k_0 < k_2$. Now from the study of the asymptotic behavior of $\mu_{k,q}$, we can find $k_\star$ and $k_{\star\star}$ independent of q with $k_\star < k_1 < k_0 < k_2 < k_{\star\star}$ such that $\mu_{k,q} < \min\{\mu_{k_\star,q}, \mu_{k_{\star\star},q}\}$. Then we define

$$\mu_{k_1,q} = \inf_{u \in \mathscr{D}_{k,q}} F_q(u)$$

for each q fixed, where

$$\mathscr{D}_{k,q} = \{u \in \mathbf{S}_1^2(M) : k_\star \leqslant \|u\|_q^q \leqslant k_{\star\star}\}.$$

At this point, we can apply similar argument in the proof of the existence of the first solution in Theorem 1.1 to obtain a positive solution of Eq. (1.4). □

To conclude Theorem 1.2, it remains to verify the uniqueness of positive smooth solutions of (1.4). This is the content of the following lemma whose proof makes use of conformal changes.

Lemma 5.2. *Eq. (1.4) admits at most one positive smooth solution.*

Proof. Assume that u_1 and u_2 are positive smooth solution of (1.4). Then we denote $\widehat{\theta} = u_1^{2/n}\theta$ and $u = u_2/u_1$. It is well-known that the sub-Laplacian $\Delta_{\widehat{\theta}}$ with respect to contact form $\widehat{\theta}$ verifies

$$\Delta_{\widehat{\theta}} u = u_1^{-2/n}\left(\Delta_\theta u + \frac{2}{u_1}\langle \nabla_\theta u_1, \nabla_\theta u\rangle_\theta\right) \tag{5.1}$$

on M. Then, a simple calculation shows that

$$\begin{aligned}
&-\Delta_\theta u_2 + \frac{n}{2(n+1)}\mathrm{Scal}_\theta u_2 \\
&\quad = -\Delta_\theta(uu_1) + \frac{n}{2(n+1)}\mathrm{Scal}_\theta uu_1 \\
&\quad = -u\Delta_\theta u_1 - 2\langle\nabla_\theta u, \nabla_\theta u_1\rangle_\theta - u_1\Delta_\theta u + \frac{n}{2(n+1)}\mathrm{Scal}_\theta uu_1 \\
&\quad = \frac{n}{2(n+1)} f u_1^{1+2/n} u - 2\langle\nabla_\theta u, \nabla_\theta u_1\rangle_\theta - u_1\Delta_\theta u.
\end{aligned} \tag{5.2}$$

Using (5.1) and (5.2), we have further

$$-\Delta_\theta u_2 + \frac{n}{2(n+1)}\mathrm{Scal}_\theta u_2 = \frac{n}{2(n+1)} f u_1^{1+2/n} u - u_1^{1+2/n}\Delta_{\widehat{\theta}} u.$$

Since u_2 solves (1.4) and by canceling the common term $u_1^{1+2/n}$, we arrive at

$$-\Delta_{\widehat{\theta}} u + \frac{n}{2(n+1)} f u = \frac{n}{2(n+1)} f u^{1+2/n},$$

or equivalently,

$$-\Delta_{\widehat{\theta}}(u-1) = \frac{n}{2(n+1)} f(u^{1+2/n} - u). \tag{5.3}$$

Using the test function $(u-1)^\pm$ together with the smoothness of $u-1$ and the non-positivity of f, we conclude that the only possibility for (5.3) to hold is that $u \equiv 1$, which completes the proof. □

5.2. *Proof of Theorem 1.3*

To prove the theorem, we use the method of sub- and super-solutions. By using the change of variable $u = \exp(v)$, we get that

$$-\Delta_\theta u + Ru - f u^{1+2/n} = e^v(-\Delta_\theta v - |\nabla_\theta v|^2) + Re^v - f e^{(1+2/n)v}.$$

Therefore, to find a super-solution $\bar{u}$ for (1.4), it suffices to find some v in such a way that

$$-\Delta_\theta v - |\nabla_\theta v|^2 + R - f e^{2v/n} \geqslant 0.$$

In order to do this, thanks to $\int_M f\,\theta \wedge (d\theta)^n < 0$, we can pick $b > 0$ small enough such that

$$\sup_M \left| f\left(e^{nb\varphi/2} - 1\right)\right| \leqslant -\frac{1}{4}\int_M f\,\theta \wedge (d\theta)^n$$

and

$$b|\nabla_\theta \varphi|^2 < -\frac{1}{4}\int_M f\,\theta \wedge (d\theta)^n,$$

where φ is a positive smooth solution of the following equation $-\Delta_\theta \varphi = f - \int_M f\,\theta \wedge (d\theta)^n$. We now find the function v of the form $v = b\varphi + (n \log b)/2$. Indeed, by calculations, we have

$$\begin{aligned}
-\Delta_\theta v - |\nabla_\theta v|^2 + R - f e^{2v/n} &= -b\Delta_\theta \varphi - b^2 |\nabla_\theta \varphi|^2 + R - b f e^{2b\varphi/n} \\
&= -b \int\limits_M f\,\theta \wedge (d\theta)^n - b^2 |\nabla_\theta \varphi|^2 + R - bf(e^{2b\varphi/n} - 1) \\
&\geqslant -\frac{3b}{4} \int\limits_M f\,\theta \wedge (d\theta)^n + R - b \sup_M \left| f\left(e^{2b\varphi/n} - 1\right)\right| \\
&\geqslant -\frac{b}{2} \int\limits_M f\,\theta \wedge (d\theta)^n + R > 0
\end{aligned}$$

provided

$$|R| < -\frac{b}{2} \int\limits_M f\,\theta \wedge (d\theta)^n.$$

Therefore, if we set $\bar{u} = \exp(v)$ and set

$$\mathcal{C}_3 = -\frac{b}{2} \int\limits_M f\,\theta \wedge (d\theta)^n, \tag{5.4}$$

then we conclude that $\bar{u}$ is a super-solution of (1.4) provided $|R| < \mathcal{C}_3$. We now turn to the existence of a sub-solution $\underline{u}$. Before doing so, we can easily check that

$$\bar{u} = \exp\left(b\varphi + (n \log b)/2\right) > \exp\left((n \log b)/2\right) = b^{n/2}.$$

Since $\bar{u}$ has a strictly positive lower bound and thanks to the fact that a suitable small positive constant is a sub-solution for (1.4), we can easily construct a sub-solution $\underline{u}$ with $\underline{u} < \bar{u}$. The proof of the theorem is now complete.

5.3. *Proof of Theorem 1.1(c)*

Before closing the present paper, we prove that in fact the condition $|R| < \lambda_f$ is not sufficient for the solvability of (1.4). To do so, for any given constant $R \in (-\lambda_f, 0)$, we construct a smooth function f such that $\sup_M f > 0$ and $\int_M f\,\theta \wedge (d\theta)^n < 0$ in such a way that (1.4) admits no solution. The following result, in the spirit of [19], is needed.

Proposition 5.3. *Suppose that $R < 0$ is constant, if a positive solution to (1.4) exists, then the unique solution ϕ of*

$$\Delta_\theta \phi + \frac{2R}{n}\phi = \frac{2f}{n} \tag{5.5}$$

must be positive.

Proof. Assume that u is a positive solution to (1.4). Using the substitution $v = u^{-2/n}$, one can easily see that

$$\Delta_\theta v = \frac{2}{n}\Big(1 + \frac{2}{n}\Big)u^{-2-2/n}|\nabla_\theta u|^2 - \frac{2}{n}u^{-1-2/n}\Delta_\theta u$$

and that

$$|\nabla_\theta v|^2 = \frac{4}{n^2}u^{-2-4/n}|\nabla_\theta u|^2.$$

Therefore,

$$\Delta_\theta v = \frac{n+2}{2}\frac{|\nabla_\theta v|^2}{v} - \frac{2}{n}v\frac{\Delta_\theta u}{u}.$$

Making use of (1.4), we further have

$$-\Delta_\theta v - \frac{2R}{n}v = -\frac{n+2}{2}\frac{|\nabla_\theta v|^2}{v} - \frac{2f}{n}.$$

Set $w = \phi - v$ where ϕ is the unique solution of (5.5). Then

$$-\Delta_\theta w - \frac{2R}{n}w = \frac{n+2}{2}\frac{|\nabla_\theta v|^2}{v}.$$

A standard application of maximum principle shows that w must be non-negative. Hence $\phi \geqslant v > 0$ as claimed. □

Now for given function f satisfying $\sup_M f > 0$ and $\int_M f\,\theta \wedge (d\theta)^n < 0$ we pick a constant R arbitrary which satisfies $R \in (-\lambda_f, 0)$. We now construct a new function, say h, having the following three properties $\sup_M h > 0$, $\int_M h\,\theta \wedge (d\theta)^n < 0$, and $\lambda_h \geqslant \lambda_f$, but Eq. (1.4) with f replaced by h admit no solution, that is to say the following equation

$$-\Delta_\theta u + Ru = hu^{1+2/n} \tag{5.6}$$

has no positive smooth solution. Once we can construct such a function h, we can conclude that the condition $R \in (-\lambda_f, 0)$ is not sufficient for the solvability of (1.4). For simplicity, let us denote by $M_\pm$ and M_0 the following

$$M_\pm = \left\{x \in M : f^\pm(x) \neq 0\right\},$$
$$M_0 = \{x \in M : f(x) = 0\}.$$

Our construction for h depends on the following two simple equations

$$\begin{aligned} -\Delta_\theta \psi_+ &= 1 \quad \text{in } M_-,\\ \psi_+ &= 0 \quad \text{on } \partial M_-, \end{aligned} \tag{5.7}$$

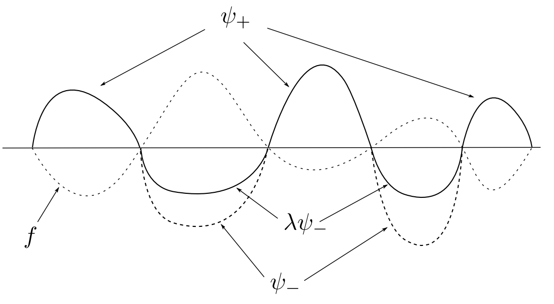

Fig. 3. The construction of the functions $\psi_\pm$.

and

$$\begin{aligned} -\Delta_\theta \psi_- &= -1 \quad \text{in } M_+, \\ \psi_- &= 0 \quad \text{on } \partial M_+. \end{aligned} \tag{5.8}$$

It is well-known that these functions $\psi_\pm$ exist and are smooth, see Fig. 3. In addition, $\psi_+ > 0$ in M_- and $\psi_- < 0$ in M_+. Then we construct a continuous function ψ as follows

$$\psi = \begin{cases} \psi_+, & \text{in } M_-, \\ 0, & \text{in } M_0, \\ \lambda\psi_-, & \text{in } M_+, \end{cases} \tag{5.9}$$

where the constant $\lambda > 0$ is determined later. Clearly, the function ψ always changes sign in M. Then, we simply set

$$h = \frac{n}{2}\Delta_\theta \psi + R\psi.$$

First, we calculate to obtain

$$\begin{aligned} \int_M h\,\theta \wedge (d\theta)^n &= \frac{n}{2}\Big(\int_{M_-} \Delta_\theta \psi_+ \theta \wedge (d\theta)^n + R\int_{M-} \psi_+ \theta \wedge (d\theta)^n\Big) \\ &\quad + \lambda\Big(\int_{M_+} \Delta_\theta \psi_- \theta \wedge (d\theta)^n + R\int_{M_+} \psi_- \theta \wedge (d\theta)^n\Big) \\ &= -\frac{n}{2}\Big(\mathrm{vol}(M_-) + |R|\int_{M_-} \psi_+ \theta \wedge (d\theta)^n\Big) \\ &\quad + \lambda\Big(\mathrm{vol}(M_+) + |R|\int_{M_+} |\psi_-|\, \theta \wedge (d\theta)^n\Big). \end{aligned}$$

Therefore, by selecting suitable small $\lambda > 0$, we obtain $\int_M h\,\theta \wedge (d\theta)^n < 0$. Now, it suffices to show that the above function h verifies the following two conditions: $\sup_M h > 0$ and $\lambda_h \geqslant \lambda_f$. To this purpose, we first observe that since $R < 0$ in M, it is clear to see that $h|_{M_+} > 0$ which implies the first condition. For the second condition, we note that in the region M_-

$$h|_{M_-} = \frac{n}{2}(\Delta_\theta \psi)\Big|_{M_-} + R\psi\big|_{M_-}$$
$$= -\frac{n}{2} + R\psi_+ < 0,$$

thanks to the positivity of ψ_+, $R < 0$, and (5.7). Hence,

$$\{x \in M : h(x) \geqslant 0\} \subset M_0 \cup M_+$$

which immediately implies $\lambda_h \geqslant \lambda_f$. Since the sign-changing function ψ solves (5.5) with f replaced by the above h, it is clear that Eq. (5.6) admits no solution by means of Lemma 5.3.

Before closing this section, we would like to mention that conclusions in Theorems 1.3 and 1.1(c) do not contradict each other. The reason is as follows: The constant $\mathcal{C}_3(f)$ appearing in Theorem 1.3 basically depends on the full function of f which is being kept fixed. However, the constant λ_f basically depends only on the negative part f^- and therefore, a suitable change in f^+, which also affects $\mathcal{C}_3$, could lead to a non-existence result. Mimicking the results in [29,30], we believe that there is a constant $\mathcal{C} \in (0, \lambda_f)$ strongly depending on f such that Eq. (1.4) has no solution if $|R| > \mathcal{C}$ while it has at least one solution if $|R| \leqslant \mathcal{C}$. Hence, Theorems 1.3 and 1.1(c) is nevertheless a one step to understand relation between these two constants.

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Appendix A. Construction of a function Φ satisfying (1.7) and (1.8)

In this section, we construct an example of functions Φ satisfying the assumptions (1.7) and (1.8). To this purpose, we first pick a smooth non-negative function Θ such as $\int_M f\Theta\theta \wedge (d\theta)^n > 0$. For example, one can choose Θ whose support is contained in the support of the positive part f^+. Then, we consider the function $\Phi = \alpha\exp(\beta\Theta)$ for suitable $\alpha, \beta > 0$. The elementary inequality $e^x > x$ for $x \geqslant 0$ implies that

$$\int_M f\Phi^N \theta \wedge (d\theta)^n = \alpha^N \int_M f\exp(N\beta\Theta)\,\theta \wedge (d\theta)^n$$
$$> N\alpha^N \beta \int_M f\Theta\,\theta \wedge (d\theta)^n > 0$$

for any $\alpha, \beta > 0$. A direct calculation then shows

$$\frac{\|\nabla_\theta \Phi\|_2}{\|\Phi\|_N} = \beta\frac{\left(\int_M e^{2\beta\Theta}|\nabla_\theta\Theta|^2\,\theta \wedge (d\theta)^n\right)^{1/2}}{\left(\int_M e^{N\beta\Theta}\,\theta \wedge (d\theta)^n\right)^{1/N}}. \tag{A.1}$$

While the denominator of the right hand side of (A.1) can be bounded from below using

$$\int\limits_M e^{N\beta\Theta}\,\theta\wedge(d\theta)^n \geqslant 1,$$

its numerator is also bounded from above using

$$\int\limits_M e^{2\beta\Theta}|\nabla_\theta\Theta|^2\,\theta\wedge(d\theta)^n \leqslant \int\limits_M e^{2\Theta}|\nabla_\theta\Theta|^2\,\theta\wedge(d\theta)^n$$

provided $\beta < 1$. Therefore, for fixed Θ, a suitable choice of $\beta \in (0,1)$ will lead to the condition (1.8). Finally, we note that if one choose α small enough, the condition $\|\Phi\|_N \leqslant 1$ is also satisfied.

Appendix B. Solvability of the equation $-\Delta_\theta u = f$

In this section, we provide a proof of the following result.

Proposition B.1. *Let $f \in L^2(M)$, then the equation*

$$-\Delta_\theta u = f(x) \tag{B.1}$$

in M possesses a solution if and only if $\int_M f\,\theta\wedge(d\theta)^n = 0$.

It is clear that the above proposition is well-known in the context of Riemannian manifolds, however, we find no such a result for CR manifolds. The proof we provide here is adapted from the context of Riemannian manifolds.

Proof of Proposition B.1. Clearly, the "only if" is obvious by simply integrating both sides of (B.1). To prove the "if" part, we consider the functional

$$\Xi(u) = \frac{1}{2}\int\limits_M |\nabla_\theta u|^2\,\theta\wedge(d\theta)^n - \int\limits_M f(x)u\,\theta\wedge(d\theta)^n$$

under the constraint

$$G = \Big\{u \in \mathbf{S}_1^2(M) : \int\limits_M u\,\theta\wedge(d\theta)^n = 0\Big\}.$$

In view of the Friedrichs inequality, we can use in G the equivalent norm

$$\|u\|_G = \Bigg(\int\limits_M |\nabla_\theta u|^2\,\theta\wedge(d\theta)^n\Bigg)^{1/2},$$

which tells us that there is a positive constant C such that $\|u\|_{L^2} \leqslant C\|u\|_G$ for all $u \in G$. Using the Hölder and Poincaré inequalities, we obtain

$$\begin{aligned}\Xi(u) &= \frac{1}{2}\|u\|_G^2 - \|f\|_{L^2}\|u\|_{L^2} \\ &\geqslant \frac{1}{2}\|u\|_G^2 - C\|f\|_{L^2}\|u\|_G \\ &\geqslant \frac{1}{2}\big(\|u\|_G - C\|f\|_{L^2}\big)^2 - \frac{C^2}{2}\|f\|_{L^2}^2 > -\infty.\end{aligned}$$

Thus Ξ is bounded from below in the set G. Then, we can define

$$\xi = \inf_{u\in G} \Xi(u)$$

which is finite. Let $(u_k)_k \subset G$ be a minimizing sequence for ξ. Since f is fixed, the previous estimate implies that $(u_k)_k$ is bounded in $\mathbf{S}_1^2(M)$. A standard argument shows that there is some $u \in \mathbf{S}_1^2(M)$ such that u_k converges to u strongly in $L^1(M)$, $L^2(M)$ and weakly in $\mathbf{S}_1^2(M)$. Then it is easy to show that u weakly solves

$$-\Delta_\theta u = f(x) + \lambda$$

for some constant $\lambda \in \mathbb{R}$. By simply integrating both sides of this equation and using the condition $\int_M f\,\theta \wedge (d\theta)^n = 0$ we conclude that $\lambda = 0$ and this completes the present proof. □

Appendix C. The method of sub- and super-solutions on CR manifolds

In this section, we recover the method of sub- and super-solutions for Eq. (1.4) in the context of compact strictly pseudo-convex CR manifolds. We say that $\bar{u} \in \mathbf{S}_1^2(M)$ (resp. $\underline{u} \in \mathbf{S}_1^2(M)$) is a (weak) super-solution (resp. a weak sub-solution) of Eq. (1.4) if

$$\int_M \langle\nabla_\theta \bar{u}, \nabla_\theta v\rangle_\theta\, \theta \wedge (d\theta)^n + R\int_M \bar{u} v\, \theta \wedge (d\theta)^n \geqslant \int_M f\bar{u}^{1+2/n} v\, \theta \wedge (d\theta)^n$$

for each $v \geqslant 0$, $v \in \mathbf{S}_1^2(M)$ (resp.

$$\int_M \langle\nabla_\theta \underline{u}, \nabla_\theta v\rangle_\theta\, \theta \wedge (d\theta)^n + R\int_M \underline{u} v\, \theta \wedge (d\theta)^n \leqslant \int_M f\underline{u}^{1+2/n} v\, \theta \wedge (d\theta)^n$$

for each $v \geqslant 0$, $v \in \mathbf{S}_1^2(M)$). Following is the main result in this section.

Proposition C.1. *Assume that there exist a weak super-solution $\bar{u}$ and a weak sub-solution $\underline{u}$ of (1.4) satisfying $\underline{u} \leqslant \bar{u}$ a.e. in M, then there exists a weak solution u of (1.4) such that*

$$\underline{u} \leqslant u \leqslant \bar{u}$$

a.e. in M.

Proof. We fix a number $\lambda > 0$ sufficiently large such that the mapping

$$t \mapsto (\lambda - R)t + ft^{1+2/n}$$

is non-decreasing. For simplicity, we set $u_0 = \underline{u}$ and inductively define $u_{k+1} \in \mathbf{S}_1^2(M)$ with $k \geqslant 0$ to be the unique solution of the following problem

$$-\Delta_\theta u_{k+1} + \lambda u_{k+1} = (\lambda - R)u_k + f u_k^{1+2/n}. \tag{C.1}$$

We claim that the sequence $(u_k)_{k\geqslant 0}$ is non-decreasing in M. Indeed, using (C.1) with $k = 0$, we obtain

$$\begin{aligned}
&\int_M \langle \nabla_\theta u_1, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + \lambda \int_M u_1 v \, \theta \wedge (d\theta)^n \\
&\qquad = (\lambda - R) \int_M u_0 v \, \theta \wedge (d\theta)^n + \int_M f u_0^{1+2/n} v \, \theta \wedge (d\theta)^n \\
&\qquad \geqslant \lambda \int_M u_0 v \, \theta \wedge (d\theta)^n + \int_M \langle \nabla_\theta u_0, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n
\end{aligned}$$

for each $0 \leqslant v \in \mathbf{S}_1^2(M)$. Setting $v = (u_0 - u_1)^+ \in \mathbf{S}_1^2(M)$, we obtain

$$\int_{\{u_0 \geqslant u_1\}} |\nabla_\theta (u_0 - u_1)|^2 \, \theta \wedge (d\theta)^n + \lambda \int_{\{u_0 \geqslant u_1\}} (u_0 - u_1)^2 \, \theta \wedge (d\theta)^n \leqslant 0,$$

which immediately implies $u_0 \leqslant u_1$ a.e. in M. We now assume inductively that

$$u_{k-1} \leqslant u_k$$

a.e. in M, we claim that $u_k \leqslant u_{k+1}$ also holds a.e. in M. To this purpose, we notice from (C.1) and the monotonicity of the mapping $t \mapsto (\lambda - R)t + ft^{1+2/n}$ that

$$\begin{aligned}
&\int_M \langle \nabla_\theta u_{k+1}, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + \lambda \int_M u_{k+1} v \, \theta \wedge (d\theta)^n \\
&\qquad = (\lambda - R) \int_M u_k v \, \theta \wedge (d\theta)^n + \int_M f u_k^{1+2/n} v \, \theta \wedge (d\theta)^n \\
&\qquad \geqslant (\lambda - R) \int_M u_{k-1} v \, \theta \wedge (d\theta)^n + \int_M f u_{k-1}^{1+2/n} v \, \theta \wedge (d\theta)^n \\
&\qquad = \lambda \int_M u_k v \, \theta \wedge (d\theta)^n + \int_M \langle \nabla_\theta u_k, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n
\end{aligned}$$

for each $0 \leqslant v \in \mathbf{S}_1^2(M)$. In particular, there holds

$$\int_M \langle \nabla_\theta (u_k - u_{k+1}), \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + \lambda \int_M (u_k - u_{k+1}) v \, \theta \wedge (d\theta)^n \leqslant 0.$$

By using the test function $v = (u_k - u_{k+1})^+ \in \mathbf{S}_1^2(M)$, we easily conclude that $u_k \leqslant u_{k+1}$ a.e. in M. Finally, we prove that $u_k \leqslant \bar{u}$ for all k. Clearly, this holds for $k = 0$ by our assumption. By induction, we assume that $u_k \leqslant \bar{u}$ and our aim is to prove that $u_{k+1} \leqslant \bar{u}$ also holds a.e. in M. Using (C.1) and by means of the super-solution $\bar{u}$, we find that

$$\begin{aligned}
&\int_M \langle \nabla_\theta u_{k+1}, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + \lambda \int_M u_{k+1} v \, \theta \wedge (d\theta)^n \\
&\qquad = (\lambda - R) \int_M u_k v \, \theta \wedge (d\theta)^n + \int_M f u_k^{1+2/n} v \, \theta \wedge (d\theta)^n \\
&\qquad \leqslant (\lambda - R) \int_M \bar{u} v \, \theta \wedge (d\theta)^n + \int_M f \bar{u}^{1+2/n} v \, \theta \wedge (d\theta)^n \\
&\qquad \leqslant \lambda \int_M \bar{u} v \, \theta \wedge (d\theta)^n + \int_M \langle \nabla_\theta \bar{u}, \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n
\end{aligned}$$

for each $0 \leqslant v \in \mathbf{S}_1^2(M)$. In particular,

$$\int_M \langle \nabla_\theta (u_{k+1} - \bar{u}), \nabla_\theta v\rangle_\theta \, \theta \wedge (d\theta)^n + \lambda \int_M (u_{k+1} - \bar{u}) v \, \theta \wedge (d\theta)^n \leqslant 0.$$

Using the test function $v = (u_{k+1} - \bar{u})^+ \in \mathbf{S}_1^2(M)$ we conclude that $u_{k+1} \leqslant \bar{u}$ a.e. in M. Hence, we have just shown that the sequence $(u_k)_k$ obeys

$$\underline{u} \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant u_k \leqslant \cdots \leqslant \bar{u}$$

a.e. in M. Therefore, the pointwise limit

$$u_\infty(x) = \lim_{k \to +\infty} u_k(x)$$

exists a.e. in M. By the Sobolev embedding $\mathbf{S}_1^2(M) \hookrightarrow L^2(M)$ and the fact that $\bar{u} \in \mathbf{S}_1^2(M)$, $u_k \to u_\infty$ in $L^2(M)$ by the Dominated Convergence Theorem. Standard L^p-estimates [10, Theorem 3.16 (2)] implies from (C.1) that

$$\|u_{k+1}\|_{\mathbf{S}_2^2(M)} \leqslant C\Big(|R| \|u_k\|_{L^2} + (\sup_M f) \big\|u_k^{1+2/n}\big\|_{L^2}\Big)$$

since (C.1) always admits a unique solution. Since $u_k \in [\underline{u}, \bar{u}]$, we have $\|u_k\|_{L^2}$ and $\|u_k^{1+2/n}\|_{L^2}$ are uniformly bounded. Hence, the sequence $(u_k)_k$ is bounded in $\mathbf{S}_2^2(M)$. Since the embedding

$\mathbf{S}_2^2(M) \hookrightarrow \mathbf{S}_1^2(M)$ is compact, $u_k \to u_\infty$ strongly in $\mathbf{S}_1^2(M)$. Thus u_∞ is a weak solution of (1.4) as claimed. $\square$

References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.

[2] A. Bahri, J.M. Coron, The scalar curvature problem on the standard three-dimensional sphere, J. Funct. Anal. 95 (1991) 106–172.

[3] M. Ben Ayed, M. Ould Ahmedou, Multiplicity results for the prescribed scalar curvature on low spheres, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) VII (2008) 1–26.

[4] S.A. Chang, M.J. Gursky, P. Yang, The scalar curvature equation on 2- and 3-spheres, Calc. Var. Partial Differential Equations 1 (1993) 205–229.

[5] S.A. Chang, P. Yang, Prescribing Gaussian curvature on S^2, Acta Math. 159 (1987) 215–259.

[6] S.A. Chang, P. Yang, Conformal deformation of metrics on S^2, J. Differential Geom. 27 (1988) 256–296.

[7] S.A. Chang, P. Yang, A perturbation result in prescribing scalar curvature on S^n, Duke Math. J. 64 (1991) 27–69.

[8] X.Z. Chen, X. Xu, The scalar curvature flow on S^n-perturbation theorem revisited, Invent. Math. 187 (2012) 395–506.

[9] H. Chtioui, M. Ould Ahmedou, R. Yacoub, Topological methods for the prescribed Webster Scalar Curvature problem on CR manifolds, Differential Geom. Appl. 28 (2010) 264–281.

[10] S. Dragomir, G. Tomassini, Differential Geometry and Analysis on CR Manifolds, Progress in Mathematics, vol. 246, Birkhäuser, Boston, 2006.

[11] V. Felli, F. Uguzzoni, Some existence results for the Webster scalar curvature problem in presence of symmetry, Ann. Mat. Pura Appl. 183 (2004) 469–493.

[12] R.L. Frank, E.H. Lieb, Sharp constants in several inequalities on the Heisenberg group, Ann. of Math. 176 (2012) 349–381.

[13] N. Gamara, The CR Yamabe conjecture the case $n = 1$, J. Eur. Math. Soc. 3 (2001) 105–137.

[14] N. Gamara, R. Yacoub, CR Yamabe conjecture the conformally flat case, Pacific J. Math. 201 (2001) 121–175.

[15] P.T. Ho, Results related to prescribed pseudo-Hermitian curvature, Int. J. Math. 24 (2013) 29pp.

[16] D. Jerison, J.M. Lee, The Yamabe problem on CR manifolds, J. Differential Geom. 25 (1987) 167–197.

[17] D. Jerison, J.M. Lee, Intrinsic CR normal coordinates and the CR Yamabe problem, J. Differential Geom. 29 (1989) 303–343.

[18] D. Jerison, J.M. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. Amer. Math. Soc. 1 (1988) 1–13.

[19] J. Kazdan, F. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geom. 10 (1975) 113–134.

[20] Y.Y. Li, Prescribing scalar curvature on S^n and related topics. I, J. Differential Equations 120 (1995) 319–410.

[21] Y.Y. Li, Prescribing scalar curvature on S^n and related problems. II. Existence and compactness, Comm. Pure Appl. Math. 49 (1996) 437–477.

[22] A. Malchiodi, The scalar curvature problem on S^n: an approach via Morse theory, Calc. Var. Partial Differential Equations 14 (2002) 429–445.

[23] A. Malchiodi, F. Uguzzoni, A perturbation result for the Webster scalar curvature problem on the CR sphere, J. Math. Pures Appl. 81 (2002) 983–997.

[24] Q.A. Ngô, The Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds, 2012, PhD thesis, available online at http://scholarbank.nus.edu.sg/handle/10635/36344.

[25] Q.A. Ngô, X. Xu, Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds, Adv. Math. 230 (2012) 2378–2415.

[26] Q.A. Ngô, X. Xu, Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds in the null case, Comm. Math. Phys. 334 (2015) 193–222.

[27] Q.A. Ngô, X. Xu, Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds in the positive case, Bull. Inst. Math. Acad. Sin. (N.S.) 9 (2014) 451–485.

[28] R. Schoen, D. Zhang, Prescribed scalar curvature on the n-sphere, Calc. Var. Partial Differential Equations 4 (1996) 1–25.

[29] T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on compact manifolds. II, Indiana Univ. Math. J. 40 (1991) 1083–1141.

[30] T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds, Trans. Amer. Math. Soc. 331 (1992) 503–527.
[31] A. Rauzy, Courbures scalaires des variétés d'invariant conforme négatif, Trans. Amer. Math. Soc. 347 (1995) 4729–4745.
[32] M. Riahi, N. Gamara, Multiplicity results for the prescribed Webster scalar curvature on the three CR sphere under flatness condition, Bull. Sci. Math. 136 (2012) 72–95.